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Galois points for a plane curve and its dual curve

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ABSTRACT - A point P in projective plane is said to be Galois for a plane curve of degree at least three if the function field extension induced by the projection from P is Galois. Further we say that a Galois point is extendable if any birational transformation by the Galois group can be extended to a linear transformation of the projective plane. In this article, we propose the following problem: *If a plane curve has a Galois point and its dual curve has one, what is the curve?* We give an answer. We show that the dual curve of a smooth plane curve does not have a Galois point. On the other hand, we settle the case where both a plane curve and its dual curve have extendable Galois points. Such a curve must be defined by $X^d - Y^e Z^{d-e} = 0$, which is a famous self-dual curve.

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1. Introduction

Let the base field K be an algebraically closed field of characteristic $p = 0$ and let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d \geq 3$. We recall the notions of *dual curve* and *Galois point*.

Let $\mathbb{P}^{2*}$ be the dual projective plane which parameterizes projective lines of $\mathbb{P}^2$ and let $(X : Y : Z)$, $(U : V : W)$ be systems of homogeneous coordinates of $\mathbb{P}^2$ and of $\mathbb{P}^{2*}$ respectively. We denote by Sing(C) the singular locus of C. If C is defined by a homogeneous polynomial $F(X, Y, Z)$,

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we have a rational map $\gamma = \gamma_C : C \dashrightarrow \mathbb{P}^{2*}$, which sends a smooth point $Q \in C \setminus \mathrm{Sing}(C)$ to the point $\left(\frac{\partial F}{\partial X}(Q) : \frac{\partial F}{\partial Y}(Q) : \frac{\partial F}{\partial Z}(Q)\right) \in \mathbb{P}^{2*}$ parameterizing the projective tangent line T_QC to C at Q. This rational map is called the *dual map* of C and (the closure of) the image of C is called the *dual curve*, which is denoted by C^*. It is well-known that *projective duality* $C^{**} = C$ holds (see, for example, [7, 11]).

If the function field extension $K(C)/K(\mathbb{P}^1)$ induced by the projection $\pi_P : C \dashrightarrow \mathbb{P}^1$ from a point $P \in \mathbb{P}^2$ is Galois, then the point P is said to be *Galois*. Moreover, denoting by G_P the Galois group associated to the projection π_P, we say that a Galois point P is *extendable* if any birational transformation of C induced by the Galois group G_P can be extended to a linear transformation of $\mathbb{P}^2$.

The notion of Galois point was introduced by H. Yoshihara (see e.g. [4, 9, 13]) and it is an interesting topic on plane curves. For instance, a well-known theorem of Noether and later results assure that if $C \subset \mathbb{P}^2$ is a smooth curve of degree d, then the minimum degree of a morphism $C \to \mathbb{P}^1$ is $d-1$, and all the maps of degree $d-1 \geq 2$ and $d \geq 5$ are projections $\pi_P : C \dashrightarrow \mathbb{P}^1$ from some point $P \in C$ and $P \in \mathbb{P}^2 \setminus C$ respectively (cf. [3, 5, 10]). Thus describing Galois points on a smooth plane curve is equivalent to detect all the Galois coverings $C \to \mathbb{P}^1$ having minimal degrees.

The singular curve defined by $X^d - Y^eZ^{d-e} = 0$, where $e \geq 1$ and d, e are coprime, has lovely properties. Its dual curve is defined by the same equation (up to a projective equivalence, see Lemma 2.5). Therefore, this curve is a "self-dual" curve. It has an (extendable) Galois point $(1 : 0 : 0)$ (Proposition 2.3) and its dual curve has one, by the self-duality. In the light of this fact, we propose the following problem.

PROBLEM 1.1. *If a plane curve has a Galois point and its dual curve has one, what is the curve? Is it a self-dual curve?*

In this article, we give an answer. We show the following for smooth curves.

THEOREM 1.2. *Let C be a smooth plane curve of degree $d \geq 3$. Then, there exist no Galois points for the dual curve of C.*

By projective duality, we have the following.

Corollary 1.3. *Let C be a plane curve of degree $d \geq 3$ and let C^* be the dual curve. If both C and C^* have Galois points, then they are singular.*

We say that a Galois point P is *inner* (resp. *outer*) if $P \in C \setminus \mathrm{Sing}(C)$ (resp. $P \in \mathbb{P}^2 \setminus C$). In the case where Galois points are extendable, we show the following characterization theorems.

Theorem 1.4. *Let C be a plane curve of degree $d \geq 3$ and let C^* be the dual curve of C.*

- (I) *The following conditions are equivalent.*
  - (1) *C and C^* have extendable outer Galois points.*
  - (2) *C is projectively equivalent to the plane curve defined by $X^d - Y^e Z^{d-e} = 0$ for some $e \geq 1$.*
- (II) *The following conditions are equivalent.*
  - (1) *C has an extendable outer Galois point and C^* has an extendable inner Galois point.*
  - (2) *C and C^* have extendable inner Galois points.*
  - (3) *C is projectively equivalent to the plane curve defined by $X^d - Y^{d-1}Z = 0$.*

To prove our result we will connect Galois points for a plane curve C and its dual curve. In particular, we will show that – given an extendable Galois point P for C – the Galois group G_P has a natural action on the dual curve C^*, and such an action preserves the fibers of a certain projection $\pi_{\overline{P}} : C^* \to \mathbb{P}^1$. Namely,

Proposition 1.5. *Let C be a plane curve with an extendable Galois point $P \in \mathbb{P}^2$ and let G_P be the Galois group. Any $\sigma \in G_P$ induces a natural linear transformation $\overline{\sigma} : \mathbb{P}^{2*} \to \mathbb{P}^{2*}$ (see Lemma 2.2 for details). Then, there exists a unique point $\overline{P} \in \mathbb{P}^{2*}$ such that the map $\sigma \mapsto \overline{\sigma}$ induces an injective homomorphism*

$$G_P \hookrightarrow G[\overline{P}] := \{\tau \in \mathrm{Bir}(C^*) \mid \tau(C^* \cap \ell \setminus \{\overline{P}\}) \subset \ell \ \text{ for a general line } \ell \ni \overline{P}\},$$

where $\mathrm{Bir}(C^*)$ is the group of all birational transformations of C^*. In particular, the degree of the projection $\pi_{\overline{P}} : C^* \dashrightarrow \mathbb{P}^1$ from $\overline{P}$ is at least the order of G_P.

In the next Section we will recall some preliminary facts about projective duality and Galois group, and we will achieve Proposition 1.5. Section 3 and Section 4 will be devoted to prove Theorems 1.2 and 1.4 respectively.

2. Preliminaries

Let C be an irreducible plane curve of degree $d \geq 3$. If a point $P \in \mathbb{P}^2$ is not in C, we define the multiplicity of C at P as zero. We denote by d^* the degree of the dual curve C^*.

We recall duality principle between $\mathbb{P}^2$ and $\mathbb{P}^{2*}$. For a projective line $\ell \subset \mathbb{P}^2$, we denote by $[\ell]$ the point in $\mathbb{P}^{2*}$ corresponding to ℓ. If ℓ is defined by $uX + vY + wZ = 0$ for some $u, v, w \in K$, then the point $[\ell] \in \mathbb{P}^{2*}$ is given by $(u : v : w)$. For a point $P \in \mathbb{P}^2$, we denote by $[P] \subset \mathbb{P}^{2*}$ the line corresponding to P, which parameterizes the star of lines through P. If P is defined by $u_1X + v_1Y + w_1Z = u_2X + v_2Y + w_2Z = 0$ for some $u_1, u_2, v_1, v_2, w_1, w_2 \in K$, then $[P]$ is the line passing through the two points $(u_1 : v_1 : w_1), (u_2 : v_2 : w_2) \in \mathbb{P}^{2*}$. Then, we have the following elementary facts.

Lemma 2.1. *Let $P \in \mathbb{P}^2$ be a point and let $\ell \subset \mathbb{P}^2$ be a line. Then, $[[P]] = P$ and $[[\ell]] = \ell$ hold. Furthermore,*

$$P \in \ell \Leftrightarrow [\ell] \in [P]$$

holds.

Lemma 2.2. *Let ϕ be a linear transformation of $\mathbb{P}^2$ and let A_ϕ be a matrix representing ϕ, i.e. $\phi(X : Y : Z) = (X, Y, Z)A_\phi$. Then the induced map $\overline{\phi} : \mathbb{P}^{2*} \to \mathbb{P}^{2*}; [\ell] \mapsto [\phi(\ell)]$ is represented by the transpose of the matrix A_ϕ^{-1}, that is*

$$\overline{\phi}(U : V : W) = (U, V, W)\ {}^tA_\phi^{-1}.$$

Moreover, the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma_C} & \mathbb{P}^{2*} \\ \phi \downarrow & & \downarrow \overline{\phi} \\ \phi(C) & \xrightarrow{\gamma_{\phi(C)}} & \mathbb{P}^{2*} \end{array}$$

commutes, where the horizontal arrows are the dual maps of C and $\phi(C)$.

We often use the *standard form* of the defining equation for a plane curve with an extendable Galois point, which is given by the following.

Proposition 2.3 (see [8, 13, 15]). *Let $P \in \mathbb{P}^2$ be a point with multiplicity $m \geq 0$. The point P is extendable Galois for C if and only if there* *exists a linear transformation* ϕ *on* $\mathbb{P}^2$ *such that* $\phi(P) = (1:0:0)$ *and* $\phi(C)$ *is given by*

$$X^{d-m}G_m(Y,Z) + G_d(Y,Z) = 0,$$

where $G_i(Y,Z)$ *is a homogeneous polynomial of degree* i *in variables* Y, Z. *In this case, the Galois group* $G_{\phi(P)}$ *is cyclic and there exists a primitive* $(d-m)$*-th root* ζ *of unity such that a generator* $\sigma \in G_{\phi(P)}$ *is represented by* $\sigma(X:Y:Z) = (\zeta X:Y:Z)$.

Firstly, we prove the first assertion of Proposition 1.5 for the curves with the standard form as in Proposition 2.3.

LEMMA 2.4. *Let* $P = (1:0:0) \in \mathbb{P}^2$, *let* C *be defined by*

$$X^{d-m}G_m(Y,Z) + G_d(Y,Z) = 0,$$

and let a generator $\sigma \in G_P$ *be represented by* $\sigma(X:Y:Z) = (\zeta X:Y:Z)$. *Then, there exists a unique point* $\overline{P} \in \mathbb{P}^{2*}$ *such that the map* $G_P \to G[\overline{P}]; \sigma^i \mapsto \overline{\sigma^i}$ *is a well-defined and injective homomorphism.*

PROOF. By Lemma 2.2, $\overline{\sigma}$ is represented by $\overline{\sigma}(U:V:W) = (\zeta^{-1}U:V:W)$ and $\overline{\sigma}(C^*) = C^*$. Let $\overline{P} = (1:0:0) \in \mathbb{P}^{2*}$. By the representation of $\overline{\sigma}$, we have

$$\overline{\sigma} \in G[\overline{P}] = \{\tau \in \mathrm{Bir}(C^*) \mid \tau(C^* \cap \ell \setminus \{\overline{P}\}) \subset \ell \ \text{ for a general line } \ell \ni \overline{P}\}.$$

We prove the uniqueness of $\overline{P}$. By contradiction we assume that $\overline{\sigma} \in G[\overline{P}] \cap G[Q]$, where $Q \neq \overline{P}$. Let $R \in C^*$ be a point not contained in the line passing through $\overline{P}$ and Q such that $\overline{\sigma}(R) \neq R$. By $\overline{\sigma} \in G[\overline{P}] \cap G[Q]$, $\overline{\sigma}(R)$ is contained in the lines passing through R and $\overline{P}$, R and Q respectively. This implies that $\overline{\sigma}(R) = R$. This is a contradiction. □

PROOF OF PROPOSITION 1.5. By Proposition 2.3, there exists a linear transformation ϕ such that $\phi(P) = (1:0:0)$, $\phi(C)$ is given by $X^{d-m}G_m(Y,Z) + G_d(Y,Z) = 0$ and a generator $\sigma \in G_{\phi(P)}$ is represented by $\sigma(X:Y:Z) = (\zeta X:Y:Z)$. By Lemma 2.4, we have an injection $f: G_{\phi(P)} \hookrightarrow G[\overline{\phi(P)}]; \sigma^i \mapsto \overline{\sigma^i}$. Let $\overline{P} = \overline{\phi}^{-1}(\overline{\phi(P)})$. We also have isomorphisms $g: G_P \to G_{\phi(P)}; \tau \mapsto \phi\tau\phi^{-1}$ and $h: G[\overline{\phi(P)}] \to G[\overline{P}]; \eta \mapsto \overline{\phi}^{-1}\eta\overline{\phi}$. Since $({}^tA_{\phi}^{-1})\,{}^t(A_{\phi}^{-1}A_{\tau}A_{\phi})^{-1}({}^tA_{\phi}^{-1})^{-1} = ({}^tA_{\phi}^{-1})\,{}^tA_{\phi}\,{}^tA_{\tau}^{-1}\,{}^tA_{\phi}^{-1}({}^tA_{\phi}) = {}^t\,A_{\tau}^{-1}$, we have $(h \circ f \circ g)(\tau) = \overline{\tau}$. Therefore, $G_P \to G[\overline{P}]; \tau \mapsto \overline{\tau}$ is a well-defined and injective homomorphism. The uniqueness of the point $\overline{P}$ follows, similarly to the proof of Lemma 2.4.

We prove the second assertion. Let m^* be the multiplicity of C^* at $\overline{P}$. Then, the order of $G[\overline{P}]$ is at most $d^* - m^*$, which is equal to the degree of the projection $\pi_{\overline{P}} : C^* \dashrightarrow \mathbb{P}^1$. By the first assertion, we have $d - m \le d^* - m^*$. □

Here, we summarize the properties of the curve defined by $X^d - Y^e Z^{d-e} = 0$.

LEMMA 2.5. *Let C be defined by $X^d - Y^e Z^{d-e} = 0$. Then, we have the following.*

(1) *The dual curve C^* is given by $(-1)^d e^e (d-e)^{d-e} U^d - d^d V^e W^{d-e} = 0$.*
(2) *Points $(1:0:0) \in \mathbb{P}^2$ and $(1:0:0) \in \mathbb{P}^{2*}$ are extendable outer Galois for C and C^* respectively.*
(3) *If $e = d-1$, then points $(0:1:0) \in \mathbb{P}^2$ and $(0:1:0) \in \mathbb{P}^{2*}$ are extendable inner Galois for C and C^* respectively.*

PROOF. Since the dual map γ is given by

$$\begin{aligned} &(dX^{d-1} : -eY^{e-1}Z^{d-e} : -(d-e)Y^e Z^{d-e-1}) \\ &= (dX^d : -eXY^{e-1}Z^{d-e} : -(d-e)XY^e Z^{d-e-1}) \\ &= (dY^e Z^{d-e} : -eXY^{e-1}Z^{d-e} : -(d-e)XY^e Z^{d-e-1}) \\ &= (dYZ : -eXZ : -(d-e)XY), \end{aligned}$$

we have (1). We have assertions (2) and (3) by Proposition 2.3. □

3. Galois points for the dual curve of a smooth curve

Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d \ge 3$ and let C^* be the dual curve of C. The dual map $\gamma : C \to C^*$ is birational ([7], [11, Theorem 1.5.3]) and C^* is of degree $d^* = d(d-1)$, since the linear system $\left\langle \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \right\rangle$ has no base point. Note that any birational transformation of C is extendable when $d \ge 4$ (see [1, Appendix A, 17 and 18] or [2]). Therefore any Galois point is extendable in this case.

LEMMA 3.1. *Let C be a smooth plane curve of degree $d \ge 4$. Then, any birational map $C^* \dashrightarrow C^*$ can be extended to a linear transformation of $\mathbb{P}^{2*}$. Therefore, any Galois point for C^* is extendable.*

PROOF. Let $\tau : C^* \dashrightarrow C^*$ be a birational map. Then the rational map $\tau_0 := \gamma^{-1}\tau\gamma : C \dashrightarrow C$ is a birational transformation of C. Since C is smooth of degree $d \geq 4$, τ_0 can be extended to a linear transformation of $\mathbb{P}^2$ as observed above. Since the diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma_C} & C^* \\ {\scriptstyle\tau_0}\downarrow & & \downarrow{\scriptstyle\tau} \\ C & \xrightarrow{\gamma_C} & C^* \end{array}$$

commutes, we have $\overline{\tau_0}|_{C^*} = \tau$. Hence τ can be extended to a linear transformation of $\mathbb{P}^{2*}$. □

PROOF OF THEOREM 1.2. Let C be a smooth plane curve of degree $d \geq 3$. Throughout we treat the cases $d = 3$ and $d \geq 4$ separately.

Firstly, we assume that $d \geq 4$. Aiming for a contradiction, let $R \in \mathbb{P}^{2*}$ be a Galois point for C^*, and let m be the multiplicity of C^* at R. It follows from Lemma 3.1 and Proposition 1.5 that the degree of the projection from R is at most d. Therefore, we have $d^* - m \leq d$. Since $R \in C^*$, there exists a point $R_0 \in C$ with $\gamma(R_0) = R$. Let $\ell \subset \mathbb{P}^{2*}$ be a general line with $R \in \ell$. Then, $C^* \cap \ell \setminus \{R\}$ consists of exactly $d^* - m$ smooth points of C^*. Since $C^* \cap \ell \setminus \{R\}$ consists of only smooth points, there are exactly $d^* - m$ points in $\gamma^{-1}(C^* \cap \ell \setminus \{R\})$ and such points are not flexes (i.e. the intersection multiplicity of C and the tangent line at the point is two). It follows from Lemma 2.1 that $[\ell] \in [\gamma(Q_0)] = T_{Q_0}C \subset \mathbb{P}^2$ for any point $Q_0 \in \gamma^{-1}(C^* \cap \ell)$. This implies that, for a general point $P \in [R] = [\gamma(R_0)] = T_{R_0}C$, there exist exactly $d^* - m$ points not in $\gamma^{-1}(R)$ which are not flexes and the tangent lines contain P. Then, by Hurwitz formula for the projection $\pi_P : C \to \mathbb{P}^1$ from a general point $P \in T_{R_0}C \setminus C$, we have $2g(C) - 2 + 2\deg(\pi_P) = \deg B$, where $g(C) = \dfrac{(d-1)(d-2)}{2}$ is the genus of C and B is the ramification divisor. Since the sum of the degrees of B given by points in $C \cap T_{R_0}C$ is at most $d - 1$, we have $\deg B \leq (d-1) + (d^* - m)$. Thus Hurwitz formula leads to the following inequality

$$d^2 - 3d + 2d \leq (d-1) + (d^* - m) \leq 2d - 1.$$

This implies that $d \leq 3$. This is a contradiction to the assumption $d \geq 4$.

So we assume that $d = 3$. Then, we have $d^* = 6$. We recall the well-known fact that there are nine flexes for C. Let $\mathrm{Flex}(C) \subset C$ be the set of all flexes of C. Then $\gamma(\mathrm{Flex}(C)) = \mathrm{Sing}(C^*)$ as C does not admit bitangent lines. Firstly we would like to determine lines $\ell \subset \mathbb{P}^{2*}$ such that

$\ell \cap \mathrm{Sing}(C^*) \neq \emptyset$. Let $Q \in \mathrm{Flex}(C)$ and let $P \in T_QC$, which is maybe Q. Using Hurwitz formula for the projection π_P, the ramification divisor has degree 6 (resp. 4) if $P \neq Q$ (resp. $P = Q$). Hence one of the following conditions holds.

(1) There exist three flexes whose tangent lines contain P.

(2) There exist exactly two flexes and two other points whose four tangent lines contain P.

(3) There exist exactly one flex and four other points whose five tangent lines contain P.

(4) $P = Q$ and there exist three points which are not flexes and tangent lines at them contain P.

It follows from Lemma 2.1 that one of the following conditions holds if $\ell \cap \mathrm{Sing}(C^*) \neq \emptyset$ for a line $\ell \subset \mathbb{P}^{2*}$.

(1^*) $C^* \cap \ell$ consists of three singular points.

(2^*) $C^* \cap \ell$ consists of two singular points and two smooth points.

(3^*) $C^* \cap \ell$ consists of one singular point and four smooth points.

(4^*) $C^* \cap \ell$ consists of exactly one singular point and three smooth points.

Let $R \in \mathbb{P}^{2*}$ be a point and let $\hat{\pi}_R := \pi_R \circ \gamma : C \to \mathbb{P}^1$. We denote by e_P the ramification index at $P \in C$ for $\hat{\pi}_R$. Note that if $Q \in \mathrm{Flex}(C)$ and $\gamma(Q) \neq R$, then $\hat{\pi}_R$ is ramified at Q, since the differential of γ at Q is zero. The above conditions represent types of ramification indices of flexes for $\hat{\pi}_R$. For example, condition (1^*) implies that $e_{Q_1} = e_{Q_2} = e_{Q_3} = 2$, where $\gamma(Q_1), \gamma(Q_2), \gamma(Q_3) \in C^* \cap \ell$ if $R \in \ell$ and $R \notin C^*$.

Assume by contradiction that $R \in \mathbb{P}^{2*}$ is a Galois point for C^*. Then, the field extension induced by $\hat{\pi}_R$ is Galois. We often use the property of Galois extensions that the ramification index e_{Q_1} is equal to the one e_{Q_2} if $\hat{\pi}_R(Q_1) = \hat{\pi}_R(Q_2)$ ([12, Corollary 3.7.2]). Moreover, we separate the cases $R \in C^* \setminus \mathrm{Sing}(C^*)$, $R \in \mathbb{P}^{2*} \setminus C^*$ and $R \in \mathrm{Sing}(C^*)$.

Firstly we suppose that R is an inner Galois point, that is $R \in C^* \setminus \mathrm{Sing}(C^*)$. Then, the projection $\hat{\pi}_R$ is of degree five. Then, any ramification index is five, by the property mentioned above. Therefore, the set $\hat{\pi}_R^{-1}(\hat{\pi}_R(Q)) = \{Q\}$ if $Q \in \mathrm{Flex}(C)$. Since this condition does not coincide with (1), ..., (4) we have a contradiction.

Then we assume that R is an outer Galois point, that is $R \in \mathbb{P}^{2*} \setminus C^*$, so that $\deg \hat{\pi}_R = 6$. Note that there exist no tangent lines with two contact points for C^*, by the smoothness of C and projective duality. By the property of Galois extensions, any line containing R and a singular point must satisfy (1^*). Since the number of singular points is nine, there exists at most three lines containing R and satisfying (1^*). Applying Hurwitz formula to the projection $\hat{\pi}_R$, we have that the ramification divisor must have degree 12. Thus there exists another ramification point which is not a flex of C, but this contradicts the property of Galois extensions.

Finally, we assume that $R \in \mathrm{Sing}(C^*)$. Let $R_0 \in \mathrm{Flex}(C)$ with $\gamma(R_0) = R$. By the property of Galois extensions, any line containing R and another singular point must satisfy (1^*). Let points $P \in \mathbb{P}^2$ and $Q_1, Q_2, Q_3 \in \mathrm{Flex}(C)$ satisfy $P \in T_{Q_i}C$ for any i, as in condition (1). We may assume that $P = (1:0:0), Q_1 = (0:1:0), Q_2 = (0:0:1)$ and C is defined by

$$X^3 + G_1(Y,Z)X^2 + G_2(Y,Z)X + G_3(Y,Z) = 0,$$

where G_i is a homogeneous polynomial of degree i. Since $Q_1, Q_2 \in \mathrm{Flex}(C)$ and $T_{Q_1}C$ (resp. $T_{Q_2}C$) is defined by $Z = 0$ (resp. $Y = 0$), $G_1 = 0$ and G_2, G_3 is divisible by YZ. Since $T_{Q_3}C$ is defined by $\alpha Y + \beta Z = 0$ for some $\alpha, \beta \in K \setminus 0$, we also have $G_2 = 0$. We have an equation $X^3 + G_3(Y,Z) = 0$. We can take a linear transformation ϕ of $\mathbb{P}^2$ such that $\phi(P) = P$ and the three points given by $X = G_3(Y,Z) = 0$ move to $(0:1:-1)$, $(0:1:\omega)$, $(0:1:\omega^2)$ by ϕ, where ω is a cubic root of -1 different from -1. Then, $\phi(C)$ is given by $X^3 + \alpha(Y^3 + Z^3) = 0$ for some $\alpha \in K$. We may assume that C is defined by $X^3 + Y^3 + Z^3 = 0$. Note that the nine flexes are contained in the union of lines $X = 0$, $Y = 0$ and $Z = 0$. For this Fermat curve, if $P_1, P_2 \in \mathrm{Flex}(C)$, then there exists a linear transformation ψ such that $\psi(P_1) = P_2$ and $\psi(C) = C$. Therefore, we may assume that $R_0 = (0:1:-1)$. The dual map is given by $(X^2 : Y^2 : Z^2)$. We find that $\gamma(\mathrm{Flex}(C) \cap \{X = 0\}) = C^* \cap \{U = 0\}$, and analogous equalities holds for the pairs Y, V and Z, W. Here, we have $R = (0:1:1)$. Let $R' = (1:0:1)$ and let ℓ be the line passing through $R, R' \in \mathrm{Sing}(C^*)$, which is defined by $U + V - W = 0$. Since $R \in C^* \cap \{U = 0\}$, $R' \in C^* \cap \{V = 0\}$ and ℓ satisfies condition (1^*), we have that ℓ must meet C^* at another singular point lying on the line $\{W = 0\}$. Thus we have a contradiction as ℓ intersects $\{W = 0\}$ at $(1:-1:0) \notin \mathrm{Sing}(C^*)$. $\square$

4. Curves with extendable Galois points

Before proving Theorem 1.4, we present a preliminary lemma involved in the proof.

LEMMA 4.1. *Let $G(Y,Z), H(V,W)$ be homogeneous polynomials of degree d. Assume that $d^d(-G)^{d-1} = -H(G_Y, G_Z)$ holds and $m \geq 1$ is the maximal number such that Y^m divides G. Then, W^{d-m} divides H.*

PROOF. Let $G = Y^m F$. Then F is not divisible by Y and, by Euler formula $(d-m)F = YF_Y + ZF_Z$, F_Z also is not. Then, $G_Y = mY^{m-1}F + Y^m F_Y = Y^{m-1}(mF + YF_Y)$ and $G_Z = Y^m F_Z$. We denote by $H = \sum_{i=0}^{d} \alpha_i V^i W^{d-i}$. Then, by the assumption,

$$\begin{aligned} d^d(-1)^{d-1} Y^{m(d-1)} F^{d-1} &= -H(G_Y, G_Z) \\ &= -\sum_i \alpha_i Y^{i(m-1)+m(d-i)} (mF + YF_Y)^i F_Z^{d-i}. \end{aligned}$$

Since $(mF + YF_Y)^i F_Z^{d-i}$ is not divisible by Y for any i, we have $\alpha_i = 0$ for any i such that $m(d-1) > i(m-1) + m(d-i)$, i.e. $i > m$. Therefore, $H = \sum_{i=0}^{m} \alpha_i V^i W^{d-i} = W^{d-m} \sum_{i=0}^{m} \alpha_i V^i W^{m-i}$. □

PROOF OF THEOREM 1.4(I). The assertion (2) ⇒ (1) is nothing but Lemma 2.5(2). We prove (1) ⇒ (2). By using projective duality, we may assume that $d \geq d^*$. By Proposition 2.3, we may assume that $P = (1:0:0)$ be an extendable outer Galois point, C is given by

$$X^d + G(Y,Z) = 0,$$

where $G(Y,Z)$ is a homogeneous polynomial of degree d, and any birational transformation $\sigma \in G_P$ is represented by $\sigma(X:Y:Z) = (\zeta^i X : Y : Z)$ for some i. By Lemma 2.4, we have an injection $G_P \hookrightarrow G[\overline{P}]$, where $\overline{P} = (1:0:0)$. Then, $d^* = d$ and $G_P = G[\overline{P}]$. Therefore $\overline{P} \in \mathbb{P}^{2*}$ is extendable outer Galois for C^*. Furthermore, C^* is given by

$$U^d + H(V,W) = 0,$$

where H is a homogeneous polynomial of degree d. Since the dual map $\gamma : C \dashrightarrow C^*$ is given by $\gamma = (dX^{d-1} : G_Y : G_Z)$, we have $(dX^{d-1})^d + H(G_Y, G_Z) = 0$. Since $X^d = -G$ on C, we have an equation

$$d^d(-G)^{d-1} + H(G_Y, G_Z) = 0$$

on C. Since the variable X does not appear, this equation holds as polynomials. For a suitable system of coordinates, by Lemma 2.2, $G(Y,Z)$ is divisible by YZ. Let $m \geq 1$ (resp. $n \geq 1$) be the maximal number such that Y^m (resp. Z^n) divides G. Note that $m + n \leq d$. Then, by Lemma 4.1, H is divisible by $V^{d-n}W^{d-m}$. Since $(d-n)+(d-m)=2d-(n+m)\geq d=\deg H$, we have $H=\alpha V^{d-n}W^{d-m}$ for some $\alpha\in K$. Therefore, C^* is projectively equivalent to the curve given by $U^d-V^eW^{d-e}=0$ for some $e\geq 1$. The curve C is also by Lemma 2.5(1). □

Proof of Theorem 1.4(II). The assertion (3) ⇒ (1) is nothing but Lemma 2.5. We prove (1) ⇒ (2). Let P be an extendable outer Galois point for C and let Q be an extendable inner Galois point for C^*. By Proposition 1.5, $d^*\geq d$ and $d\geq d^*-1$. Then we treat the cases $d=d^*-1$ and $d=d^*$ separately.

Firstly, we assume that $d=d^*-1$. By Proposition 2.3, we may assume that $P=(1:0:0)$, a generator $\sigma\in G_P$ is represented by $\sigma(X:Y:Z)=(\zeta X:Y:Z)$ and C is given by

$$X^d+G(Y,Z)=0.$$

By Lemma 2.4, we have an injection $G_P\hookrightarrow G[\overline{P}]$, where $\overline{P}=(1:0:0)$. Considering the orders of G_P and $G[\overline{P}]$, we have $G_P=G[\overline{P}]$. Hence $\overline{P}\in C^*$ is smooth and extendable inner Galois. Considering the action, C^* is given by

$$U^dH_1(V,W)+H_{d+1}(V,W)=0,$$

where H_i is a homogeneous polynomial of degree i. For a suitable system of coordinates, by Lemma 2.2, we may assume that $H_1=W$. We denote by $H_{d+1}=H$. Then H is not divisible by W as C^* cannot contain lines, and also H_V is not by Euler formula. Since the dual map $\gamma_{C^*}:C^*\dashrightarrow C^{**}=C$ is given by

$$\begin{aligned}\gamma_{C^*}&=(dU^{d-1}W:H_V:U^d+H_W)=(dU^{d-1}W^2:WH_V:U^dW+WH_W)\\&=(dU^{d-1}W^2:WH_V:-H+WH_W)\end{aligned}$$

by using $U^dW=-H$, we have

$$(dU^{d-1}W^2)^d+G(WH_V,-H+WH_W)=0.$$

Since $U^dW=-H$ on C^*, we have an equation

$$d^d(-H)^{d-1}W^{d+1}=-G(WH_V,-H+WH_W)$$

on C^*. Since the variable U does not appear, this equation holds as polynomials. Let $G(Y,Z)=\sum_{i=0}^{d}\alpha_iY^iZ^{d-i}$. Then, we have

$$d^d(-H)^{d-1}W^{d+1}=-\sum_i\alpha_iW^iH_V^i(-H+WH_W)^{d-i}.$$

Since H_V and $-H + WH_W$ are not divisible by W, $\alpha_i = 0$ for any i. This is a contradiction.

Then we assume that $d^* = d$. By Proposition 1.5, we have an injection $G_Q \hookrightarrow G[\overline{Q}]$. Considering the orders, we have $G_Q = G[\overline{Q}]$. Hence $\overline{Q} \in C$ is smooth and extendable inner Galois. We have assertion (2).

Then it remains to prove (2) $\Rightarrow$ (3). Let P be an extendable inner Galois point for C and let Q be an extendable inner Galois point for C^*. By projective duality, we may assume that $d \geq d^*$, and Proposition 1.5 assures that $d^* \geq d - 1$. Hence we deal with the cases $d^* = d - 1$ and $d^* = d$ separately.

Assume that $d^* = d - 1$. By Proposition 1.5 and considering the orders, we have $G_P = G[\overline{P}]$. Then, $\overline{P}$ is extendable outer Galois for C^*. Similarly to the above discussion to prove (1) $\Rightarrow$ (2), we have a contradiction.

Therefore, we have $d^* = d$. By Proposition 2.3, we may assume that $P = (1:0:0)$ and C is given by

$$X^{d-1}Z + G(Y,Z) = 0.$$

By Lemma 2.4, we have an injection $G_P \hookrightarrow G[\overline{P}]$, where $\overline{P} = (1:0:0)$. Considering the orders of G_P and $G[\overline{P}]$, we have $G_P = G[\overline{P}]$. Therefore, $\overline{P} \in \mathbb{P}^{2*}$ is extendable inner Galois for C^*. Furthermore, C^* is given by

$$U^{d-1}(\alpha V + \beta W) + H(V,W) = 0,$$

where $\alpha, \beta \in K$. Since the dual map $\gamma : C \dashrightarrow C^*$ is given by

$$\begin{aligned}\gamma &= ((d-1)X^{d-2}Z : G_Y : X^{d-1} + G_Z) = ((d-1)X^{d-2}Z^2 : ZG_Y : X^{d-1}Z + ZG_Z)\\ &= ((d-1)X^{d-2}Z^2 : ZG_Y : -G + ZG_Z)\end{aligned}$$

by using $X^{d-1}Z = -G$, we have

$$((d-1)X^{d-2}Z^2)^{d-1}(\alpha ZG_Y + \beta(-G + ZG_Z)) + H(ZG_Y, -G + ZG_Z) = 0.$$

Since $X^{d-1}Z = -G$ on C, we have an equation

$$(d-1)^{d-1}(-G)^{d-2}Z^d(\alpha ZG_Y + \beta(-G + ZG_Z)) = -H(ZG_Y, -G + ZG_Z)$$

on C. Since the variable X does not appear, this equation holds as polynomials. Let $H(V,W) = \sum_{i=0}^{d} \alpha_i V^i W^{d-i}$. Then, we have

$$(d-1)^{d-1}(-G)^{d-2}Z^d(\alpha ZG_Y + \beta(-G + ZG_Z)) = -\sum_i \alpha_i Z^i G_Y^i(-G + ZG_Z)^{d-i}.$$

Since G_Y and $-G + ZG_Z$ are not divisible by Z, $\alpha_i = 0$ for any $i < d$.

Therefore, $H(V, W) = cV^d$ for some $c \in K$. We have an equation

$$U^{d-1}(\alpha V + \beta W) + cV^d = 0$$

for C^*. The dual curve C^* is projectively equivalent to the curve given by $U^d - V^{d-1}W = 0$. Therefore, C is also by Lemma 2.5(1). □

REMARK 4.2. In order to deal with Problem 1.1, the condition $P \notin \mathrm{Sing}(C)$ in the definition of inner Galois point is crucial. In particular, Theorem 1.4 fails to hold true as we extend the assertion to (non-extendable) outer Galois point for C and Galois point lying on $\mathrm{Sing}(C^*)$.

For example, let $C \subset \mathbb{P}^2$ be curve defined by $(X + Y + Z)^3 - 27XYZ = 0$. Then $(1 : 0 : 0)$ is an outer Galois point for C ([14, Example 2]). Since C is a rational nodal curve with three flexes, the dual curve C^* is a quartic plane curve with three singular double points. Thus C is not a self-dual curve. However, any singular point $Q \in \mathrm{Sing}(C^*)$ is a Galois point for C^* as the projection $\pi_Q : C^* \dashrightarrow \mathbb{P}^1$ has degree two.

REMARK 4.3. According to [6], the set of all Galois points for the plane curve defined by $X^d - Y^e Z^{d-e} = 0$ is equal to $\{(1:0:0), (0:1:0), (0:0:1)\}$ (if $d \geq 4$).

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