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Connectedness of the Tannakian group attached to semi-stable multiple filtrations

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Abstract - We show that the affine group scheme whose category of finite dimensional representations is equivalent to a tensor category of finite dimensional vector spaces equipped with semi-stable (multiple) filtrations of slope zero is connected.

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1. Introduction

Let K be an arbitrary field, L a separable algebraic closure of K, and $\mathfrak{M}$ any set of indices. We denote by $\mathcal{C}(K, L, \mathfrak{M})$ the tensor category of finite dimensional vector spaces over K equipped with multiple filtrations over L indexed by the set $\mathfrak{M}$. We write $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ for the full subcategory of $\mathcal{C}(K, L, \mathfrak{M})$ composed of semi-stable objects of slope zero and of zero objects. We quickly recall the definitions [6, Definition 1.9 & Definition 1.16] of $\mathcal{C}(K, L, \mathfrak{M})$ and $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.

For a finite dimensional vector space V over K, a family of subspaces F^iV ($i \in \mathbb{R}$) over L of $L \otimes_K V$ is called a filtration over L of V if the relations

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$$F^iV \supset F^jV \ (i \le j), \qquad \bigcup_{i\in\mathbb{R}} F^iV = L\otimes_K V,$$

$$\bigcap_{i\in\mathbb{R}} F^iV = 0, \qquad F^iV = \bigcap_{j<i} F^jV \tag{1}$$

are enjoyed. An object of $\mathcal{C}(K, L, \mathfrak{M})$ is a finite dimensional vector space V over K equipped with a family of filtrations $F_v^{\cdot}V$ $(v \in \mathfrak{M})$ over L of V such that for except a finite number of indices v, the filtrations are trivial:

$$F_v^iV = \begin{cases} L\otimes_K V & (i \le 0) \\ 0 & (i > 0) \end{cases} \tag{2}$$

A morphism between two objects of $\mathcal{C}(K, L, \mathfrak{M})$ is a linear map over K between the underlying vector spaces over K which respects all their filtrations when linearly extended over L. For two objects $\left(V, \left(F_v^{\cdot}V\right)_{v\in\mathfrak{M}}\right)$ and $\left(W, \left(F_v^{\cdot}W\right)_{v\in\mathfrak{M}}\right)$, their tensor product is the vector space $V\otimes_K W$ over K equipped with filtrations over L

$$F_v^i(V\otimes_K W) = \sum_{j+q=i} F_v^jV\otimes_L F_v^qW \quad (i\in\mathbb{R}). \tag{3}$$

The category $\mathcal{C}(K, L, \mathfrak{M})$ thus defined is a K-linear additive tensor category.

For a non-zero object $\left(V, \left(F_v^{\cdot}V\right)_{v\in\mathfrak{M}}\right)$ (below, we call it V for short) of $\mathcal{C}(K, L, \mathfrak{M})$, the slope $\mu(V) = \mu\left(V, \left(F_v^{\cdot}V\right)_{v\in\mathfrak{M}}\right)$ is a real number given by

$$\mu(V) = \sum_{v\in\mathfrak{M}} \frac{1}{\dim_K V} \sum_{w\in\mathbb{R}} w \dim_L \mathrm{gr}^w\left(F_v^{\cdot}V\right), \tag{4}$$

where $\mathrm{gr}^w\left(F_v^{\cdot}V\right) = F_v^wV/F_v^{w+}V$, $F_v^{w+}V = \cup_{j>w}F_v^jV$. A non-zero object V of $\mathcal{C}(K, L, \mathfrak{M})$ is semi-stable if it satisfies the condition that for any monomorphism $W\to V$ in $\mathcal{C}(K, L, \mathfrak{M})$ of a non-zero object W, we have $\mu(W) \le \mu(V)$.

A fundamental fact about the full subcategory $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ of $\mathcal{C}(K, L, \mathfrak{M})$ consisting of semi-stable objects of slope zero and of zero objects is the following:

THEOREM 1.1 (Faltings [4], Totaro [7], cf. André [1]). *Let $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ be the forgetful functor of $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ onto the tensor category of finite dimensional vector spaces over K. The category $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is canonically equivalent to the tensor category of finite dimensional representations over K of an affine group scheme* Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ *over K of natural equivalences of the functor $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.*

In our previous paper [6], we have particularly shown that any connected reductive group over K appears (up to isomorphism) in many ways as a quotient group scheme of the affine group scheme Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ when the cardinality of $\mathfrak{M}$ is infinite. Denoting by $\omega_K(G)$ the forgetful tensor functor of the tensor category $\mathrm{Rep}_K(G)$ of finite dimensional representations over K of an affine group scheme G over K onto the tensor category of finite dimensional vector spaces over K, this result is differently stated that when G is connected reductive, there exists a fully faithful tensor functor $\imath$ of $\mathrm{Rep}_K(G)$ into $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ such that $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M}) \circ \imath = \omega_K(G)$.

In our present paper, we prove the next:

THEOREM 1.2. *Let $\imath$ be any tensor functor of* $\mathrm{Rep}_K(G)$ *to* $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ *such that $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M}) \circ \imath = \omega_K(G)$. If the group scheme G over K is finite, then for any object V of* $\mathrm{Rep}_K(G)$*, the image $\imath(V)$ must be an object with all the filtrations trivial. In particular, the functor $\imath$ cannot be full unless $G = 1$.*

Since the group scheme $\pi_0\big(\mathrm{Aut}\ \omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})\big)$ of connected components of the affine group scheme Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is pro-étale (cf., e.g., [3, III, § 3, 7.7]), our theorem implies the following:

COROLLARY 1.3. *The affine group scheme Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ over K is connected.*

In Section 2, a proof of Theorem 1.2 is given for an arbitrary (finite or infinite) GALOIS extension L. It might be useful and natural to bring in a finite étale K-algebra L. But we would like in the present paper to stick to our original setting [5, 6] bearing Diophantine approximation in mind. A little deviation is that the index set $\mathfrak{M}$ is any non-empty set. In Section 3, we make a few observations on the problem to find all (algebraic) quotients of Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.

2. Finite groups

Let K be an arbitrary field, G a finite group scheme over K, and $K[G]$ the K-algebra of global functions on G. Note that as G is finite over K, the group scheme G is affine (over K). The key to the result of the present paper is that the dual vector space $K[G]^*$ over K of $K[G]$ is a *finite* dimensional representation of G by (left) translation. An immediate well-known consequence is the following:

Lemma 2.1. *Each finite dimensional representation over K of G is (canonically) a quotient representation of a finite direct sum of copies of* $K[G]^*$.

We recall its proof for the convenience of readers.

Proof. Let V be a finite dimensional representation over K of G. By the very definition of representation, the comorphism

$$V^* \hookrightarrow K[G] \otimes_K V^*$$

of the action of G on V is an injective comorphism between representation spaces when the tensor product $K[G]^* \otimes_K V$ is regarded as a representation with the trivial action on V. This tells us that the representation V is a quotient of $K[G]^* \otimes_K V$ which is isomorphic to a finite direct sum of copies of $K[G]^*$. □

Let L be a (finite or infinite) Galois extension field of K and $\mathfrak{M}$ any non-empty set of indices. For finite dimensional vector spaces over K equipped with descending exhaustive separated left-continuous (multiple) filtrations defined over L (called (multiple) filtrations over L for simplicity [2][6], cf. (1) in Section 1), we consider the following quantities:

Definition 2.2. *Let V be a finite dimensional non-zero vector space over K equipped with a family of filtrations $F_v^{\cdot}V$ over L indexed by $\mathfrak{M}$. Writing* $\mathrm{gr}^{\cdot}$ *for the graduation derived from a filtration, we set for all $v \in \mathfrak{M}$*

$$m_v(V) = \min\{w \in \mathbb{R} \mid \mathrm{gr}^w(F_v^{\cdot}V) \neq 0\}.$$

Remember that when saying multiple filtrations, we are tacitly assuming for except a finite number of indices $v \in \mathfrak{M}$, the filtrations $F_v^{\cdot}V$ are trivial in the sense of (2) *in Section* 1. *In particular, we see that $m_v(V) = 0$ for almost all $v \in \mathfrak{M}$. We put*

$$m(V) = \sum_{v \in \mathfrak{M}} m_v(V).$$

Remark 2.3. By the definition of slopes μ_v ($v \in \mathfrak{M}$) of filtrations [6, Definition 1.12], we have for each $v \in \mathfrak{M}$

$$m_v(V) \leq \frac{1}{\dim_K V} \sum_{w \in \mathbb{R}} w \dim_L \mathrm{gr}^w(F_v^{\cdot}V) = \mu_v(V)$$

and

$$m(V) \le \sum_{v \in \mathfrak{M}} \mu_v(V) = \mu(V).$$

For finite dimensional vector spaces V and W over K equipped with multiple filtrations over L indexed by $\mathfrak{M}$, the direct sum of filtrations was defined as

$$F_v^i(V \oplus W) = F_v^i V \oplus F_v^i W \quad (i \in \mathbb{R},\ v \in \mathfrak{M}).$$

We obtain

$$m_v(V \oplus W) = \min\{m_v(V), m_v(W)\} \quad (v \in \mathfrak{M}),$$

hence

(5) $$m_v(V \oplus \cdots \oplus V) = m_v(V).$$

From the definition of the tensor product of filtrations ((3) in Section 1), we get

$$m_v(V \otimes W) = m_v(V) + m_v(W) \quad (v \in \mathfrak{M}),$$

in particular,

(6) $$m_v(V^{\otimes n}) = n \cdot m_v(V).$$

LEMMA 2.4. *Let V and W be finite dimensional non-zero vector spaces over K equipped with multiple filtrations over L indexed by $\mathfrak{M}$. If there is a filtered homomorphism of V to W which is surjective as a linear map between vector spaces, then we have $m_v(V) \le m_v(W)$.*

PROOF. Call f the filtered homomorphism in the statement. By the definition of filtered homomorphism, we have

$$f(L \otimes_K V) = f\big(F_v^i V\big) \subset F_v^i W \subset L \otimes_K W \quad (i \le m_v(V)).$$

Since f is surjective, we see

$$F_v^i W = L \otimes_K W \quad (i \le m_v(V)),$$

which means $m_v(W) \ge m_v(V)$. □

Let $\mathrm{Rep}_K(G)$ be the tensor category of finite dimensional representations over K of G, Vec_K the tensor category of finite dimensional vector spaces over K, and $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ the tensor category of finite dimensional vector spaces over K equipped with semi-stable multiple filtrations over L indexed by $\mathfrak{M}$ of slope zero ([6, Definition 1.16], cf. Section 1). We denote respectively by $\omega_K(G)$ and by $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ the forgetful tensor functors of $\mathrm{Rep}_K(G)$ and of $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ to Vec_K.

Proof of Theorem 1.2 (for general L). We may suppose V is not zero. Let n be an arbitrary positive integer. By Lemma 2.1, there exists a surjective G-homomorphism f onto the n-times tensor product $V^{\otimes n}$ of V of a G-representation W which is isomorphic to a finite direct sum of copies of $K[G]^*$. The assumption $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M}) \circ \imath = \omega_K(G)$ says that the morphism $\imath(f)$ as a linear map between vector spaces is f itself. Due to Lemma 2.4, we see $m_v(W) \leq m_v(V^{\otimes n})$. On the other hand, thanks to (5) and (6), we have $m_v(W) = m_v(K[G]^*)$ and $m_v(V^{\otimes n}) = n \cdot m_v(V)$. Hence we get

$$\frac{1}{n}\, m_v(K[G]^*) \leq m_v(V).$$

Making n large, we obtain

$$0 \leq m_v(V).$$

As $\mu(V) = 0$ by the definition of $\imath$, Remark 2.3 forces

$$m(V) = \mu(V),$$

which is possible only when all the filtrations are trivial. □

Remark 2.5. In general, the condition $m(V) = \mu(V)$ is not sufficient to assure that the filtrations are trivial. For example, let $\mathfrak{M} = \{0, \infty\}$, $V = K$,

$$F_0^i V = \begin{cases} L \otimes_K V & (i \leq -1) \\ 0 & (i > -1) \end{cases},$$

and

$$F_\infty^i V = \begin{cases} L \otimes_K V & (i \leq 1) \\ 0 & (i > 1) \end{cases}.$$

We have $m_0(V) = \mu_0(V) = -1$, $m_\infty(V) = \mu_\infty(V) = 1$, and $m(V) = \mu(V) = 0$. The point of the proof of Theorem 1.2 is that we can show $m_v(V) \geq 0$ for all $v \in \mathfrak{M}$.

3. Algebraic groups which do not or do appear

Let K be an arbitrary field, L a (finite or infinite) GALOIS extension, and $\mathfrak{M}$ any non-empty set of indices. We consider in this section a connected linear algebraic group G over K in general.

PROPOSITION 3.1. *Let $\imath$ be any tensor functor of* $\mathrm{Rep}_K(G)$ *to* $\mathcal{C}_0^{\mathrm{ss}}(K,L,\mathfrak{M})$ *such that* $\omega_0^{\mathrm{ss}}(K,L,\mathfrak{M})\circ\imath=\omega_K(G)$. *If the group scheme G over K is* unipotent, *then for any object V of* $\mathrm{Rep}_K(G)$, *the image* $\imath(V)$ *must be an object with all the filtrations trivial.*

PROOF. Denote by U a one-dimensional trivial representation space over K of G. Since $\imath(U)\otimes\imath(U)\simeq\imath(U\otimes U)\simeq\imath(U)$, the filtrations of $\imath(U)$ must be trivial, i.e., unit objects go to unit objects.

Let V be any non-zero finite dimensional representation over K of G. On the assumption that G is unipotent, there exists an injective G-homomorphism of U into V. The other assumption $\omega_0^{\mathrm{ss}}(K,L,\mathfrak{M})\circ\imath=\omega_K(G)$ implies that the functor $\imath$ sends kernels to kernels and cokernels to cokernels (cf. [5, Lemma 1.8]). We have particularly an exact sequence

$$0\to\imath(U)\to\imath(V)\to\imath(V/U)\to 0.$$

When the filtrations of $\imath(V/U)$ are trivial, we see readily that those of $\imath(V)$ are also trivial. By the induction on the dimensions of representation spaces, we are done. □

COROLLARY 3.2. *If G is isomorphic to a quotient of the affine group scheme Aut* $\omega_0^{\mathrm{ss}}(K,L,\mathfrak{M})$, *then semi-simple elements generate a dense subgroup of G.*

PROOF. Let N be the (ZARISKI) closure of the subgroup generated by semi-simple elements of G. The variety N is a (closed) normal subgroup defined over K of G. By definition, all the elements of the quotient group G/N are unipotent, hence G/N is unipotent. Since G/N is isomorphic to a quotient of Aut $\omega_0^{\mathrm{ss}}(K,L,\mathfrak{M})$, Proposition 3.1 means that $G/N=1$. □

REMARK 3.3. When L is a separable closure of the base field K and $\mathfrak{M}$ is infinite, we have shown in the appendix of our previous paper [6] that any affine algebraic group scheme a dense subgroup of which is generated by tori defined over K appears (up to isomorphism) as a quotient of Aut $\omega_0^{\mathrm{ss}}(K,L,\mathfrak{M})$. Applying the method of [6] to tori defined over L, we observe that any affine algebraic group which fills the necessity in Corollary 3.2 really appears (up to isomorphism) as a quotient of Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$. In this way, Corollary 3.2 presents a kind of characterization for linear algebraic groups which can be isomorphic to quotients of Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$.

REMARK 3.4. Let S be the linear algebraic group of upper triangular matrices of degree 2 with determinant 1. The group S is defined over an arbitrary field K, solvable, and generated by two split tori. According to [6, Theorem A.14], the solvable group S appears (up to isomorphism) as a quotient of Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ provided at least the cardinality of the index set $\mathfrak{M}$ is greater than three. Thus the affine group scheme Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is not pro-reductive and the Tannakian category $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ is not poly-stable in that case.

On the other hand, if the cardinality of $\mathfrak{M}$ is one, then the group S cannot be isomorphic to any quotient of Aut $\omega_0^{\mathrm{ss}}(K, L, \mathfrak{M})$, because the one-dimensional multiplicative group $\mathbb{G}_{\mathrm{m}}$ is isomorphic to a quotient of S but one-dimensional objects of $\mathcal{C}_0^{\mathrm{ss}}(K, L, \mathfrak{M})$ are all units in this case.

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REFERENCES

[1] Y. ANDRÉ, *On nef and semistable hermitian lattices, and their behaviour under tensor product*. Tôhoku Math. J., 63(4): pp. 629–649, 2011.

[2] J.-F. DAT - S. ORLIK - M. RAPOPORT, *Period Domains over Finite and p-adic Fields*, volume 183 of Cambridge Tracts in Math. Cambridge Univ. Press, New York, 2010.

[3] M. DEMAZURE - P. GABRIEL, *Groupes Algébriques*, volume 1. North-Holland Publishing Company, Amsterdam, The Netherlands, 1970.

[4] G. FALTINGS, *Mumford-Stabilität in der algebraischen Geometrie*. In *Proceedings of the International Congress of Mathematicians*, Zürich, Switzerland 1994, pp. 648–655, Basel, Switzerland, 1995. Birkhäuser Verlag.

[5] M. FUJIMORI, *On systems of linear inequalities*. Bull. Soc. Math. France, 131(1): pp. 41–57, 2003. Corrigenda. *ibid.*, 132(4): pp. 613–616, 2004.

[6] M. FUJIMORI, *The algebraic groups leading to the Roth inequalities*. J. Théor. Nombres Bordeaux, 24: pp. 257–292, 2012.

[7] B. TOTARO, *Tensor products in p-adic Hodge theory*. Duke Math. J., 83: pp. 79–104, 1996.

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