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Localizations of tensor products

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Abstract - A homomorphism $\lambda : A \to B$ between R-modules is called a localization if for all $\varphi \in Hom_R(A, B)$ there is a unique $\psi \in Hom_R(B, B)$ such that $\varphi = \psi \circ \lambda$. We investigate localizations of tensor products of torsion-free abelian groups. For example, we show that the natural multiplication map $\mu : R \otimes R \to R$ is a localization if and only if R is an E-ring.

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1. Introduction

Let R denote some ring. Many notions in the theory of R-modules can be stated in terms of some universal property. Here are some examples:

(1) The R-module F is **free** with basis B if B is a subset of F such that for any R-module X and any function $f : B \to X$ there exists a unique homomorphism $\varphi : F \to X$ such that $\varphi\restriction_B = f$.

(2) The R-module G is **small**, if for each family $\{X_i : i \in I\}$ of R-modules the abelian group $Hom_R\big(G, \bigoplus_{i\in I} X_i\big)$ is naturally isomorphic to $\bigoplus_{i\in I} Hom_R(G, X_i)$.

(3) The R-module G is **strongly slender** [9] if for each family $\{X_i : i \in I\}$ of R-modules the abelian group $Hom_R\big(\prod_{i\in I} X_i, G\big)$ is naturally isomorphic to $\bigoplus_{i\in I} Hom_R(X_i, G)$.

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(4) Let $\lambda \in Hom_R(A,B)$ be a homomorphism. We call λ a **split** homomorphism if for any R-module X and any $\varphi \in Hom_R(A,X)$ there exists some $\psi \in Hom_R(B,X)$ such that $\varphi = \psi \circ \lambda$. If one considers the case of $X = A$ and $\varphi = id_A$, then it is easy to see that λ is indeed a splitting homomorphism, i.e. λ is injective and $\lambda(A)$ is a direct summand of B.

(5) The R-module G is **injective** if for any R-module X and any submodule K of X and any $\varphi \in Hom_R(K,G)$ there exists some $\psi \in Hom_R(X,G)$ such that $\varphi = \psi\restriction_K$. Note that **"projective"** is the dual of **"injective".**

(6) Tensor products: To simplify notation, let us assume $R = \mathbb{Z}$. Let A, B and T be abelian groups and $\tau : A \times B \to T$ a bilinear map. The pair (T, τ) is a **tensor product** of A and B, if for any abelian group X and any bilinear map $\sigma : A \times B \to X$, there exists a unique $\psi \in Hom(T,X)$ such that $\sigma = \psi \circ \tau$. It is well known, of course, that tensor products exist and are unique up to isomorphism.

It would be easy to continue this list. All these definitions can be modified by restricting the X's. Let us do this. We get

(1^s) In (1), replace "X" by "F". A module F satisfying (1^s) is called **self-free** with basis B, c.f. [5].

(2^s) In (2), replace all the "X_i" by "G". Such a module G is called **self-small**. This notion has been studied by many authors, c.f. [3] and the literature referenced there.

(3^s) In (3), replace all the "X_i" by "G". Such a module is called **strongly self-slender**, c.f. [9].

(4^s) In (4) replace "X" by "B" and add the condition that the map ψ is unique. Such a homomorphism $\lambda : A \to B$ is called a **localization** of A. There is a large amount of literature on localizations. See for instance [6] and [7] and the papers referenced there.

(5^s) In (5), replace "X" by "G". Then G is called **quasi-injective**. If $R = \mathbb{Z}$ and K is restricted to p-pure subgroups of G, then G is called **quasi-p-pure-injective**, or qppi for short. Again, there is a lot of literature on this topic dating back to the 1970's, c.f. [1] or [13].

In this paper we focus on a specification (6^s) of (6):

Definition 1. *Let R be a ring, $A = A_R$ a right R-module, $B = {}_R B$ a left R-module and T some Abelian group. Following* [12, page 207], *we call a map $\tau : A \times B \to T$ a* ***middle linear map*** *if τ is bilinear and $\tau(ar, b) = \tau(a, rb)$ for all $a \in A, b \in B$ and $r \in R$. Let $Midlin_R(A, B; T)$ denote the set of all middle linear maps from $A \times B$ into T. We call the pair (T, τ) a* ***qutensor product*** *of A, B over R if for all $\sigma \in Midlin_R(A, B; T)$, there exists a* ***unique*** *homomorphism $\psi \in Hom_{\mathbb{Z}}(T, T)$ such that $\sigma = \psi \circ \tau$. The map $\otimes : A \times B \to A \otimes_R B$ is a middle linear map with $\otimes(a, b) = a \otimes b$ for all $a \in A, b \in B$.*

It follows from the definitions that qutensor products are a combination of the notions in (4^s) and (6):

The pair (T, τ) is a qutensor product of the modules A, B if and only if there exists a localization $\lambda : A \otimes_R B \to T$ such that $\tau = \lambda \circ \otimes$, where $\otimes : A \times B \to A \otimes_R B$ is the natural map with $\otimes(a, b) = a \otimes b$ for all $a \in A$ and $b \in B$.

In this paper we concern ourselves with localizations of tensor products of torsion-free abelian groups A, B.

After some preliminaries in Section 2, we study localizations of arbitrary direct sums of modules in Section 3. In Section 4, we look at surjective localizations and find conditions for p-reduced torsion-free abelian groups having a p-reduced tensor product. In Section 5, we use the absolute E-rings constructed in [10] and [11] to obtain absolute localizations of torsion-free abelian groups that are p-reduced for infinitely many primes p. In Section 6, we present some examples of some surprising properties of tensor products of reduced torsion-free abelian groups. For example, there exists a strongly indecomposable reduced abelian group G such that $G \otimes G$ is reduced and completely decomposable. In Section 7, we consider torsion-free abelian groups of finite rank whose p-rank is less than their rank. We find an example of such a group G of rank 4 and p-rank 2 such that $G \otimes G$ is not reduced, but G has no pure subgroups of rank at least 2 but of p-rank 1. (It is well known that $G \otimes G$ is not p-reduced if G has rank at least 2 and p-rank 1.) We also find a quasi-isomorphism invariant for such groups. In the last section, we take a glimpse at zero product determinated algebras, c.f. [4]. Let $1 \in R$ be a ring. Then there exists an epimorphism $\mu : R \otimes_{\mathbb{Z}} R \to R$ with $\mu(a \otimes b) = ab$ for all $a, b \in R$. We show that μ is a localization if and only if R is an E-ring.

2. Definitions and First Results

Recall the following, well established

DEFINITION 2. *Let M, L, X be modules over some ring and $\lambda \in Hom(M, L)$. Then $\lambda \perp X$ provided that for any $\alpha \in Hom(M, X)$ there is a* ***unique*** *$\beta \in Hom(L, X)$ such that $\alpha = \beta \circ \lambda$. If $\lambda \perp L$, then λ is called a localization of M. Abusing notations, sometimes L is called a localization of M.*

Next we show that qutensors are exactly the localizations of the ordinary tensor product of the modules.

PROPOSITION 1. *Let A_R and $_RB$ be modules and $\tau \in Midlin_R(A, B; D)$. Then τ factors through $\otimes$, i.e. there exists $\lambda : A \otimes_R B \to D$ such that $\tau = \lambda \circ \otimes$. Moreover, (D, τ) is a qutensor product of A, B if and only if $\lambda : A \otimes_R B \to D$ is a localization of $A \otimes_R B$.*

PROOF. By the universal property of the tensor product, there is a unique $\lambda \in Hom(A \otimes_R B, D)$ such that $\tau = \lambda \circ \otimes$. Assume that λ is a localization and let $\sigma : A \times B \to D$ be a bilinear map. By the universal property of the tensor product, there exists a unique map $\delta \in Hom(A \otimes_R B, D)$ such that $\sigma = \delta \circ \otimes$. Then there is a unique $\beta \in End(D)$ such that $\delta = \beta \circ \lambda$ and we have that $\sigma = \beta \circ \lambda \circ \otimes = \beta \circ \tau$. On the other hand, assume that $\sigma = \gamma \circ \tau$. We infer that $\gamma \circ \tau = \gamma \circ \lambda \circ \otimes = \beta \circ \lambda \circ \otimes$ and thus $\gamma \circ \lambda = \beta \circ \lambda$. We conclude that $\gamma = \beta$ and we have that (D, τ) is a qutensor of A, B.

For the other direction, let $\alpha \in Hom(A \otimes_R B, D)$ and put $\sigma = \alpha \circ \otimes$, a middle linear map from $A \times B$ into D. Thus there exists a unique $\beta \in End(D)$ such that $\sigma = \beta \circ \tau$ and thus $\alpha \circ \otimes = \beta \circ \lambda \circ \otimes$, which implies that $\alpha = \beta \circ \lambda$. If $\beta' \in End(D)$ is some other map with $\alpha = \beta' \circ \lambda$, then $\sigma = \alpha \circ \otimes = \beta' \circ \lambda \circ \otimes = \beta \circ \lambda \circ \otimes$ and we infer $\beta \circ \tau = \beta' \circ \tau$ and thus $\beta = \beta'$. This shows that λ is a localization. □

3. Localizations of Direct Sums of Modules

We begin with:

PROPOSITION 2. *Let $\lambda : A = \bigoplus_{i \in I} A_i \to D$ be a localization, i.e. $\lambda \perp D$. Then there exists a set $\{\gamma_i : i \in I\}$ of orthogonal idempotents in $End(D)$ such that:*

(a) *For* $D_i = \gamma_i(D)$ *we have that* $D' = \bigoplus_{i\in I} D_i \subseteq D$ *and* $\delta_i = \lambda\restriction_{A_i} \in Hom(A_i, D_i)$.

(b) $\delta_j \perp D_i$ *for all* $i,j \in I$.

(c) $\ker(\lambda) = \bigoplus_{i\in I}(A_i \cap \ker(\lambda))$.

(d) *If the index set* I *is finite, then* $D = D'$.

Proof. Define $\lambda_i : A \to D$ by $\lambda_i\restriction_{A_i} = \lambda\restriction_{A_i}$ and $\lambda_i(A_j) = \{0\}$ for all $i \neq j \in I$. Then there exist unique $\gamma_i \in End(D)$ such that

$\lambda_i = \gamma_i \circ \lambda$ for all $i \in I$.

Let $a = \sum_{j\in I} a_j \in A$ where $a_j \in A_j$. Then $(\gamma_i \circ \lambda_i)(a) = (\gamma_i \circ \lambda_i)(a_i) = (\gamma_i \circ \lambda)(a_i) = \lambda_i(a_i) = \lambda_i(a)$ and we have:

$\lambda_i = \gamma_i \circ \lambda_i$ for all $i \in I$.

Now we have $(\gamma_i^2) \circ \lambda = \gamma_i \circ (\gamma_i \circ \lambda) = \gamma_i \circ \lambda_i = \lambda_i = \gamma_i \circ \lambda$ and we infer that γ_i is an idempotent element of $End(D)$.

Let $i \neq j \in I$ and $a \in A$ as above. Then $(\gamma_i \circ \gamma_j \circ \lambda)(a) = (\gamma_i \circ \lambda_j)(a) = \gamma_i(\lambda_j(a_j)) = \gamma_i(\lambda(a_j)) = \lambda_i(a_j) = 0$. This shows that $(\gamma_i \circ \gamma_j) \circ \lambda = 0 = 0 \circ \lambda$ and it follows that $\gamma_i \circ \gamma_j = 0$ for all $i \neq j \in I$. We infer that $\{\gamma_i : i \in I\}$ is a set of orthogonal idempotents and thus $D' = \bigoplus_{i\in I} D_i$ is a submodule of D where $D_i = \gamma_i(D)$.

Let $a_i \in A_i$. Then $\lambda(a_i) = \lambda_i(a_i) = \gamma_i(\lambda_i(a_i)) \subseteq \gamma_i(D) = D_i$. This shows that $\delta_i \in Hom(A_i, D_i)$.

Note that $D = D_i \oplus \ker(\gamma_i)$ and $\gamma_j(D) \subseteq \ker(\gamma_i)$ for all $i \neq j \in I$.

To show that $\delta_i = \lambda\restriction_{A_i} : A_i \to D_i$ is orthogonal to D_j, let $\eta \in Hom(A_i, D_j)$ for some $j \in I$. Then η naturally extends to a map η' from A to D by setting the map η' equal to 0 on the other summands of A. Then there exists a unique map $\theta : D \to D$ such that $\eta' = \theta \circ \lambda$. Let $a = a_i + a^{(i)} \in A$ where $a_i \in A_i$ and $a^{(i)} \in \bigoplus_{i\neq j\in I} A_j$. Then $(\theta \circ \lambda)(a) = (\theta \circ \lambda)(a_i) + (\theta \circ \lambda)(a^{(i)}) = \eta'(a) = \eta(a_i) \in D_j$. It follows that $(\theta \circ \lambda)(a^{(i)}) = 0$ by setting $a_i = 0$. Since γ_j acts as the identity map on D_j, we get $(\gamma_j \circ \theta\restriction_{D_i}) \circ (\lambda\restriction_{A_i}) = \eta$. Let $\theta_i = \gamma_j \circ \theta\restriction_{D_i} \in Hom(D_i, D_j)$. Then $\eta = \theta_i \circ \delta_i$.

Next we need to show that the map θ_i is unique with that property. Let θ_i' be another such map, and note that $\lambda(A^{(i)}) \subseteq \ker(\gamma_i)$. Let $\theta' \in End(D)$ with $\theta'\restriction_{D_i} = \theta_i'$ and $\theta'(\ker(\gamma_i)) = \{0\}$. It follows that $\eta' = \theta' \circ \lambda$ and since $\lambda \perp D$, we infer that $\theta = \theta'$ and thus $\theta_i = \theta_i'$.

Let $a = \sum_i a_i \in \ker(\lambda)$ where $a_i \in A_i$. Then $0 = \sum_i \lambda(a_i) = \sum_i \lambda_i(a_i) = \sum_i \gamma_i(\lambda_i(a_i)) \in \bigoplus_{i\in I} D_i$ and it follows that $\lambda(a_i) = 0$ for all $i \in I$. This shows that $\ker(\lambda) \subseteq \bigoplus_{i\in I}(A_i \cap \ker(\lambda)) \subseteq \ker(\lambda)$ and (c) follows.

Now assume that I is finite. Then $id_D \circ \lambda = \sum_{i \in I} \lambda_i = \sum_{i \in I} \gamma_i \circ \lambda = \left(\sum_{i \in I} \gamma_i \right) \circ \lambda$ and it follows that $id_D = \sum_{i \in I} \gamma_i$. □

Corollary 1. *Given abelian groups D and $A = \bigoplus_{i \in I} A_i$ with I finite. Let $\lambda \in Hom(A, D)$. If $\lambda \perp D$, then $D = \bigoplus_{i \in I} D_i$ and $\delta_i = \lambda\restriction_{A_i} \in Hom(A_i, D_i)$ satisfies $\delta_i \perp D_j$ for all $i, j \in I$.*

Conversely, if $D = \bigoplus_{i \in I} D_i$ and $\lambda_i \in Hom(A_i, D_i)$ with $\lambda_i \perp D_j$ for all $i, j \in I$, then $\lambda = \bigoplus_{i \in I} \lambda_i \perp D$.

Corollary 2. *Let $A = \bigoplus_{i \in I} A_i$ with finite index set I and K a subgroup of A. The canonical map $\pi : A \to A/K$ is a localization if and only if*

(1) *The canonical maps $\pi_i : A_i \to A_i/(K \cap A_i)$ are localizations for all $i \in I$ and*
(2) *$K = \bigoplus_{i \in I} (K \cap A_i)$ and $\pi_i \perp (A_j/(K \cap A_j))$ for all $i, j \in I$.*

Proof. It follows from Proposition 2 that (1) and (2) are necessary. Suppose (1) and (2) hold. Let $\varphi \in Hom(A, A/K)$ and note that $A/K \equiv \bigoplus_{i \in I} (A_i/(A_i \cap K))$ because of (2). There exist $\varphi_{ji} : A_i \to A_j/(A_i \cap K)$ such that φ is represented by the matrix $[\varphi_{ji]}]$. For each φ_{ji} there is a unique $\psi_{ji} : A_i/(A_i \cap K) \to A_j/(A_j \cap K)$ such that $\varphi_{ji} = \psi_{ji} \circ \pi_i$. It follows from the definitions that $\psi = [\psi_{ji}] \in End(A/K)$ is the desired map. □

Corollary 3. *With the above notation, if $A_i \cong A_j$ then $D_i \cong D_j$ for any $i, j \in I$.*

Proof. Let $\sigma_{ji} : A_i \to A_j$ be an isomorphism with $\sigma_{ij} : A_j \to A_i$ the inverse map. Extend σ_{ji} to $\sigma'_{ji} : A \to A$ by setting $\sigma'_{ji}(A_\alpha) = \{0\}$ for all $i \neq \alpha \in I$. Then $\lambda \circ \sigma'_{ji} \in Hom(A, D)$ and there exists a unique $\delta_{ji} \in End(D)$ such that

(1) $\lambda \circ \sigma'_{ji} = \delta_{ji} \circ \lambda$. In a similar fashion we get
(2) $\lambda \circ \sigma'_{ij'} = \delta_{ij} \circ \lambda$.

It follows that $\gamma_j \circ \lambda = \lambda_j = (\lambda \circ \sigma'_{ji}) \circ \sigma'_{ij} = (\delta_{ji} \circ \lambda) \circ \sigma'_{ij} = \delta_{ji} \circ (\lambda \circ \sigma'_{ij}) = (\delta_{ji} \circ \delta_{ij}) \circ \lambda$. We infer

(3) $\gamma_j = \delta_{ji} \circ \delta_{ij}$ as well as $\gamma_i = \delta_{ij} \circ \delta_{ji}$. It follows that
(4) $\gamma_i \circ \delta_{ij} = \delta_{ij} \circ \delta_{ji} \circ \delta_{ij} = \delta_{ij} \circ \gamma_j$ and $\gamma_j \circ \delta_{ji} = \delta_{ji} \circ \delta_{ij} \circ \delta_{ji} = \delta_{ji} \circ \gamma_i$.

Taking the restrictions to D_j, D_i we get

(5) $\delta_{ij}\restriction_{D_j} = \gamma_i \circ \delta_{ij}\restriction_{D_j} : D_j \to D_i$ as well as $\delta_{ji}\restriction_{D_i} = \gamma_j \circ \delta_{ji}\restriction_{Di} : D_i \to D_j$.

The equations (3) now imply that $\delta_{ij}\restriction_{D_j}$and $\delta_{ji}\restriction_{D_i}$ are a pair of inverse isomorphisms and we have $D_i \cong D_j$. □

Proposition 3. *Same notations as above. If D is slender, then $D = \bigoplus_{i\in I} D_i$.*

Proof. Since λ is a localization, we have that $\prod_{i\in I} Hom(A_i, D) \cong Hom\big(\bigoplus_{i\in I} A_i, D\big) = Hom(A, D) \cong End(D)$. The latter isomorphism $^*: End(D) \to Hom(A, D)$ is given by $\varphi^* = \varphi \circ \lambda$ for all $\varphi \in End(D)$. Let $^\#$ denote the inverse of *.

Pick an element $d \in D$.

Define a map $\sigma_d : End(D) \to D$ by $\sigma_d(\varphi) = \varphi(d)$ for all $\varphi \in End(D)$. This gives rise to a homomorphism $\tau_d : \prod_{i\in I} Hom(A_i, D) \to D$. Since D is slender, there exists a cofinite subset J of I such that $\tau_d\big(\prod_{i\in J} Hom(A_i, D)\big) = \{0\}$. Note that $(\lambda_i)^\# = \gamma_i$ for all $i \in I$ and $\tau_d(\lambda_j) = 0$ for all $j \in J$. But now we have $0 = \tau_d(\lambda_j) = \gamma_j(d)$ for all $j \in J$. We infer that $\sum_{i\in I} \gamma_i(d)$ is a finite sum for all $d \in D$ and thus $\gamma = \sum_{i\in I} \gamma_i \in End(D)$ with $\gamma \circ \lambda = \lambda = id_D \circ \lambda$. It follows that $\gamma = id_D$ and thus $D = \bigoplus_{i\in I} D_i$. □

Definition 3. *Let (A, D) be a pair of abelian groups. We call this pair* ***relatively semi-rigid****, if each $0 \neq \varphi \in Hom(A, D)$ is injective.*

For example, if $A \cong \mathbb{Z}$ and D is torsion-free, then the pair (A, D) is relatively semi-rigid.

Proposition 4. *Let I be an index set and A_i, D_i be abelian groups for all $i \in I$. Moreover, let $\lambda_i \in Hom(A_i, D_i)$ for all $i \in I$ such that:*

(1) *$\lambda_i \perp D_j$ for all $i,j \in I$.*

(2) *For all $i \in I$, we have that the pair (A_i, D_j) is relatively semi-rigid for all but finitely many $j \in I$.*

Then $\lambda = \bigoplus_{i\in I} \lambda_i : A = \bigoplus_{i\in I} A_i \to D = \bigoplus_{i\in I} D_i$ is a localization, i.e. $\lambda \perp D$.

Proof. Let $\varphi \in Hom(A,D)$. Then $\varphi = [\varphi_{ji}]$ where $\varphi_{ji} \in Hom(A_i, D_j)$ and for all $i \in I$ and $a_i \in A_i$ we have $\varphi_{ji}(a_i) = 0$ for all but finitely many $j \in I$. By (2) we infer that $\varphi_{ji} = 0$ for all but finitely many $j \in I$. By (1), there exist unique $\gamma_{ji} \in Hom(D_i, D_j)$ such that $\varphi_{ji} = \gamma_{ji} \circ \lambda_i$. Note that $\gamma_{ji} = 0$ whenever $\varphi_{ji} = 0$. This implies that $\gamma = [\gamma_{ji}] \in End(D)$ and $\varphi = \gamma \circ \lambda$. The uniqueness of γ with that property follows immediately. □

Corollary 4. *Let I be an index set and A_i, D_i be abelian groups for all $i \in I$. Moreover, let $\lambda_i \in Hom(A_i, D_i)$ for all $i \in I$ such that:*

(1) *$\lambda_i \perp D_j$ for all $i,j \in I$.*

(2) *For all $i \in I$, we have that $Hom(A_i, D_j) = 0$ for all but finitely many $j \in I$.*

Then $\lambda = \bigoplus_{i \in I} \lambda_i : A = \bigoplus_{i \in I} A_i \to D = \bigoplus_{i \in I} D_i$ is a localization, i.e. $\lambda \perp D$.

4. Surjective Localization

Proposition 5. *Let B be a subgroup of the abelian group A such that:*

(1) *B is fully invariant in A.*

(2) *The natural map $End(A) \to Hom(A, A/B)$ is surjective.*

Then the canonical map $\pi : A \to A/B$ is a localization.

Proof. Let $\varphi : A \to A/B$ be a homomorphism. By (2), there is some $\theta \in End(A)$ such that $\varphi = \pi \circ \theta$. Define $\psi : A/B \to A/B$ by $\psi(a + B) = \theta(a) + B$. Note that ψ is well defined by (1) and it follows that $\varphi = \psi \circ \pi$. Since π is surjective, we infer that ψ is unique with that property. □

Let G be a torsion-free abelian group. For $a \in G$, let $\|a\|$ denote the type of $a \in G$ and $G(\tau) = \{g \in G : \|g\| \geq \tau\}$. A type $\tau = (\tau_p)_{p \in \mathbb{P}}$ is called idempotent (or ring) type, if $\tau_p \in \{0, \infty\}$ for all primes $p \in \mathbb{P}$. Moreover $\|G\|$ denotes the set of all types of the elements of G.

Proposition 6. *Let G be a torsion-free abelian group. Suppose the type τ is an element in $\|A\|$ and of ring type, i.e. for $\tau = (\tau_p)_{p \in \mathbb{P}}$ we have $\tau_p \in \{0, \infty\}$. Then $(G/G(\tau))(\tau) = \{0\}$ and for all $\varphi \in Hom(G, G/G(\tau))$ we have $G(\tau) \subseteq \ker(\varphi)$. Moreover, the natural map $\pi : G \to G/G(\tau)$ is a localization.*

PROOF. Let $x \in G$ such that $\|x + G(\tau)\| \geq \tau$. Let p be a prime such that $\tau_p = \infty$. We have that for all $n \in \mathbb{N}$, there is some $x_n \in G$ such that $p^n x_n - x = k_n \in G(\tau)$. Now k_n is p-divisible in G and it follows that x is p-divisible in G. This implies that $x \in G(\tau)$ and thus $x + G(\tau) = 0$. Note that $\varphi(G(\tau)) \subseteq (G/G(\tau))(\tau) = \{0\}$. Thus φ induces a map $\widetilde{\varphi} \in End(G/G(\tau))$ by $\widetilde{\varphi}(x + G(\tau)) = \varphi(x)$. It follows that $\varphi = \widetilde{\varphi} \circ \pi$ and $\widetilde{\varphi}$ is unique with this property because π is surjective. This shows that π is a localization. □

Let p be a prime integer. Then $\mathbb{Z}_p = \{\frac{z}{n} : z \in \mathbb{Z}, n \in \mathbb{N}, \gcd(p, n) = 1\}$ denotes the ring of integers localized at p. The torsion-free abelian group is called p-locally free if $G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module.

REMARK 1. *Let A', B' be pure subgroups of the torsion-free abelian groups A, B respectively. Then $A' \otimes B'$ is a pure subgroup of $A \otimes B$.*

To see this, note that $0 \to A' \otimes B \to A \otimes B \to (A/A') \otimes B \to 0$ is pure-exact, and $0 \to A' \otimes B' \to A' \otimes B \to A' \otimes (B/B') \to 0$ is pure-exact and purity is transitive. This shows:

PROPOSITION 7. *Let A, B be torsion-free abelian groups. Then $A \otimes B$ is p-reduced if and only if $A' \otimes B'$ is p-reduced for all pure, finite rank subgroups A', B' of A, B respectively.*

THEOREM 1. *Let A, B be torsion-free, p-reduced abelian groups such that all pure, finite rank subgroups of A are p-locally free. Then $A \otimes B$ is p-reduced.*

PROOF. If there exists a non-zero element w in $A \otimes B$ of infinite p-height, then there exists a pure, finite subgroup A' of A such that w is an element of $A' \otimes B$. Now consider the localization $(A' \otimes B)_p$ of $A' \otimes B$ at the prime p. We have $(A' \otimes B)_p \cong A'_p \otimes B_p \cong \sum_k B_p$ as $\mathbb{Z}_p$-modules since A'_p is a free $\mathbb{Z}_p$-module of some rank k. Since B is p-reduced, the $\mathbb{Z}_p$-module B_p has no elements of infinite p-height, which shows that no such element w exists. □

By an assertion of Warfield's, a torsion-free group G of finite rank m is p-locally free if and only if $m = r_p(G) := \dim_{\mathbb{Z}/p\mathbb{Z}}(G/pG)$. For the convenience of the reader, here is an outline of the proof:

Let $\{a_i + pG : 1 \leq i \leq m\}$ be a basis of G/pG and $B = \sum_{1 \leq i \leq m} a_i\mathbb{Z}$. Then $\{a_i : 1 \leq i \leq m\}$ is p-independent and B is a p-basic subgroup of G, i.e. B is a free, p-pure subgroup of G and G/B is p-divisible and a torsion group with $(G/B)[p] = \{0\}$. It follows that $G \otimes \mathbb{Z}_p = B \otimes \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module.

Corollary 5. *Let A, B be torsion-free, p-reduced abelian groups such that for all pure, finite rank subgroups A' of A the rank of A' is equal to the p-rank $r_p(A')$ of A'. Then $A \otimes B$ is p-reduced.*

5. Absolute Localizations

Let A be some algebraic structure that has some property $\mathcal{P}$. Then A has property $\mathcal{P}$ **absolutely** if A has property $\mathcal{P}$ in any generic extension of the set-theoretic universe in which A was originally constructed. Let $\kappa(\omega)$ denote the first ω-Erdös cardinal. In a remarkable paper [10], Göbel, Herden and Shelah constructed absolute E-rings R of cardinality λ for any infinite cardinal $\lambda < \kappa(\omega)$. Inspecting their proof, one realizes that the following result was shown:

Theorem 2. [10] *Let $\lambda < \kappa(\omega)$ be a cardinal and $\mathbb{Z}[X]$ (resp. $\mathbb{Q}[X]$) the polynomial ring in λ commuting variable over $\mathbb{Z}$ (resp. $\mathbb{Q}$). Then there exists a countable family $\{L_i : i < \omega\}$ of ideals of $\mathbb{Z}[X]$ such that*

(1) *Each L_i is a direct summand of the abelian group $\mathbb{Z}[X]$ and*
(2) $\{\varphi \in End_{\mathbb{Q}}(\mathbb{Q}[X]) : \varphi(\mathbb{Q}L_i) \subseteq \mathbb{Q}L_i$ *for all* $i < \omega\} = \mathbb{Q}[X]\cdot$ *absolutely.*

This version will appear in [11].

We will use this result to construct absolute localizations. First we show:

Lemma 1. *Let A be a commutative $\mathbb{Q}$-algebra and $\mathcal{F}$ a family of ideals of A such that $\{\varphi \in End_{\mathbb{Q}}(A) : \varphi(J) \subseteq J$ for all $J \in \mathcal{F}\} = A\cdot$. Let V be a $\mathbb{Q}$-vector space. Then $\{\varphi \in Hom_{\mathbb{Q}}(A, V \otimes_{\mathbb{Q}} A) : \varphi(J) \subseteq V \otimes_{\mathbb{Q}} J$ for all $J \in \mathcal{F}\} = Hom_A(A, V \otimes_{\mathbb{Q}} A)$.*

Proof. Let B be a basis of the vector space V. Let $\pi_b : V \otimes_{\mathbb{Q}} A \to b \otimes A$ be the natural projection with $\pi_b(c \otimes A) = \{0\}$ for all $b \neq c \in B$. Let $\varphi \in \{\varphi \in Hom_{\mathbb{Q}}(A, V \otimes_{\mathbb{Q}} A) : \varphi(J) \subseteq V \otimes_{\mathbb{Q}} J$ for all $J \in \mathcal{F}\}$. Then $\pi_b \circ \varphi : A \to b \otimes A \cong A$ and $(\pi_b \circ \varphi)(J) \subseteq \pi_b(V \otimes_{\mathbb{Q}} J) = b \otimes J \cong J$. Thus, by hypothesis, $(\pi_b \circ \varphi)(x) = ((\pi_b \circ \varphi)(1))x$ for all $x \in A$. It follows that $\varphi(x) = \sum_{b \in B} \pi_b(\varphi(x)) = \sum_{b \in B} ((\pi_b \circ \varphi)(1))x = \varphi(1)x$ and φ is A-linear. $\square$

Let G be a torsion-free abelian group and $P = \{p_i : i < \omega\}$ an infinite set of prime integers such that G is p-reduced for all $p \in P$. Let τ_i denote the type of the subring $\mathbb{Z}[\frac{1}{p_i}]$ of $\mathbb{Q}$. With the notations of Theorem 2, let $R = \mathbb{Z}[X] + \sum_{i<\omega} \mathbb{Z}[\frac{1}{p_i}]L_i \subseteq \mathbb{Q}[X]$. Then R is the absolute E-ring constructed in [10]. Note that $R/\mathbb{Z}[X]$ is a torsion abelian group. By (1) we have $\mathbb{Z}[X] = L_i \oplus T_i$ as abelian groups. Then $R/\mathbb{Z}[X] = \sum_{i<\omega} (\mathbb{Z}[X] + \mathbb{Z}[\frac{1}{p_i}]L_i)/\mathbb{Z}[X] = \sum_{i<\omega} \big(T_i \oplus \mathbb{Z}[\frac{1}{p_i}]L_i\big)/(T_i \oplus L_i) \cong \sum_{i<\omega} \big(\mathbb{Z}[\frac{1}{p_i}]L_i\big)/L_i$ where the i-th summand is a divisible p_i-group. This shows that $R(\tau_i) = \mathbb{Z}[\frac{1}{p_i}]L_i$ for all $i < \omega$. Now let $\varphi \in End_{\mathbb{Z}}(R)$. Then $\varphi(\mathbb{Z}[\frac{1}{p_i}]L_i\big) = \varphi(R(\tau_i)) \subseteq R(\tau_i) = \mathbb{Z}[\frac{1}{p_i}]L_i$. Let $\psi \in End_{\mathbb{Q}}(\mathbb{Q}[X])$ be the unique homomorphism induced by φ. Then $\psi(\mathbb{Q}L_i) \subseteq \mathbb{Q}L_i$ for all $i < \omega$ and Theorem 2 supplies some $d \in \mathbb{Q}[X]$ with $\psi = d\,\cdot$. Since $d = \psi(1) = \varphi(1) \in R$, we have $\varphi = d\cdot$ for some $d \in R$, i.e. R is an E-ring.

Let us call such a ring R an (GHS)-E-ring with prime number set P.

Lemma 2. *Let B be a torsion-free abelian group and R, τ_i as above. If $B(\tau_i) = \{0\}$, then $(B \otimes R)(\tau_i) = B \otimes R(\tau_i)$. (Here "$\otimes$" is understood to mean "$\otimes_{\mathbb{Z}}$".)*

Proof. Consider the short exact sequence $0 \to \mathbb{Z}[X] \to R \to R/\mathbb{Z}[X] \cong \bigoplus_{i<\omega} \Big((\mathbb{Z}[\frac{1}{p_i}]L_i)/L_i\Big) \to 0$, which gives rise to the sequence $0 \to B \otimes \mathbb{Z}[X] \to B \otimes R \to \bigoplus_{i<\omega} B \otimes \Big((\mathbb{Z}[\frac{1}{p_i}]L_i)/L_i\Big) \to 0$. Note that $\mathbb{Z}[X]$ is a free abelian group and thus $B \otimes \mathbb{Z}[X]$ is isomorphic to a direct sum of copies of B and thus has no elements of infinite p_i-height. Let $x \in (B \otimes R)(\tau_i)$, i.e. x has infinite p_i-height in $B \otimes R$. We may assume that $x \in B \otimes \mathbb{Z}[X]$. Then $\mathbb{Z}[\frac{1}{p_i}]x \subseteq B \otimes R$ and $\Big((\mathbb{Z}[\frac{1}{p_i}]x + B) \otimes \mathbb{Z}[X]\Big)/(B \otimes \mathbb{Z}[X])$ is a p_i-torsion group. We infer that $\mathbb{Z}[\frac{1}{p_i}]x \subseteq B \otimes \big(\mathbb{Z}[X] + \mathbb{Z}[\frac{1}{p_i}]L_i\big) = B \otimes \big(T_i \oplus \mathbb{Z}[\frac{1}{p_i}]L_i\big)$ with T_i free abelian. It follows that $\mathbb{Z}[\frac{1}{p_i}]x \subseteq B \otimes \mathbb{Z}[\frac{1}{p_i}]L_i$ and the claim follows. $\square$

We are now ready for the following:

THEOREM 3. *Let B be a torsion-free abelian group and P an infinite set of prime integers such that B is p-reduced for all $p \in P$. Let R be a (GHS)-E-ring with prime number set P. Then the natural map $\alpha : B \to B \otimes_{\mathbb{Z}} R$ is an absolute localization of B.*

PROOF. Note that $\alpha(x) = x \otimes 1$ for all $x \in B$. It follows from Lemma 2 that $B \otimes R$ is an E(R)-module. By Proposition 1.2 in [6] we obtain that $End_{\mathbb{Z}}(B \otimes R) = End_R(B \otimes R)$. By Proposition 1.1 in [6] the map α is a localization of B.

6. Examples

The following is a well-known and highly instructive example.

EXAMPLE 1. *Even if A is p-reduced, the tensor product $A \otimes A$ might not be p-reduced:*

Let p be prime, J_p the ring of p-adic numbers, Z_p the ring of integers localized at p, and $\pi \in J_p - Z_p$, a unit. Thus

$$\pi = \lim_{n\to\infty} z_n,$$

$z_n \in Z_p$, in the p-adic topology with $p \nmid z_n$ and $p^n | (\pi - z_n)$ in J_p.

Let $A = \langle 1, \pi \rangle_* \subseteq J_p$, the pure subgroup of J_p generated by 1 and π. Then $A \otimes A = \langle 1 \otimes 1, 1 \otimes \pi, \pi \otimes 1, \pi \otimes \pi \rangle_*$ is a pure subgroup of rank 4 of $J_p \otimes J_p$.

Since $\pi \otimes \pi - \pi \otimes z_n = \pi \otimes (\pi - z_n)$ we have $p^n | (\pi \otimes \pi - \pi \otimes z_n)$. Similarly $p^n | (\pi \otimes \pi - z_n \otimes \pi)$. Hence $p^n | [(\pi \otimes \pi - \pi \otimes z_n) - (\pi \otimes \pi - z_n \otimes \pi)] = (1 \otimes \pi - \pi \otimes 1) z_n$. Then there must exist α_n and β_n in the integers such that $1 = z_n \alpha_n + p^n \beta_n$. Hence $p^n | (1 \otimes \pi - \pi \otimes 1) z_n \alpha_n = (1 \otimes \pi - \pi \otimes 1)(1 - p^n \beta_n)$ which implies $p^n | (1 \otimes \pi - \pi \otimes 1)$ for all $n < \omega$.

Since A is a Z_p-module this means $0 \neq 1 \otimes \pi - \pi \otimes 1$ is a divisible element. Hence $A \otimes A$ is not reduced.

EXAMPLE 2. *Even if A is strongly indecomposable, the tensor product $A \otimes A$ might be completely decomposable:*

Given distinct primes p, q, r, define $A = e_1 \mathbb{Z}[\frac{1}{p}] \oplus e_2 \mathbb{Z}[\frac{1}{q}] + (e_1 + e_2)\mathbb{Z}[\frac{1}{r}]$.

Then A is strongly indecomposable of rank 2 with $End(A) = \mathbb{Z}$. We have

$$A \otimes A = b_1\mathbb{Z}[\frac{1}{p}] \oplus b_2\mathbb{Z}[\frac{1}{pq}] \oplus b_3\mathbb{Z}[\frac{1}{pq}] \oplus b_4\mathbb{Z}[\frac{1}{q}] +$$
$$+(b_1+b_3)\mathbb{Z}[\frac{1}{rp}] + (b_2+b_4)\mathbb{Z}[\frac{1}{rq}] + (b_1+b_2)\mathbb{Z}[\frac{1}{pr}] +$$
$$+(b_3+b_4)\mathbb{Z}[\frac{1}{qr}] + (b_1+b_2+b_3+b_4)\mathbb{Z}[\frac{1}{r}]$$

where $b_1 = e_1 \otimes e_1,\ b_2 = e_1 \otimes e_2,\ b_3 = e_2 \otimes e_1,\ b_4 = e_2 \otimes e_2$.

Define ρ by $\rho(b_1) = b_1 + b_2$, and $\rho(b_i) = 0$ for $2 \le i \le 4$. Then

$$\rho(A \otimes A) = (b_1+b_2)\mathbb{Z}[\frac{1}{p}] + (b_1+b_2)\mathbb{Z}[\frac{1}{rp}] +$$
$$+(b_1+b_2)\mathbb{Z}[\frac{1}{pr}] + (b_1+b_2)\mathbb{Z}[\frac{1}{r}] = (b_1+b_2)\mathbb{Z}[\frac{1}{pr}].$$

and thus $\rho(A \otimes A) \subseteq A \otimes A$, i.e. $\rho \in End(A \otimes A)$ with $\rho^2 = \rho$.

Define σ by $\sigma(b_4) = b_3 + b_4$, and $\sigma(b_i) = 0$ for $1 \le i \le 3$. Then

$$\sigma(A \otimes A) = (b_3+b_4)\mathbb{Z}[\frac{1}{qr}] + (b_3+b_4)\mathbb{Z}[\frac{1}{qr}] +$$
$$+(b_3+b_4)\mathbb{Z}[\frac{1}{q}] + (b_3+b_4)\mathbb{Z}[\frac{1}{r}] =$$
$$= (b_3+b_4)\mathbb{Z}[\frac{1}{qr}] \subseteq A \otimes A \text{ and thus}$$

$\sigma \in End(A \otimes A),\ \sigma^2 = \sigma.\ \sigma \circ \rho = \rho \circ \sigma = 0.$

Note that $(1-\rho-\sigma)(b_i) = \begin{cases} -b_2 & \text{for } i = 1 \\ b_2 & \text{for } i = 2 \\ b_3 & \text{for } i = 3 \\ -b_3 & \text{for } i = 4 \end{cases}$

and $(1-\rho-\sigma)(A \otimes A) = -b_2\mathbb{Z}[\frac{1}{p}] + b_2\mathbb{Z}[\frac{1}{pq}] + b_3\mathbb{Z}[\frac{1}{pq}] - b_3\mathbb{Z}[\frac{1}{q}] +$

$$+(-b_2+b_3)\mathbb{Z}[\frac{1}{rp}] + (b_2-b_3)\mathbb{Z}[\frac{1}{rq}] + 0 + 0 + 0 =$$
$$= b_2\mathbb{Z}[\frac{1}{pq}] + b_3\mathbb{Z}[\frac{1}{pq}] + (b_2-b_3)\mathbb{Z}[\frac{1}{pqr}] =: C.$$

Observe that $C = (b_2 - b_3)\mathbb{Z}[\frac{1}{pqr}] \oplus b_2\mathbb{Z}[\frac{1}{pq}]$, and thus

$$A \otimes A = \underbrace{(b_1 + b_2)\mathbb{Z}[\frac{1}{pr}]}_{im(\rho)} \oplus \underbrace{(b_3 + b_4)\mathbb{Z}[\frac{1}{qr}]}_{im(\sigma)} \oplus \underbrace{(b_2 - b_3)\mathbb{Z}[\frac{1}{pqr}] \oplus b_2\mathbb{Z}[\frac{1}{pq}]}_{im(1-\rho-\sigma)=C}$$ is

completely decomposable.

7. Torsion-free Abelian Groups of Finite Rank with Small p-rank

Again, let J_p denote the ring of p-adic integers and "$\otimes$" means "$\otimes_{\mathbb{Z}}$".

We view $J_p \otimes J_p$ as a right J_p-module. (Of course, $J_p \otimes J_p$ is also a left J_p-module, but we need to pick a side.)

Let D denote the divisible part of the torsion-free group $J_p \otimes J_p$. There is a natural (surjective) map $\mu : J_p \otimes J_p \to J_p$ with $\mu(a \otimes b) = ab$. Since J_p is p-reduced, we have $D \subseteq \ker(\mu)$.

Let $w = \sum_{1 \le i \le n} a_i \otimes b_i = \sum_{1 \le i \le n} (a_i \otimes 1)b_i \in \ker(\mu)$ and $v = \sum_{1 \le i \le n} (1 \otimes a_i - a_i \otimes 1)b_i$. Then $v \in D$ as shown in Example 1. Note that $0 = 1 \otimes \mu(w) = 1 \otimes \left(\sum_{1 \le i \le n} a_i b_i \right) = \sum_{1 \le i \le n} (1 \otimes a_i)b_i = w + v$. This shows that $w = -v \in D$ and thus $D = \ker(\mu)$.

We record this as:

Remark 2. *We have $J_p \otimes J_p = D \oplus (1 \otimes J_p)$ and $\mu : J_p \otimes J_p \to J_p$ is a surjective homomorphism such that $D = \ker(\mu)$ is the divisible part of $J_p \otimes J_p$.*

Let G be a torsion-free group and $\mathbb{Z}_p$ the ring of integers localized at the prime p. Then each $y \in G \otimes \mathbb{Z}_p$ has the form $y = a \otimes \frac{1}{n}$ for some $a \in G$ and integer n relatively prime to p. An easy argument shows that G is p-reduced if and only if $G \otimes \mathbb{Z}_p$ is p-reduced.

Let A be a torsion-free, p-reduced group of, say, rank 2 and p-rank 1. It is easy to see that there exists a p-pure subgroup A' of J_p such that $1 \in A'$ and $A \cong A'$.

Notation 1. *Let G be a p-reduced, torsion-free $\mathbb{Z}_p$-module of finite rank $n + m$ and $n = r_p(G) = \dim_{\mathbb{Z}/p\mathbb{Z}}(G/pG)$ for some prime p. Then G*

can be represented as a pure subgroup of $\bigoplus_{1\le i\le n} e_iJ_p$ *generated by* $B = \bigoplus_{1\le i\le n} e_i\mathbb{Z}_p$ *together with elements* $u_j = \sum_{1\le i\le n} e_i\pi_{ij}$ *for* $1 \le j \le m$ *and* $\Pi = [\pi_{ij}] \in Mat_{n\times m}(J_p)$. *We call* Π *a representing matrix for G. Of course, the matrix* Π *is not uniquely determined by G. Moreover, we may, and will, make the convention that* $\pi_{ij} = 0$ *whenever* $\pi_{ij} \in \mathbb{Z}_p$.

Fomin [8] has shown that if the entries of the matrix Π are algebraically independent (over $\mathbb{Z}_p$) and $G \otimes G$ is not p-reduced but G is p-reduced, then $r_p(G) = 1$. The following example shows that this rather strong hypothesis on Π is needed:

Example 3. *There exists a p-reduced torsion-free* $\mathbb{Z}_p$*-module G such that* $rank(G) = 4$, $r_p(G) = 2$ *and* $G \otimes G$ *is not p-reduced. Moreover, G does not contain a pure subgroup A with* $r_p(A) = 1$ *and* $rank(A) = 2$.

Proof. Pick an odd prime p and $\alpha, \beta, \gamma \in \mathbb{N}$. such that $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma} \in J_p - \mathbb{Z}_p$ and the field extensions $\mathbb{Q}[\sqrt{\beta}, \sqrt{\alpha}]$ and $\mathbb{Q}[\sqrt{\beta}, \sqrt{\gamma}]$ have dimension 4 over $\mathbb{Q}$ with $\sqrt{\gamma} \notin \mathbb{Q}[\sqrt{\beta}, \sqrt{\alpha}]$. Let $u = e_1\sqrt{\alpha} \otimes e_2\sqrt{\beta\gamma}$ and $v = e_1\sqrt{\alpha\beta} \otimes e_2\sqrt{\gamma}$. Let $G = \langle e_1\mathbb{Z}_p \oplus e_2\mathbb{Z}_p, u, v\rangle_* \subset e_1 J_p \oplus e_2 J_p$. Let $w = e_1 \otimes e_1\alpha(\beta-1) + e_2 \otimes e_2\gamma(1-\beta) + u \otimes u - v \otimes v \in G \otimes G \subset \bigoplus_{1\le i,j\le 2} J_p(e_i \otimes e_j)J_p$. We claim that

w is a p-divisible element of G:

Note that $w = \left[e_1 \otimes e_1\alpha(\beta-1) + \sqrt{\alpha}e_1 \otimes e_1\sqrt{\alpha} - \sqrt{\alpha\beta}e_1 \otimes e_1\sqrt{\alpha\beta}\,\right] +$

$+\left[\sqrt{\alpha}e_1 \otimes e_2\sqrt{\beta\gamma} - \sqrt{\alpha\beta}e_1 \otimes e_2\sqrt{\gamma}\,\right] +$

$+\left[\sqrt{\beta\gamma}e_2 \otimes e_1\sqrt{\alpha} - \sqrt{\gamma}e_2 \otimes e_1\sqrt{\alpha\beta}\,\right] +$

$+\left[e_2 \otimes e_2\gamma(1-\beta) + \sqrt{\beta\gamma}e_2 \otimes e_2\sqrt{\beta\gamma} - \sqrt{\gamma}e_2 \otimes \sqrt{\gamma}e_2\right].$

Each term in a square bracket is in the divisible part of $J_pe_i \otimes e_jJ_p$ and thus $w \neq 0$ is a divisible element of $G \otimes G$.

Now let $0 \neq g = e_1a_1 + e_2a_2 + ub + vc \in G$. We claim that $g\pi \notin G$ for all $\pi \in J_p - \mathbb{Z}_p$. This allows us to infer that G has no pure subgroup A with $rank(A) = 2$ and p-rank $r_p(A) = 1$.

Let $g' = e_1a_1' + e_2a_2' + ub' + vc' \in G$ such that $g\pi = g'$.

W.l.o.g. we may assume that all coefficients are integers. Using matrix notation, we get

$$\begin{bmatrix} a_1 + b\sqrt{\alpha} + c\sqrt{\alpha\beta} \\ a_2 + b\sqrt{\beta\gamma} + c\sqrt{\gamma} \end{bmatrix} \pi = \begin{bmatrix} a_1' + b'\sqrt{\alpha} + c'\sqrt{\alpha\beta} \\ a_2' + b'\sqrt{\beta\gamma} + c'\sqrt{\gamma} \end{bmatrix}.$$

Case 1: $a_1 + b\sqrt{\alpha} + c\sqrt{\alpha\beta} \neq 0 \neq a_2 + b\sqrt{\beta\gamma} + c\sqrt{\gamma}$.

In this case we have $\pi \in \mathbb{Q}[\sqrt{\alpha}, \sqrt{\beta}] \cap \mathbb{Q}[\sqrt{\beta}, \sqrt{\gamma}] = \mathbb{Q}[\sqrt{\beta}]$ and w.l.o.g. we may assume that $\pi = \sqrt{\beta}$ and we get

$$\begin{bmatrix} a_1 + b\sqrt{\alpha} + c\sqrt{\alpha\beta} \\ a_2 + b\sqrt{\beta\gamma} + c\sqrt{\gamma} \end{bmatrix} \sqrt{\beta} =$$

$$\begin{bmatrix} \sqrt{\beta}a_1 + b\sqrt{\alpha\beta} + c\beta\sqrt{\alpha} \\ \sqrt{\beta}a_2 + c\sqrt{\beta\gamma} + b\beta\sqrt{\gamma} \end{bmatrix} = \begin{bmatrix} a_1' + b'\sqrt{\alpha} + c'\sqrt{\alpha\beta} \\ a_2' + b'\sqrt{\beta\gamma} + c'\sqrt{\gamma} \end{bmatrix}$$ and we infer the equations:

$a_1 = 0 = a_1', b = c', c\beta = b'$ as well as

$a_2 = 0 = a_2', c = b', b\beta = c'$ and thus $\beta b = b$ and $c\beta = c$. We now have that $b = 0 = c$ and $g = 0$, a contradiction.

Case 2: $a_1 + b\sqrt{\alpha} + c\sqrt{\alpha\beta} = 0$ but $a_2 + b\sqrt{\beta\gamma} + c\sqrt{\gamma} \neq 0$ (or vice versa).

In this case we have that $a_1' = b' = c' = 0 = a_1 = b = c$ and thus $a_2\pi = a_2'$, a contradiction to $\pi \notin \mathbb{Z}_p$. ☐

A possible choice for p, α, β, γ satisfying our hypotheses would be $p = 19$, $\alpha = 7$, $\beta = 11$ and $\gamma = 17$, because these are distinct primes and $8^2 \equiv 7 \bmod 19$, $7^2 \equiv 11 \bmod 19$ and $6^2 \equiv 17 \bmod 19$. Therefore, by Hensel's Lemma, we have that $\sqrt{7}, \sqrt{11}, \sqrt{17} \in J_{19}$.

Next we will introduce a quasi-isomorphism invariant for our group G.

Let $G = \langle e_i\mathbb{Z}_p, u_j, 1 \leq i \leq n, 1 \leq j \leq k\rangle_* \subset \bigoplus_{1\leq i\leq n} e_i J_p$ be a p-reduced group of rank $n + k$ and $r_p(G) = n$ where $u_j = \sum_{1\leq i\leq n} e_i\pi_{ij}$ with $\pi_{ij} \in J_p$. Let $G' = \left\langle e_i'\mathbb{Z}_p, u_j', 1 \leq i \leq n', 1 \leq j \leq k'\right\rangle_* \subset \bigoplus_{1\leq i\leq n} e_i' J_p$ be another such group with $u_j' = \sum_{1\leq i\leq n'} e_i'\pi_{ij}'$ for $1 \leq j \leq k'$.

Assume that for all β, $1 \leq \beta \leq n'$, the sets $\Pi_\beta = \{1, \pi_{\beta j}' : 1 \leq j \leq k'\}$ are linearly independent over the field extension $K = \mathbb{Q}(\pi_{ij}, 1 \leq i \leq n, 1 \leq j \leq k)$. Then $G \otimes G'$ is p-reduced:

We have $G \otimes G' = \bigoplus_{i,j} e_i \otimes e_j'\mathbb{Z}_p + \sum_{i,j} e_i \otimes u_j'\mathbb{Z}_p + \sum_{i,j} u_i \otimes e_j'\mathbb{Z}_p + \sum_{i,j} u_i \otimes u_j'\mathbb{Z}_p$.

Let $y = \sum_{1\leq i,j\leq n} e_i \otimes e_j' a_{ij} + \sum_{i,j} e_i \otimes u_j' b_{ij} + \sum_{i,j} u_i \otimes e_j' c_{ij} + \sum_{i,j} u_i \otimes u_j' d_{ij} \in G \otimes G'$.

Note that the $e_\alpha \otimes e'_\beta$ entry of y is

$$y_{\alpha\beta} = e_\alpha \otimes e'_\beta a_{\alpha\beta} + \sum_j e_\alpha \otimes e'_\beta \pi'_{\beta j} b_{\alpha j} + \sum_i \pi_{\alpha i} e_\alpha \otimes e'_\beta c_{i\beta} + \sum_{i,j} \pi_{\alpha i}(e_\alpha \otimes e'_\beta)\pi'_{\beta j} d_{ij}.$$

If y is a divisible element, then, by Remark 2, we have

$$0 = \mu(y_{\alpha\beta}) = a_{\alpha\beta} + \sum_j \pi'_{\beta j} b_{\alpha j} + \sum_i \pi_{\alpha i} c_{i\beta} + \sum_{i,j} \pi_{\alpha i}\pi'_{\beta j} d_{ij} =$$
$$= \Big(a_{\alpha\beta} + \sum_i \pi_{\alpha i} c_{i\beta}\Big) + \sum_j \pi'_{\beta j}\Big(b_{\alpha j} + \sum_i \pi_{\alpha i} d_{ij}\Big) \text{ for all } \alpha, \beta.$$

By our hypothesis that $\{1, \pi'_{\beta j} : 1 \le j \le k'\}$ is linearly independent over the field extension $K = \mathbb{Q}\big(\pi_{ij}, 1 \le i \le n, 1 \le j \le k\big)$, we infer that $a_{\alpha\beta} + \sum_i \pi_{\alpha i} c_{i\beta} = 0$ and it follows that $\sum_i u_i c_{i\beta} \in \bigoplus_{1 \le j \le n} e_j \mathbb{Z}_p$. This is a contradiction to $n + k = rank(G)$ unless all $a_{\alpha\beta} = 0 = c_{i\beta}$.

We infer that $\sum_j \pi'_{\beta j}(b_{\alpha j} + \sum_i \pi_{\alpha i} d_{ij}) = 0$ >for all α, β and thus $b_{\alpha j} + \sum_i \pi_{\alpha i} d_{ij} = 0$ for all α, j. Again, it follows that $\sum_i u_i d_{ij} \in \bigoplus_{1 \le \gamma \le n} e_\gamma \mathbb{Z}_p$ and thus $d_{ij} = 0$ for all i, j. Now we infer that $b_{\alpha j} = 0$ for all α, j and we have that $y = 0$.

Proposition 8. *Let $G = \langle e_i \mathbb{Z}_p, u_j, 1 \le i \le n, 1 \le j \le k\rangle_* \subset \bigoplus_{1 \le i \le n} e_i J_p$ be a p-reduced group of rank $n + k$ and $r_p(G) = n$ where $u_j = \sum_{1 \le i \le n} e_i \pi_{ij}$ with $\pi_{ij} \in J_p$. Let $G' = \langle e'_i \mathbb{Z}_p, u'_j, 1 \le i \le n, 1 \le j \le k\rangle_* \subset \bigoplus_{1 \le i \le n} e_i J_p$ be another such group with $u'_j = \sum_{1 \le i \le n} e_i \pi'_{ij}$. Let K (K') be the field extension of $\mathbb{Q}$ generated by the elements π_{ij} (π'_{ij}).*

If G is quasi-isomorphic to G', then $K = K'$.

Proof. Note that $G = \Big\langle\Big(\bigoplus_{1 \le i \le n} e_i \mathbb{Z}_p\Big) \oplus \Big(\bigoplus_{1 \le i \le k} u_i \mathbb{Z}_p\Big)\Big\rangle_* \subset \bigoplus_{1 \le i \le n} e_i J_p$ where $u_i = \sum_{1 \le j \le n} e_j \pi_{ji}$ for some $\pi_{ji} \in J_p$. Put $\Pi = [\pi_{ji}] \in Mat_{n \times k}(J_p)$.

Let $\varphi : G \to G'$ be a quasi-isomorphism. Then there exist matrices $A = [a_{ij}] \in Mat_{n \times n}(\mathbb{Q}), B = [b_{ij}] \in Mat_{k \times n}(\mathbb{Q})$ such that

$\varphi(e_i) = \sum_j e'_j a_{ji} + \sum_j u'_j b_{ji}$. Now

$$\varphi(u_i) = \sum_j \varphi(e_j)\pi_{ji} = \sum_j \Big(\sum_\alpha e'_\alpha a_{\alpha j} + \sum_\beta u'_\beta b_{\beta j}\Big)\pi_{ji} =$$
$$= \sum_j \Big(\sum_\alpha e'_\alpha a_{\alpha j} + \sum_\beta \sum_\alpha (e'_\alpha \pi'_{\alpha\beta}) b_{\beta j}\Big)\pi_{ji} =$$
$$= \sum_\alpha e'_\alpha \Big(\sum_j \Big(a_{\alpha j}\pi_{ji} + \sum_\beta \pi'_{\alpha\beta} b_{\beta j}\pi_{ji}\Big)\Big) =$$
$$= \sum_\alpha e'_\alpha([A\Pi]_{\alpha i} + [\Pi' B\Pi]_{\alpha i}).$$

Since $\varphi(u_i) \in G'$ for all $1 \le i \le k$, we have that

$$\varphi(u_i) = \sum_j e'_j x_{ji} + \sum_j u'_j y_{ji} = \sum_\alpha e'_\alpha x_{\alpha i} + \sum_j \big(\sum_\alpha e'_\alpha \pi'_{\alpha j}\big) y_{ji} =$$
$= \sum_\alpha e'_\alpha \Big(x_{\alpha i} + \sum_j \pi'_{\alpha j} y_{ji}\Big)$. Let $X = [x_{\alpha i}] \in Mat_{n\times k}(\mathbb{Q})$ and

$Y = [y_{\alpha i}] \in Mat_{k\times k}(\mathbb{Q})$. We now have a matrix equation

$A\Pi + \Pi' B\Pi = X + \Pi' Y$ of $n \times k$ matrices, which is equivalent to

$\Pi'(B\Pi - Y) = X - A\Pi$ where $B\Pi - Y$ is a square $k \times k$ matrix.

Note that

$$\varphi(e_i) = \sum_j e'_j a_{ji} + \sum_j u'_j b_{ji} = \sum_\alpha e'_\alpha a_{\alpha i} + \sum_j \sum_\alpha e'_\alpha \pi'_{\alpha j} b_{ji} = \sum_\alpha e'_\alpha (A + \Pi' B)_{\alpha i}.$$

Since φ is injective, the square matrix $\begin{bmatrix} A & X \\ B & Y \end{bmatrix} \in Mat_{(n+k)\times(n+k)}(\mathbb{Q})$ is invertible.

Assume that there is some $T \in Mat_{k\times k}(J_p)$ such that $(B\Pi - Y)T = 0$. Then $0 = \Pi'(B\Pi - Y)T = (X - A\Pi)T$.

This implies that $\begin{bmatrix} A & X \\ B & Y \end{bmatrix}\begin{bmatrix} \Pi T \\ -T \end{bmatrix} = \begin{bmatrix} A\Pi T - XT \\ B\Pi T - YT \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $\begin{bmatrix} A & X \\ B & Y \end{bmatrix}$ is invertible, we infer that $T = 0$. It follows that $B\Pi - Y$ is invertible.

Let $K = \mathbb{Q}\big(\pi_{ij}, 1 \le i \le n, 1 \le j \le k\big)$ and $K' = \mathbb{Q}\big(\pi'_{ij}, 1 \le i \le n, 1 \le j \le k\big)$. Then $B\Pi - Y \in Mat_{m\times m}(K)$ and so is $(B\Pi - Y)^{-1}$, which implies $\Pi' \in Mat_{k\times k}(K)$ and thus $K' \subseteq K$. By symmetry it follows that $K = K'$. □

Let A be an E-ring. Let $\varphi \in Hom(A \otimes A, A)$. The group $A \otimes A$ contains subgroups of the form $A \otimes b$ and $a \otimes A$. There exists a surjective map $i_b : A \to A \otimes b$ with $i_b(x) = x \otimes b$ for all $x \in A$. Then $\psi_b = \varphi \circ i_b : A \to A$

and since A is an E-ring, there is some $\sigma_b \in A$ such that $\psi_b(x) = \varphi(x \otimes b) = x\sigma_b$ for all $x \in A$. By a similar argument, there are elements $\pi_a \in A$ such that $\varphi(a \otimes y) = y\pi_a$ for all $y, a \in A$.

Note that $\varphi(a \otimes b) = b\pi_a = \sigma_b a$. It follows that $\sigma_1 = \pi_1$. For $a = 1$ we get $\sigma_b = b\pi_1$ and thus $\varphi(a \otimes b) = ab\pi_1$ for all $a, b \in A$.

We conclude that $\varphi = \pi_1\mu$ where $\mu \in Hom(A \otimes A, A)$ is the map with $\mu(a \otimes b) = ab$ for all $a, b \in A$. We have shown:

PROPOSITION 9. *If A is an E-ring, then the map $\mu : A \otimes A \to A$ with $\mu(a \otimes b) = ab$ for all $a, b \in A$ is a localization.*

8. Zero Product Determined Algebras

The following definition can be found in [4] and elsewhere:

DEFINITION 4. *Let A be an algebra over the commutative ring C. Then A is called **zero product determined** if for any C-module X and every C-bilinear map $\langle \_ \mid \_ \rangle : A \times A \to X$ the following holds:*

If for all $a, b \in A$, $ab = 0$ implies $\langle a \mid b \rangle = 0$, then there exists some $T \in Hom_C(A^2, X)$ with $\langle x \mid y \rangle = T(xy)$ for all $x, y \in A$. As usual A^2 denotes the C-submodule of A generated by the set of products xy for $x, y \in A$.

REMARK 3. *Let C be a commutative ring and X some C-module. We will always assume that X is a C-bimodule with $sx = xs$ for all $x \in X$ and all $s \in C$. Let X, Y, W be C-modules. Then each C-linear map $\langle \_ \mid \_ \rangle : X \times Y \to W$ is automatically a middle linear map, i.e. $\langle xs \mid y \rangle = \langle x \mid sy \rangle$ for all $x \in X, y \in Y$ and $s \in C$. This means that $\langle \_ \mid \_ \rangle$ factors through $X \otimes_C Y$. We will write "$\otimes$" instead of "$\otimes_C$".*

PROPOSITION 10. *Let A be a C-algebra. Consider the maps $A \times A \xrightarrow{\otimes} A \otimes A \xrightarrow{\mu} A^2 = span_C\{xy : x, y \in A\}$ where $\mu(x \otimes y) = xy$.*

The following are equivalent:

(*a*) *Let X be some C-module and $\theta \in Hom_C(A \otimes A, X)$. If $\ker(\mu \circ \otimes) \subseteq \ker(\theta \circ \otimes)$, then there exists some $T \in Hom_C(A^2, X)$ making the following diagram commutative:*

$$\begin{array}{ccccc} A \times A & \xrightarrow{\otimes} & A \otimes A & \xrightarrow{\mu} & A^2 \\ & & \downarrow \theta & \swarrow T & \\ & & X & & \end{array}$$

. Note that if $1 \in A$, then $A = A^2$.

(*b*) *A is zero product determined.*

(c) *Let X be some C-module and $\theta \in Hom_C(A \otimes A, X)$. If* $\ker(\mu \circ \otimes) \subseteq \ker(\theta \circ \otimes)$*, then* $\ker(\mu) \subseteq \ker(\theta)$.

PROOF. Assume that (a) holds and let $\langle \_ \mid \_ \rangle : A \times A \to X$ be a C-bilinear map such that $\langle a \mid b \rangle = 0$ whenever $ab = 0$. Then there exists a unique $\theta \in Hom_C(A \otimes A, X)$ such that $\langle a \mid b \rangle = \theta(a \otimes b)$ for all $a, b \in A$. Thus $\theta(a \otimes b) = 0$ whenever $ab = 0$, i.e. $\ker(\mu \circ \otimes) \subseteq \ker(\theta \circ \otimes)$. By clause (a), there is some $T \in Hom_C(A^2, X)$ such that $\theta = T \circ \mu$. It follows that $\langle a \mid b \rangle = (T \circ \mu)(a \otimes b) = T(ab)$ for all $a, b \in A$ and (b) holds.

For the converse, assume that (b) holds. With the notations in (a), define $\langle \_ \mid \_ \rangle : A \times A \to X$ by $\langle a \mid b \rangle = \theta(a \otimes b)$ for all $a, b \in A$. Since $\ker(\mu \circ \otimes) \subseteq \ker(\theta \circ \otimes)$ we have that $ab = 0$ implies $\langle a \mid b \rangle = 0$ for all $a, b \in A$. By clause (b), there exists some $T \in Hom_C(A^2, X)$ such that $\langle a \mid b \rangle = T(ab)$ for all $a, b \in A$. Therefore $\theta(a \otimes b) = \langle a \mid b \rangle = T(ab) = T(\mu(a \otimes b))$ for all $a, b \in A$. This shows that $\theta = T \circ \mu$ and (a) follows.

That (a) implies (c), is a trivial consequence of the commutative diagram in (a). For the converse, if $\ker(\mu) \subseteq \ker(\theta)$, the map T may be defined by $T\big(\sum_i a_i b_i\big) = \theta\big(\sum_i a_i \otimes b_i\big)$, since $\sum_i a_i b_i = 0$ means $\mu\big(\sum_i a_i \otimes b_i\big) = 0$. □

Note that the map T in clause (a) is unique since the map μ is surjective.

DEFINITION 5. *Let C be a commutative ring and A a C-algebra and X a C-module. Then $Hom_C^0(A, X) = \{\theta \in Hom_C(A \otimes A, X) :$ If $x, y \in A$ with $xy = 0$, then $\theta(x \otimes y) = 0\}$.*

Then A is zero product determined if and only if for any $\theta \in Hom_C^0(A, X)$ there exists a (unique) $T \in Hom_C(A^2, X)$ such that $\theta = T \circ \mu$.

Note that for any $T \in Hom_C(A^2, X)$ we have that $T(\mu(x \otimes y)) = T(xy) = T(0) = 0$ whenever $xy = 0$, i.e. $T \circ \mu \in Hom_C^0(A, X)$. This shows that

$Hom_C(A^2, X) \overset{\_\circ\mu}{\to} Hom_C^0(A, X) \to 0$ is exact. We have shown:

PROPOSITION 11. *The C-algebra A is zero product determined if and only if $Hom_C(A^2, X) \cong Hom_C^0(A, X)$ via the natural map $\_ \circ \mu$.*

Assume that $1 \in A$ has no zero divisors and is zero product determined. Then $Hom_C^0(A, X) = Hom_C(A \otimes A, X) \cong Hom_C(A, X)$ via the natural map $\_ \circ \mu$.

PROPOSITION 12. *Let C be an integral domain and A a torsion-free C-algebra with $1 \in A$ and without zero divisors. Then A is a zero product determined algebra if and only if A is a subring of the field of fractions of C.*

PROOF. Note that $\ker(\mu \circ \otimes) = \{(a, 0) : a \in A\} \cup \{(0, a) : a \in A\} \subseteq \ker(\theta \circ \otimes)$ for all $\theta \in Hom_C(A \otimes A, X)$. Consider the identity map $id : A \otimes A \to A \otimes A$. If A is zero-product determined, then there exists a map $T : A \to A \otimes A$ such that $id = T \circ \mu$ and thus μ is an isomorphism. This shows that the C-rank of A is 1 and thus A is a subring of the field of fractions of C. The converse is obvious. □

COROLLARY 6. *The ring J_p of p-adic integers is not zero product determined.*

This motivates a weaker condition:

DEFINITION 6. *Let C be a commutative ring and A a C-algebra. We call A zero product self-determined, if A has the property (a) of Proposition* 11 *with "X" replaced by "A^2".*

THEOREM 4. *Let A be an integral domain. The following are equivalent:*

(*a*) *The ring A is a zero product self-determined $\mathbb{Z}$-algebra.*
(*b*) *The map $\mu : A \otimes A \to A$ is a (surjective) localization of $A \otimes A$.*
(*c*) *The ring A is an E-ring.*

PROOF. Because of Proposition 9 and 10, we only have to show that (b) implies (c). To this end, pick any $\beta \in End(A)$ and consider $id_A \otimes \beta : A \otimes A \to A \otimes A$. Then there exists some $T : A \to A$ with $\mu \circ (id_A \otimes \beta) = T \circ \mu$ and thus $a\beta(b) = T(ab)$ for all $a, b \in A$. Let $a = 1$. Then $\beta(b) = T(b)$ follows for all $b \in B$ and thus $a\beta(b) = \beta(ab)$ for all $a, b \in A$. Now, for $b = 1$, we get $\beta(a) = a\beta(1)$ for all $a \in A$. Since $\beta \in End(A)$ was arbitrary, it follows that A is an E-ring. Note that the hypothesis that A has no zero divisors was not used in the previous argument. □

We may use the previous argument and Proposition 9 to obtain the following characterization of E-rings:

COROLLARY 7. *Let A be some ring (viewed as a $\mathbb{Z}$-algebra). The following are equivalent:*

(a) *The natural map $\mu : A \otimes A \to A$ is a localization.*
(b) *The ring A is an E-ring.*

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