Rend. Sem. Mat. Univ. Padova, Vol. 132 (2014)
DOI 10.4171/RSMUP/132-8

Stability of Cartan-Eilenberg Gorenstein categories

Li Liang(*) - Gang Yang(**)

Abstract - We study Cartan-Eilenberg Gorenstein categories by introducing CE-projective CE-generators and CE-injective CE-cogenerators in the paper. We give a relationship between injective cogenerators (resp., projective generators) introduced by Sather-Wagstaff, Sharif and White and CE-injective CE-cogenerators (resp., CE-projective CE-generators). As applications, we prove some stability results of Cartan-Eilenberg Gorenstein categories.

Mathematics Subject Classification (2010). 18G10, 18G25, 18G35.

Keywords. Gorenstein categories, Cartan-Eilenberg Gorenstein categories, projective generators, injective cogenerators.

1. Introduction

Throughout this work, R denotes an associative ring with identity, and by an R-module we mean a left R-module.

Auslander and Bridger [1] introduced the Gorenstein dimension for finitely generated modules over a commutative Noetherian ring. Enochs and Jenda [4] defined Gorenstien projective modules over arbitrary rings whether the modules are finitely generated or not. It is well known that, for finitely generated modules over commutative Noetherian rings, Gorenstein projective modules are actually the modules of Gorenstein dimension zero. Dually, Gorenstein injective modules were defined in [4]. Sather-Wagstaff, Sharif and White [7] introduced the concept of $\mathcal{W}$-Gorenstein modules that unifies the concepts of Gorenstein projective and

(*) Indirizzo dell'A.: School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou 730070, China.
E-mail: lliangnju@gmail.com
(**) Indirizzo dell'A.: School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou 730070, China.
E-mail: yanggang@mail.lzjtu.cn

injective modules, and they studied the stability of Gorenstein categories by using so called projective generators and injective cogenerators (see (2.3) for definitions) that are very useful tools for studying Gorenstein categories.

In his thesis Verdier [8] considered Cartan-Eilenberg projective and injective resolutions of a complex using the ideas of Cartan and Eilenberg [2]. Furthermore, Enochs [3] introduced Cartan-Eilenberg Gorenstein projective and injective complexes in terms of so called Cartan-Eilenberg complete projective and injective resolutions, respectively. Using the ideas of Sather-Wagstaff, Sharif and White [7], we introduce CE-projective CE-generators and CE-injective CE-cogenerators in this paper (see Definition 3.2), which are useful for studying Cartan-Eilenberg Gorenstein categories. In particular, we give a relationship between injective cogenerators (resp., projective generators) and CE-injective CE-cogenerators (resp., CE-projective CE-generators) as follows (see Theorem 3.3):

Theorem A. Assume that $\mathcal{W}$ and $\mathcal{X}$ are classes of R-modules such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{W}$ is closed under finite direct sums. Let $\mathrm{CE}(\mathcal{W})$ (resp., $\mathrm{CE}(\mathcal{X})$) be the class of CE-$\mathcal{W}$ (resp., CE-$\mathcal{X}$) complexes of R-modules. If $\mathcal{X}$ is closed under extensions, then the following statements hold.

(1) $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ if and only if $\mathrm{CE}(\mathcal{W})$ is a CE-injective CE-cogenerator for $\mathrm{CE}(\mathcal{X})$.
(2) $\mathcal{W}$ is a projective generator for $\mathcal{X}$ if and only if $\mathrm{CE}(\mathcal{W})$ is a CE-projective CE-generator for $\mathrm{CE}(\mathcal{X})$.

As an application of Theorem A, we get the next result that compares to [3, Theorem 8.5] (see Proposition 3.7).

Theorem B. If $\mathcal{W}$ is a class of R-modules closed under finite direct sums satisfying $\mathcal{W} \perp \mathcal{W}$, then $\mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W})) = \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W})) \subseteq \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$. Furthermore, if $\mathcal{W}$ contains R or a faithfully injective R-module, then $\mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W})) = \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$.

Sather-Wagstaff, Sharif and White gave a stability result of Gorenstein categories as follows.

Theorem. ([7, Theorem 4.9]) Assume that $\mathcal{W}$ is a class of R-modules closed under finite direct sums satisfying $\mathcal{W} \perp \mathcal{W}$. Let $\cdots \to X_1 \to X_0 \to X_{-1} \to \cdots$ be an exact complex of R-modules in $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$ such that it is $\mathrm{Hom}(\mathcal{G}_{\mathcal{M}}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathcal{G}_{\mathcal{M}}(\mathcal{W}))$-exact, then each $K_i = \mathrm{Ker}(X_i \to X_{i-1})$ is in $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$ for $i \in \mathbb{Z}$.

The next result shows that the corresponding stability result of Cartan-Eilenberg Gorenstien categories is also true (see Theorem 3.12).

THEOREM C. Assume that $\mathcal{V}$ is a class of complexes of R-modules closed under finite direct sums. Let $\cdots \to G^{-1} \to G^0 \to G^1 \to \cdots$ be a CE-exact complex of complexes in $\mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ such that it is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact. Then each $K^i = \mathrm{Ker}(G^i \to G^{i+1})$ is in $\mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ for $i \in \mathbb{Z}$. In particular, $\mathcal{G}_{\mathrm{CE}}(\mathcal{G}_{\mathrm{CE}}(\mathcal{V})) = \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$.

We notice that Theorem C does not need the condition $\mathcal{V} \perp \mathcal{V}$, and one can find a partial converse in Proposition 3.15.

2. Preliminaries

We begin with some notation and terminology for use throughout this paper.

2.1. Let $\mathcal{A}$ be an abelian category. A complex $\cdots \to X_1 \stackrel{\delta_1^X}{\to} X_0 \stackrel{\delta_0^X}{\to} X_{-1} \to \cdots$ of objects of $\mathcal{A}$ will be denoted by (X, δ^X) or simply X. The nth boundary (resp., cycle, homology) of X is defined as $\mathrm{Im}\delta_{n+1}^X$ (resp., $\mathrm{Ker}\delta_n^X$, $\mathrm{Ker}\delta_n^X/\mathrm{Im}\delta_{n+1}^X$) and it is denoted by $\mathrm{B}_n(X)$ (resp., $\mathrm{Z}_n(X)$, $\mathrm{H}_n(X)$). We let $\mathrm{Z}(X)$, $\mathrm{B}(X)$ and $\mathrm{H}(X)$ denote the complexes of cycles, boundaries and homologies of X, respectively. For any $m \in \mathbb{Z}$, $\Sigma^m X$ denotes the complex with the degree-n term $(\Sigma^m X)_n = X_{n-m}$ and whose boundary operators are $(-1)^m \delta_{n-m}^X$. We set $\Sigma M = \Sigma^1 M$.

If X and Y are both complexes of objects of $\mathcal{A}$, then by a morphism $\alpha : X \to Y$ we mean a sequence $\alpha_n : X_n \to Y_n$ such that $\alpha_{n-1}\delta_n^X = \delta_n^Y \alpha_n$ for each $n \in \mathbb{Z}$. We let $\mathrm{Hom}(X, Y)$ denote the set of morphisms of complexes from X to Y, and $\mathrm{Ext}^i(X, Y)$ $(i \geq 1)$ denote the right derived functors of Hom. The mapping cone $\mathrm{Cone}(\alpha)$ of α is defined as $\mathrm{Cone}(\alpha)_n = Y_n \oplus X_{n-1}$ with nth boundary operator $\delta_n^{\mathrm{Cone}(\alpha)} = \begin{pmatrix} \delta_n^N & \alpha_{n-1} \\ 0 & -\delta_{n-1}^M \end{pmatrix}$. For an object M of $\mathcal{A}$, we let $\mathcal{H}\mathrm{om}(M, X)$ denote the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(M, X_1) \to \mathrm{Hom}_{\mathcal{A}}(M, X_0) \to \mathrm{Hom}_{\mathcal{A}}(M, X_{-1}) \to \cdots.$$

The complex $\mathcal{H}\mathrm{om}(X, M)$ can be given dually. Given a class $\mathcal{D}$ of objects of $\mathcal{A}$, we say that a complex X is $\mathrm{Hom}(\mathcal{D}, -)$-exact if $\mathrm{Hom}_{\mathcal{A}}(M, -)$ exacts the complex X for any $M \in \mathcal{D}$, that is, the complex $\mathcal{H}\mathrm{om}(M, X)$ is exact. The term $\mathrm{Hom}(-, \mathcal{D})$-exact is defined dually.

Let M be an object of $\mathcal{A}$. $\overline{M}$ is the complex

$$\cdots \to 0 \to M \xrightarrow{1_M} M \to 0 \to \cdots$$

with all terms 0 except M in the degrees 1 and 0, and $\underline{M}$ denotes the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

with all terms 0 except M in the degree 0.

2.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ a class of objects of $\mathcal{A}$. Following from [7], an exact complex of objects in $\mathcal{D}$ is called *totally $\mathcal{D}$-acyclic* if it is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{D}, -)$-exact and $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{D})$-exact. Let $\mathcal{G}_{\mathcal{A}}(\mathcal{D})$ denote the subcategory of $\mathcal{A}$ with objects of the form $M \cong \mathrm{Ker}(\delta^X_{-1})$ for some totally $\mathcal{D}$-acyclic complex X; in this case we say that M is *$\mathcal{D}$-Gorenstein*.

2.3. Throughout the paper, $\mathcal{M}$ always denotes the category of R-modules, and $\mathcal{C}$ denotes the category of complexes of R-modules. We usually use $\mathcal{W}$ and $\mathcal{X}$ to denote two classes of R-modules such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{W}$ is closed under finite direct sums. We write $\mathcal{X} \perp \mathcal{W}$ if $\mathrm{Ext}^i_R(X, W) = 0$ for any $X \in \mathcal{X}$, $W \in \mathcal{W}$ and $i \geq 1$. Following from [7], $\mathcal{W}$ is called a *cogenerator* for $\mathcal{X}$ if, for each $X \in \mathcal{X}$, there exists an exact sequence $0 \to X \to W \to X' \to 0$ with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. $\mathcal{W}$ is called an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$. *Generators* and *projective generators* can be defined dually.

The next definition can be found in [3].

Definition 2.4. A sequence $\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots$ of complexes of R-modules is said to be CE-exact if

(1) $\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots$,
(2) $\cdots \to \mathrm{Z}(X^{-1}) \to \mathrm{Z}(X^0) \to \mathrm{Z}(X^1) \to \cdots$,
(3) $\cdots \to \mathrm{B}(X^{-1}) \to \mathrm{B}(X^0) \to \mathrm{B}(X^1) \to \cdots$,
(4) $\cdots \to X^{-1}/\mathrm{B}(X^{-1}) \to X^0/\mathrm{B}(X^0) \to X^1/\mathrm{B}(X^1) \to \cdots$,
(5) $\cdots \to X^{-1}/\mathrm{Z}(X^{-1}) \to X^0/\mathrm{Z}(X^0) \to X^1/\mathrm{Z}(X^1) \to \cdots$, and
(6) $\cdots \to \mathrm{H}(X^{-1}) \to \mathrm{H}(X^0) \to \mathrm{H}(X^1) \to \cdots$

are exact.

By [3, Lemma 5.2], if (1) and (2), or (1) and (4) of Definition 2.4 are exact then all of (1) – (6) are exact.

2.5. Let $\mathcal{W}$ be a class of R-modules. Following from [3], a complex X of R-modules is called a CE-$\mathcal{W}$ complex if X, $\mathrm{Z}(X)$, $\mathrm{B}(X)$ and $\mathrm{H}(X)$ are complexes of R-modules in $\mathcal{W}$. We denote the class of CE-$\mathcal{W}$ complexes by CE($\mathcal{W}$). We sometimes name the CE-$\mathcal{W}$ complex by the name of the class $\mathcal{W}$. For example, the CE-projective (resp. CE-injective) complexes are actually the CE-$\mathcal{P}$ (resp. CE-$\mathcal{I}$)-complexes, where $\mathcal{P}$ (resp. $\mathcal{I}$) is the class of projective (resp. injective) R-modules.

Given two complexes X and Y of R-modules, it follows from [3, Theorems 5.5 and 5.7] that there exist two CE-exact sequences $\cdots \to P^{-1} \to P^0 \to X \to 0$ and $0 \to Y \to I^0 \to I^1 \to \cdots$, where P^i is a CE-projective complex of R-modules for each $i \leq 0$ and I^k is a CE-injective complex of R-modules for each $k \geq 0$. By [3, Proposition 6.3], we can compute derived functors of Hom using either of the two sequences. We denote these derived functors by $\overline{\mathrm{Ext}}^i(X, Y)$. Note that CE-exact sequence of complexes of R-modules is exact. Thus we have $\overline{\mathrm{Ext}}^1(X, Y) \subseteq \mathrm{Ext}^1(X, Y)$. One can check easily that for any CE-exact sequence $0 \to A \to B \to C \to 0$ of complexes of R-modules, there exist exact sequences

$$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C) \to \overline{\mathrm{Ext}}^1(X,A) \to \cdots$$

and

$$0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y) \to \overline{\mathrm{Ext}}^1(C,Y) \to \cdots.$$

The next lemma was given in [3, Proposition 2.1 and Lemmas 9.1, 9.2 and 9.3].

Lemma 2.6. *If X is a complex of R-modules and M is an R-module, then for any $k \in \mathbb{Z}$ we have the following statements.*

(1) $\mathrm{Hom}(X, \Sigma^k \underline{M}) \cong \mathrm{Hom}(X_k/\mathrm{B}_k(X), M)$.

(2) $\mathrm{Hom}(\Sigma^k \underline{M}, X) \cong \mathrm{Hom}(M, \mathrm{Z}_k(X))$.

(3) $\mathrm{Hom}(X, \Sigma^k \overline{M}) \cong \mathrm{Hom}(X_k, M)$.

(4) $\mathrm{Hom}(\Sigma^k \overline{M}, X) \cong \mathrm{Hom}(M, X_{k+1})$.

(5) $\overline{\mathrm{Ext}}^1(X, \Sigma^k \underline{M}) \cong \mathrm{Ext}^1(X_k/\mathrm{B}_k(X), M)$.

(6) $\overline{\mathrm{Ext}}^1(\Sigma^k \underline{M}, X) \cong \mathrm{Ext}^1(M, \mathrm{Z}_k(X))$.

(7) $\overline{\mathrm{Ext}}^1(X, \Sigma^k \overline{M}) \cong \mathrm{Ext}^1(X_k, M)$.

(8) $\overline{\mathrm{Ext}}^1(\Sigma^k \overline{M}, X) \cong \mathrm{Ext}^1(M, X_{k+1})$.

3. CE-Gorenstein categories

The next lemma that will be used frequently throughout the paper is akin to [3, Proposition 3.3].

LEMMA 3.1. *If $\mathcal{W}\perp\mathcal{W}$, then a complex X of R-modules is a* CE-$\mathcal{W}$ *complex if and only if X can be written as the form of* $(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\overline{X_i})\times(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\underline{X'_i})$ *(or* $(\prod_{i\in\mathbb{Z}}\Sigma^i\overline{X_i})\times(\prod_{i\in\mathbb{Z}}\Sigma^i\underline{X'_i})$*) with X_i and X'_i in $\mathcal{W}$ for each $i\in\mathbb{Z}$.*

PROOF. Let $X=(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\overline{X_i})\times(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\underline{X'_i})$, where X_i and X'_i are in $\mathcal{W}$ for each $i\in\mathbb{Z}$. Since $\oplus_{i\in\mathbb{Z}}\Sigma^i\overline{X_i}$ and $\oplus_{i\in\mathbb{Z}}\Sigma^i\underline{X'_i}$ are CE-$\mathcal{W}$ complexes, we get that X is a CE-$\mathcal{W}$ complex.

Conversely, assume that X is a CE-$\mathcal{W}$ complex. Then the sequences

$$0\to\mathrm{Z}_n(X)\to X_n\to\mathrm{B}_{n-1}(X)\to 0$$

and

$$0\to\mathrm{B}_n(X)\to\mathrm{Z}_n(X)\to\mathrm{H}_n(X)\to 0$$

are split for each $n\in\mathbb{Z}$ since $\mathcal{W}\perp\mathcal{W}$, and so $X_n=(\mathrm{B}_n(X)\oplus\mathrm{B}_{n-1}(X))\oplus\mathrm{H}_n(X)$. This implies that $X=(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\overline{\mathrm{B}_i(X)})\times(\oplus_{i\in\mathbb{Z}}\,\Sigma^i\underline{\mathrm{H}_i(X)})$. By assumption $\mathrm{B}_i(X)$ and $\mathrm{H}_i(X)$ are in $\mathcal{W}$ for each $i\in\mathbb{Z}$.

DEFINITION 3.2. Let $\mathcal{V}$ and $\mathcal{Y}$ be classes of complexes of R-modules such that $\mathcal{V}\subseteq\mathcal{Y}$. $\mathcal{V}$ is said to be a CE-cogenerator for $\mathcal{Y}$ if, for any $Y\in\mathcal{Y}$, there is a CE-exact sequence $0\to Y\to V\to Y'\to 0$ such that $V\in\mathcal{V}$ and $Y'\in\mathcal{Y}$. We say that $\mathcal{V}$ is a CE-injective CE-cogenerator for $\mathcal{Y}$ if $\mathcal{V}$ is a CE-cogenerator for $\mathcal{Y}$ and $\overline{\mathrm{Ext}}^i(Y,V)=0$ for any $Y\in\mathcal{Y}$, $V\in\mathcal{V}$ and $i\geq 1$. CE-generators and CE-projective CE-generators can be defined dually.

The next result gives a relationship between injective cogenerators (resp., projective generators) and CE-injective CE-cogenerators (resp., CE-projective CE-generators).

THEOREM 3.3. *If $\mathcal{X}$ is closed under extensions, then the following statements hold.*

(1) *$\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$ if and only if* CE$(\mathcal{W})$ *is a* CE-*injective* CE-*cogenerator for* CE$(\mathcal{X})$.

(2) *$\mathcal{W}$ is a projective generator for $\mathcal{X}$ if and only if* CE($\mathcal{W}$) *is a* CE-*projective* CE-*generator for* CE($\mathcal{X}$).

Proof. We prove part (1); the proof of part (2) is dual. Assume that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$, and let $X \in \mathrm{CE}(\mathcal{X})$. Then $\mathrm{B}_i(X)$ and $\mathrm{H}_i(X)$ are in $\mathcal{X}$ for all $i \in \mathbb{Z}$, and so there are exact sequences $0 \to \mathrm{B}_i(X) \to W_i' \to T_i' \to 0$ and $0 \to \mathrm{H}_i(X) \to W_i'' \to T_i'' \to 0$ with W_i' and W_i'' in $\mathcal{W}$, and T_i' and T_i'' in $\mathcal{X}$. Note that the sequence

$$0 \to \mathrm{B}_i(X) \to \mathrm{Z}_i(X) \to \mathrm{H}_i(X) \to 0$$

is exact and $\mathrm{Hom}(-, \mathcal{W})$-exact since $\mathcal{X} \perp \mathcal{W}$. Then by [5, Remark 8.2.2], we get an exact sequence

$$0 \to \mathrm{Z}_i(X) \to W_i' \oplus W_i'' \to T_i \to 0$$

such that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{B}_i(X) & \longrightarrow & \mathrm{Z}_i(X) & \longrightarrow & \mathrm{H}_i(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_i' & \longrightarrow & W_i' \oplus W_i'' & \longrightarrow & W_i'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_i' & \longrightarrow & T_i & \longrightarrow & T_i'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns is commutative. Since $\mathcal{X}$ is closed under extensions, $T_i \in \mathcal{X}$. Note that the sequence

$$0 \to \mathrm{Z}_i(X) \to X_i \to \mathrm{B}_{i-1}(X) \to 0$$

is exact and $\mathrm{Hom}(-, \mathcal{W})$-exact. Then by [5, Remark 8.2.2], we get an exact sequence

$$0 \to X_i \to W_i' \oplus W_i'' \oplus W_{i-1}' \to X_i' \to 0$$

such that the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Z}_i(X) & \longrightarrow & X_i & \longrightarrow & \mathrm{B}_{i-1}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_i' \oplus W_i'' & \longrightarrow & W_i' \oplus W_i'' \oplus W_{i-1}' & \longrightarrow & W_{i-1}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_i & \longrightarrow & X_i' & \longrightarrow & T_{i-1}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns is commutative. Since $\mathcal{X}$ is closed under extensions, $X_i' \in \mathcal{X}$. Now consider the following commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{i+1} & \longrightarrow & \mathrm{B}_i(X) & \longrightarrow & \mathrm{Z}_i(X) & \longrightarrow & X_i & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & W_{i+1}' \oplus W_{i+1}'' \oplus W_i' & \longrightarrow & W_i' & \longrightarrow & W_i' \oplus W_i'' & \longrightarrow & W_i' \oplus W_i'' \oplus W_{i-1}' & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{i+1}' & \longrightarrow & T_i' & \longrightarrow & T_i & \longrightarrow & X_i' & \longrightarrow & \cdots . \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Then we get a CE-exact sequence

$$0 \to X \to W \to X' \to 0,$$

where $W = (\oplus_{i\in\mathbb{Z}} \Sigma^i \overline{W_i'}) \times (\oplus_{i\in\mathbb{Z}} \Sigma^i \underline{W_i''})$. From the above constructions on W and X', we get $W \in \mathrm{CE}(\mathcal{W})$ and $X' \in \mathrm{CE}(\mathcal{X})$. Thus $\mathrm{CE}(\mathcal{W})$ is a CE-cogenerator for $\mathrm{CE}(\mathcal{X})$.

Let $X \in \mathrm{CE}(\mathcal{X})$ and $M \in \mathcal{W}$, and let $0 \to M \to I_0 \to I_{-1} \to \cdots$ be an injective resolution of M. Then $0 \to \Sigma^k \underline{M} \to \Sigma^k \underline{I_0} \to \Sigma^k \underline{I_{-1}} \to \cdots$ is a CE-exact sequence with $\Sigma^k \underline{I_i}$ CE-injective for each $i \leq 0$, and so

$$\overline{\mathrm{Ext}}^i(X, \Sigma^k \underline{M}) \cong \overline{\mathrm{Ext}}^1(X, \Sigma^k \underline{C_{2-i}})$$

for each $i \geq 1$, where $C_j = \mathrm{Coker}(I_{j+1} \to I_j)$ for $j \leq -1$ with $C_0 = \mathrm{Coker}(M \to I_0)$ and $C_1 = M$. By Lemma 2.6,

$$\overline{\mathrm{Ext}}^1(X, \Sigma^k \underline{C_{2-i}}) \cong \mathrm{Ext}^1(X_k/\mathrm{B}_k(X), C_{2-i}) \cong \mathrm{Ext}^i(X_k/\mathrm{B}_k(X), M) = 0$$

since $X_k/\mathrm{B}_k(X) \in \mathcal{X}$ by the exactness of the sequence

$$0 \to \mathrm{H}_k(X) \to X_k/\mathrm{B}_k(X) \to \mathrm{B}_{k-1}(X) \to 0$$

and the fact that $\mathcal{X}$ is closed under extensions. Thus $\overline{\mathrm{Ext}}^i(X, \Sigma^k \underline{M}) = 0$ for each $i \geq 1$. Similarly, $\overline{\mathrm{Ext}}^i(X, \Sigma^k \overline{M}) = 0$. Note that $\mathcal{W} \perp \mathcal{W}$, then a CE-$\mathcal{W}$ complex W can be written as the form of $(\prod_{i \in \mathbb{Z}} \Sigma^i \overline{W_i}) \times (\prod_{i \in \mathbb{Z}} \Sigma^i \underline{W_i'})$ with W_i and W_i' in $\mathcal{W}$ by Lemma 3.1. Thus $\overline{\mathrm{Ext}}^i(X, W) = 0$ for each $W \in \mathrm{CE}(\mathcal{W})$. This implies that $\mathrm{CE}(\mathcal{W})$ is a CE-injective CE-cogenerator for $\mathrm{CE}(\mathcal{X})$.

Conversely, assume that $\mathrm{CE}(\mathcal{W})$ is a CE-injective CE-cogenerator for $\mathrm{CE}(\mathcal{X})$. Let $M \in \mathcal{X}$. Then $\underline{M} \in \mathrm{CE}(\mathcal{X})$, and so there is a CE-exact sequence

$$0 \to \underline{M} \to W \to X' \to 0$$

with $W \in \mathrm{CE}(\mathcal{W})$ and $X' \in \mathrm{CE}(\mathcal{X})$. Thus we get an exact sequence

$$0 \to M \to W_0 \to X_0' \to 0$$

with $W_0 \in \mathcal{W}$ and $W_0 \in \mathcal{X}$. This implies that $\mathcal{W}$ is a cogenerator for $\mathcal{X}$. On the other hand, for any $M \in \mathcal{X}$ and $N \in \mathcal{W}$, by Lemma 2.6, we have $\mathrm{Ext}^i(M, N) \cong \mathrm{Ext}^1(M, C) \cong \overline{\mathrm{Ext}}^1(\underline{M}, \underline{C}) \cong \overline{\mathrm{Ext}}^i(\underline{M}, \underline{N}) = 0$ since $\underline{M} \in \mathrm{CE}(\mathcal{X})$ and $\underline{N} \in \mathrm{CE}(\mathcal{W})$, where C is an R-module. This implies that $\mathcal{W}$ is an injective cogenerator for $\mathcal{X}$. □

DEFINITION 3.4. Let $\mathcal{V}$ be a class of complexes of R-modules. We call that a complex X of R-modules has a CE-complete $\mathcal{V}$-resolution if there is a CE-exact sequence $\cdots \to X^{-1} \to X^0 \to X^1 \to \cdots$ of complexes in $\mathcal{V}$ such that the sequence is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact and $X \cong \mathrm{Ker}(X^0 \to X^1)$. The class of complexes X that has a CE-complete $\mathcal{V}$-resolution is denoted by $\mathcal{G}_{\mathrm{CE}}(\mathcal{V})$.

COROLLARY 3.5. *Let $\mathcal{X}$ be closed under extensions and $\mathcal{W}$ both an injective cogenerator and a projective generator for $\mathcal{X}$. Then* $\mathrm{CE}(\mathcal{X}) \subseteq \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$.

PROOF. Suppose that $X \in \mathrm{CE}(\mathcal{X})$, then there is a CE-exact sequence

$$(*) \qquad 0 \to X \to W^0 \to X^0 \to 0$$

with $W^0 \in \mathrm{CE}(\mathcal{W})$ and $X^0 \in \mathrm{CE}(\mathcal{X})$ by Theorem 3.3. For $W \in \mathrm{CE}(\mathcal{W})$, we get an exact sequence

$$0 \to \mathrm{Hom}(X^0, W) \to \mathrm{Hom}(W^0, W) \to \mathrm{Hom}(X, W) \to \overline{\mathrm{Ext}}^1(X^0, W).$$

Since $\overline{\mathrm{Ext}}^1(X^0, W) = 0$ by Theorem 3.3, the sequence $(*)$ is $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact. Similarly, one can check that it is also $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$-exact. Using the same procedure we can construct a CE-exact sequence

$$(\dagger) \qquad 0 \to X \to W^0 \to W^1 \to \cdots$$

with $W^i \in \mathrm{CE}(\mathcal{W})$ such that it is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact.

Using the similar method we can construct another CE-exact sequence

$$(\ddagger) \qquad \cdots \to W^{-2} \to W^{-1} \to X \to 0$$

with $W^i \in \mathrm{CE}(\mathcal{W})$ such that it is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact.

Now assembling the sequences ($\dagger$) and ($\ddagger$) we get a CE-complete $\mathrm{CE}(\mathcal{W})$-resolution of X, and so $X \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. □

LEMMA 3.6. *Let $\mathcal{W} \perp \mathcal{W}$. If $X \in \mathrm{CE}(\mathcal{W})$, then $X/\mathrm{B}(X)$ is a complex of R-modules in $\mathcal{W}$.*

PROOF. If $\mathcal{W} \perp \mathcal{W}$, then $\mathcal{W}$ is closed under extensions since $\mathcal{W}$ is closed under finite direct sums. Note that $\mathrm{H}_i(X)$ and $\mathrm{B}_i(X)$ are in $\mathcal{W}$ for all $i \in \mathbb{Z}$, then $X_i/\mathrm{B}_i(X) \in \mathcal{W}$ since the sequence

$$0 \to \mathrm{H}_i(X) \to X_i/\mathrm{B}_i(X) \to \mathrm{B}_{i-1}(X) \to 0$$

is exact. This implies that $X/\mathrm{B}(X)$ is a complex of R-modules in $\mathcal{W}$. □

The next result augments Corollary 3.5 in the special case $\mathcal{X} = \mathcal{G}_{\mathcal{M}}(\mathcal{W})$.

PROPOSITION 3.7. *If $\mathcal{W} \perp \mathcal{W}$, then $\mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W})) = \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W})) \subseteq \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$. Furthermore, if $\mathcal{W}$ contains R or a faithfully injective R-module, then $\mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W})) = \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$.*

PROOF. By [7, Theorem B and Corollary 4.7], $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$ is closed under extensions and $\mathcal{W}$ is both a projective generator and an injective cogenerator for $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$. Thus $\mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W})) \subseteq \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$ by Corollary 3.5. For the inverse containment, we let $X \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. Then there is a CE-exact sequence

$$(\,*\,) \qquad \cdots \to W^{-1} \to W^0 \to W^1 \to \cdots$$

of complexes in $\mathrm{CE}(\mathcal{W})$ such that it is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact and $X = \mathrm{Ker}(W^0 \to W^1)$. Let $M \in \mathcal{W}$. Then $\Sigma^k \overline{M} \in \mathrm{CE}(\mathcal{W})$, and so $\mathrm{Hom}(\Sigma^k \overline{M}, -)$ and $\mathrm{Hom}(-, \Sigma^k \overline{M})$ exact the sequence $(\,*\,)$. Thus $\mathrm{Hom}(M, -)$ and $\mathrm{Hom}(-, M)$ exact the sequence

$$\cdots \to W_k^{-1} \to W_k^0 \to W_k^1 \to \cdots$$

for any $k \in \mathbb{Z}$ by Lemma 2.6. Note that $X_k = \mathrm{Ker}(W_k^0 \to W_k^1)$, then $X_k \in \mathcal{G}_{\mathcal{M}}(\mathcal{W})$ for each $k \in \mathbb{Z}$.

Since $\Sigma^k \underline{M} \in \mathrm{CE}(\mathcal{W})$, we get an exact sequence

$$\cdots \to \mathrm{Hom}(W^1, \Sigma^k \underline{M}) \to \mathrm{Hom}(W^0, \Sigma^k \underline{M}) \to \mathrm{Hom}(W^{-1}, \Sigma^k \underline{M}) \to \cdots,$$

and so the sequence

$$\cdots \to \mathrm{Hom}(W_k^1/\mathrm{B}_k(W^1), M) \to \mathrm{Hom}(W_k^0/\mathrm{B}_k(W^0), M) \to \\ \to \mathrm{Hom}(W_k^{-1}/\mathrm{B}_k(W^{-1}), M) \to \cdots$$

is exact by Lemma 2.6. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{Hom}(W_k^1/\mathrm{B}_k(W^1), M) & \to & \mathrm{Hom}(W_k^0/\mathrm{B}_k(W^0), M) & \to & \mathrm{Hom}(W_k^{-1}/\mathrm{B}_k(W^{-1}), M) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{Hom}(W_k^1, M) & \to & \mathrm{Hom}(W_k^0, M) & \to & \mathrm{Hom}(W_k^{-1}, M) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{Hom}(\mathrm{B}_k(W^1), M) & \to & \mathrm{Hom}(\mathrm{B}_k(W^0), M) & \to & \mathrm{Hom}(\mathrm{B}_k(W^{-1}), M) & \to \cdots . \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Note that the first and second rows of the above diagram are exact, and all columns are exact since the sequence

$$0 \to \mathrm{B}_k(W^i) \to W_k^i \to W_k^i/\mathrm{B}_k(W^i) \to 0$$

is split exact for any $i \in \mathbb{Z}$ by Lemma 3.6. Thus the last row of the above diagram is exact. This implies that the sequence

$$(\dagger) \qquad \cdots \to \mathrm{B}_k(W^{-1}) \to \mathrm{B}_k(W^0) \to \mathrm{B}_k(W^1) \to \cdots$$

is $\mathrm{Hom}(-, \mathcal{W})$-exact for any $k \in \mathbb{Z}$.

On the other hand, since the sequence

$$\cdots \to \mathrm{Hom}(\Sigma^k \underline{M}, W^{-1}) \to \mathrm{Hom}(\Sigma^k \underline{M}, W^0) \to \mathrm{Hom}(\Sigma^k \underline{M}, W^1) \to \cdots$$

is exact, we get an exact sequence

$$\cdots \to \mathrm{Hom}(M, \mathrm{Z}_k(W^{-1})) \to \mathrm{Hom}(M, \mathrm{Z}_k(W^0)) \to \mathrm{Hom}(M, \mathrm{Z}_k(W^1)) \to \cdots$$

by Lemma 2.6. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}(M, \mathrm{Z}_k(W^{-1})) & \to & \mathrm{Hom}(M, \mathrm{Z}_k(W^0)) & \to & \mathrm{Hom}(M, \mathrm{Z}_k(W^1)) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}(M, W_k^{-1}) & \to & \mathrm{Hom}(M, W_k^0) & \to & \mathrm{Hom}(M, W_k^1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}(M, \mathrm{B}_{k-1}(W^{-1})) & \to & \mathrm{Hom}(M, \mathrm{B}_{k-1}(W^0)) & \to & \mathrm{Hom}(M, \mathrm{B}_{k-1}(W^1)) & \to & \cdots . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that the first and second rows of the above diagram are exact, and all columns are exact since the sequence

$$0 \to \mathrm{Z}_k(W^i) \to W_k^i \to \mathrm{B}_{k-1}(W^i) \to 0$$

is split exact for any $i \in \mathbb{Z}$. Thus the last row of the above diagram is exact. This implies that the sequence (†) is $\mathrm{Hom}(\mathcal{W}, -)$-exact for any $k \in \mathbb{Z}$. Thus $\mathrm{B}_k(X) = \mathrm{Ker}(\mathrm{B}_k(W^0) \to \mathrm{B}_k(W^1)) \in \mathcal{G}_{\mathcal{M}}(\mathcal{W})$ for any $k \in \mathbb{Z}$. Similarly, one can check that $\mathrm{Z}_k(X)$ and $\mathrm{H}_k(X)$ are in $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$ for any $k \in \mathbb{Z}$. Thus $X \in \mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W}))$. This implies that $\mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W})) = \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$.

By definition, $\mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W})) \subseteq \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$. Let $X \in \mathcal{G}_{\mathcal{C}}(\mathrm{CE}(\mathcal{W}))$. Then there is an exact sequence

$$(\ddagger) \qquad \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots$$

with $X^i \in \mathrm{CE}(\mathcal{W})$, such that the sequence is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact and $X = \mathrm{Ker}(X^0 \to X^1)$. If $R \in \mathcal{W}$, then $\Sigma^k \underline{R} \in \mathrm{CE}(\mathcal{W})$ for any $k \in \mathbb{Z}$, and so the sequence

$$\cdots \to \mathrm{Hom}(\Sigma^k \underline{R}, X^{-1}) \to \mathrm{Hom}(\Sigma^k \underline{R}, X^0) \to \mathrm{Hom}(\Sigma^k \underline{R}, X^1) \to \cdots$$

is exact. Thus the sequence

$$\cdots \to \mathrm{Z}_k(X^{-1}) \to \mathrm{Z}_k(X^0) \to \mathrm{Z}_k(X^1) \to \cdots$$

is exact by Lemma 2.6. This implies that the sequence (‡) is CE-exact, and so $X \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. If $\mathcal{W}$ contains a faithfully injective R-module E, then $\Sigma^k \underline{E} \in \mathrm{CE}(\mathcal{W})$ for any $k \in \mathbb{Z}$, and so the sequence

$$\cdots \to \mathrm{Hom}(X^1, \Sigma^k \underline{E}) \to \mathrm{Hom}(X^0, \Sigma^k \underline{E}) \to \mathrm{Hom}(X^{-1}, \Sigma^k \underline{E}) \to \cdots$$

is exact. Thus the sequence

$$\cdots \to X_k^{-1}/\mathrm{B}_k(X^{-1}) \to X_k^0/\mathrm{B}_k(X^0) \to X_k^1/\mathrm{B}_k(X^1) \to \cdots$$

is exact by Lemma 2.6. This implies that the sequence (†) is CE-exact, and so $X \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. □

The next result extends [6, Theorem 3.6]. We prove it using a different method for our convenience.

PROPOSITION 3.8. *Assume that $\mathcal{A}$ is an abelian category and $\mathcal{D}$ is a class of objects of $\mathcal{A}$. Let*

$$(\ast) \qquad 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be a sequence in $\mathcal{A}$, and let

$$W'^{+} \equiv \cdots \to W_2' \xrightarrow{\delta_2^{W'}} W_1' \xrightarrow{\delta_1^{W'}} W_0' \xrightarrow{\alpha'} M' \to 0$$

be a sequence with $W_i' \in \mathcal{D}$, and

$$W^{+} \equiv \cdots \to W_2 \xrightarrow{\delta_2^{W}} W_1 \xrightarrow{\delta_1^{W}} W_0 \xrightarrow{\alpha} M \to 0$$

a $\mathrm{Hom}(\mathcal{D}, -)$-exact sequence. Then there is a sequence

$$W''^{+} \equiv \cdots \to W_2 \oplus W_1' \to W_1 \oplus W_0' \to W_0 \xrightarrow{\alpha''} M'' \to 0$$

satisfying the following statements.

(1) *If $(\ast)$ is exact and $\mathrm{H}_i(W^+) = 0 = \mathrm{H}_{i-1}(W'^+)$, then $\mathrm{H}_i(W''^+) = 0$, where $i \in \mathbb{Z}$.*
(2) *If $\mathrm{Hom}(A, -)$ exacts $(\ast)$ and $\mathrm{H}_i(\mathcal{H}om(A, W^+)) = 0 = \mathrm{H}_{i-1}(\mathcal{H}om(A, W'^+))$, then $\mathrm{H}_i(\mathcal{H}om(A, W''^+)) = 0$, where A is an object of $\mathcal{A}$ and $i \in \mathbb{Z}$.*
(3) *If $\mathrm{Hom}(-, A)$ exacts $(\ast)$ and $\mathrm{H}_i(\mathcal{H}om(W^+, A)) = 0 = \mathrm{H}_{i-1}(\mathcal{H}om(W'^+, A))$, then $\mathrm{H}_i(\mathcal{H}om(W''^+, A)) = 0$, where A is an object of $\mathcal{A}$ and $i \in \mathbb{Z}$.*

Furthermore, if $\mathcal{A}$ is the category of complexes of R-modules, then we have the next statement.

(4) *If $(\ast)$, W^+ and W'^+ are CE-exact, then so is W''^+.*

Proof. By hypothesis, there is a map $\lambda^+ : W'^+ \to W^+$ of complexes of R-modules with $\lambda^+_{-1} = f$. Consider the following commutative diagram

$$\begin{array}{ccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}\overline{M'} \equiv \cdots & \longrightarrow & & 0 & \longrightarrow & 0 & \longrightarrow & M' & \xrightarrow{=} & M' & \to 0 \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Cone}(\lambda^+) \equiv \cdots & \to & & W_2 \oplus W_1' & \to & W_1 \oplus W_0' & \to & W_0 \oplus M' & \to & M & \to 0 \to \cdots \\
 & \downarrow & & \downarrow{=} & & \downarrow{=} & & \downarrow & & \downarrow \\
W''^+ \equiv \cdots & \longrightarrow & & W_2 \oplus W_1' & \to & W_1 \oplus W_0' & \longrightarrow & W_0 & \xrightarrow{\alpha''} & M'' & \to 0 \to \cdots, \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where $\alpha'' = g\alpha$. In the following, we show that W''^+ satisfies the conditions desired.

(1). If $\mathrm{H}_i(W^+) = 0 = \mathrm{H}_{i-1}(W'^+)$, then $\mathrm{H}_i(\mathrm{Cone}(\lambda^+)) = 0$ since the sequence

$$(\dagger) \qquad 0 \to W^+ \to \mathrm{Cone}(\lambda^+) \to \Sigma W'^+ \to 0$$

is exact. Note that the sequence

$$0 \to \Sigma^{-1}\overline{M'} \to \mathrm{Cone}(\lambda^+) \to W''^+ \to 0$$

is exact, then $\mathrm{H}_i(W''^+) = 0$ since $\mathrm{H}_j(\Sigma^{-1}\overline{M'}) = 0$ for any $j \in \mathbb{Z}$.

(2). Note that the exact sequence (†) is degree-wise split, then the sequence

$$0 \to \mathcal{H}\mathrm{om}(A, W^+) \to \mathcal{H}\mathrm{om}(A, \mathrm{Cone}(\lambda^+)) \to \mathcal{H}\mathrm{om}(A, \Sigma W'^+) \to 0$$

of complexes is exact, and so $\mathrm{H}_i(\mathcal{H}\mathrm{om}(A, \mathrm{Cone}(\lambda^+))) = 0$ since

$$\mathrm{H}_i(\mathcal{H}\mathrm{om}(A, W^+)) = 0 = \mathrm{H}_{i-1}(\mathcal{H}\mathrm{om}(A, W'^+)).$$

On the other hand, since the sequence

$$0 \to \mathcal{H}\mathrm{om}(A, \Sigma^{-1}\overline{M'}) \to \mathcal{H}\mathrm{om}(A, \mathrm{Cone}(\lambda^+)) \to \mathcal{H}\mathrm{om}(A, W''^+) \to 0$$

of complexes is exact clearly under the hypotheses, and $\mathrm{H}_j(\mathcal{H}\mathrm{om}(A, \Sigma^{-1}\overline{M'})) = 0$ for any $j \in \mathbb{Z}$, we get

$$\mathrm{H}_i(\mathcal{H}\mathrm{om}(A, W''^+)) = 0.$$

(3) can be proved in the same way as in the proof of (2).

(4). Assume that $\mathcal{A}$ is the category of complexes of R-modules, and $(\ast), W^+$ and W'^+ are CE-exact. Then the sequences

$$\cdots \to \mathrm{Hom}(\Sigma^k \underline{R}, W_1') \to \mathrm{Hom}(\Sigma^k \underline{R}, W_0') \to \mathrm{Hom}(\Sigma^k \underline{R}, M') \to 0$$

and

$$\cdots \to \mathrm{Hom}(\Sigma^k \underline{R}, W_1) \to \mathrm{Hom}(\Sigma^k \underline{R}, W_0) \to \mathrm{Hom}(\Sigma^k \underline{R}, M) \to 0$$

are exact by Lemma 2.6. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, W_1) & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, W_0) & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow {=} & & \\
\cdots & \to & \mathrm{Hom}(\Sigma^k \underline{R}, W_1 \oplus W_0') & \to & \mathrm{Hom}(\Sigma^k \underline{R}, W_0 \oplus M') & \to & \mathrm{Hom}(\Sigma^k \underline{R}, M) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, W_0') & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, M') & \longrightarrow & 0 & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since all columns of the above diagram are exact, we get that the sequence

$$\cdots \to \mathrm{Hom}(\Sigma^k \underline{R}, W_1 \oplus W_0') \to \mathrm{Hom}(\Sigma^k \underline{R}, W_0 \oplus M') \to \mathrm{Hom}(\Sigma^k \underline{R}, M) \to 0$$

is exact. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, M') & \xrightarrow{=} & \mathrm{Hom}(\Sigma^k \underline{R}, M') & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}(\Sigma^k \underline{R}, W_1 \oplus W_0') & \to & \mathrm{Hom}(\Sigma^k \underline{R}, W_0 \oplus M') & \to & \mathrm{Hom}(\Sigma^k \underline{R}, M) & \to & 0 \\
 & & \downarrow {=} & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathrm{Hom}(\Sigma^k \underline{R}, W_1 \oplus W_0') & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, W_0) & \longrightarrow & \mathrm{Hom}(\Sigma^k \underline{R}, M'') & \to & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since the sequence $(\ast)$ is CE-exact, we get that all columns of the above diagram are exact. On the other hand, note that the first and second rows of the above diagram are exact, then so is the third row. This implies that the sequence

$$\cdots \to \mathrm{Z}_k(W_1 \oplus W'_0) \to \mathrm{Z}_k(W_0) \to \mathrm{Z}_k(M'') \to 0$$

is exact for any $k \in \mathbb{Z}$ by Lemma 2.6. The complex W''^{+} is exact by (1), and so it is CE-exact. □

The next corollary is immediate by Proposition 3.8.

COROLLARY 3.9. *Assume that $\mathcal{A}$ is an abelian category and $\mathcal{D}$ is a class of objects of $\mathcal{A}$. Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence in $\mathcal{A}$, and let

$$W'_{n-1} \to \cdots \to W'_1 \to W'_0 \to M' \to 0$$

be an exact sequence with $W'_i \in \mathcal{D}$, and

$$W_n \to W_{n-1} \to \cdots \to W_1 \to W_0 \to M \to 0$$

an exact and $\mathrm{Hom}(\mathcal{D}, -)$*-exact sequence. Then there is an exact sequence*

$$W_n \oplus W'_{n-1} \to \cdots \to W_1 \oplus W'_0 \to W_0 \to M'' \to 0$$

satisfying the statements (2), (3) *and* (4) *of Proposition* 3.8.

The following corollary can be checked easily using induction on t.

COROLLARY 3.10. *Assume that $\mathcal{A}$ is an abelian category and $\mathcal{D}$ is a class of objects of $\mathcal{A}$. Let*

$$M^{-t} \to \cdots \to M^{-1} \to M^0 \to M \to 0$$

be an exact sequence in $\mathcal{A}$, and let

$$W^i_{t+i} \to \cdots \to W^i_1 \to W^i_0 \to M^i \to 0$$

be exact and $\mathrm{Hom}(\mathcal{D}, -)$*-exact with each $W^i_j \in \mathcal{D}$. Then there is an exact sequence*

$$W_0^{-t} \oplus W_1^{-t+1} \oplus \cdots \oplus W_t^0 \to \cdots \to W_0^{-1} \oplus W_1^0 \to W_0^0 \to M \to 0$$

satisfying the statements (2), (3) *and* (4) *of Proposition* 3.8.

Furthermore, we have the next result that is a tool for the proof of Theorem 3.12.

COROLLARY 3.11. *Assume that $\mathcal{A}$ is an abelian category and $\mathcal{D}$ is a class of objects of $\mathcal{A}$. Let*

$$\cdots \to M^{-2} \to M^{-1} \to M^0 \to M \to 0$$

be an exact sequence in $\mathcal{A}$, and let

$$\cdots \to W_2^i \to W_1^i \to W_0^i \to M^i \to 0$$

be exact and $\mathrm{Hom}(\mathcal{D}, -)$*-exact with each $W_j^i \in \mathcal{D}$. Then there is an exact sequence*

$$\cdots \to W_0^{-2} \oplus W_1^{-1} \oplus W_2^0 \to W_0^{-1} \oplus W_1^0 \to W_0^0 \to M \to 0$$

satisfying the statements (2), (3) *and* (4) *of Proposition* 3.8.

The dual versions of Proposition 3.8 and Corollaries 3.9, 3.10 and 3.11 can be given easily.

The next result compares to [6, Theorem 4.1] and [7, Corollary 4.10]. We notice that this result does not need the condition $\mathcal{V} \perp \mathcal{V}$.

THEOREM 3.12. *Assume that $\mathcal{V}$ is a class of complexes of R-modules closed under finite direct sums. Let $G \equiv \cdots \to G^{-1} \to G^0 \to G^1 \to \cdots$ be a* CE*-exact sequence of complexes of R-modules. If $G^i \in \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ for each $i \in \mathbb{Z}$ and G is* $\mathrm{Hom}(\mathcal{V}, -)$ *and* $\mathrm{Hom}(-, \mathcal{V})$*-exact, then $K^i = \mathrm{Ker}(G^i \to G^{i+1}) \in \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ for each $i \in \mathbb{Z}$. In particular, we have $\mathcal{G}_{\mathrm{CE}}(\mathcal{G}_{\mathrm{CE}}(\mathcal{V})) = \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$.*

PROOF. We only need to prove that $K^1 = \mathrm{Ker}(G^1 \to G^2)$ is in $\mathcal{G}_{\mathrm{CE}}(\mathcal{V})$. Since $G^i \in \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ for each $i \in \mathbb{Z}$, there is a CE-exact sequence

$$\cdots \to W_1^i \to W_0^i \to G^i \to 0$$

of complexes of R-modules for each $i \leq 0$ with $W_j^i \in \mathcal{V}$ such that it is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact. Thus there is a CE-exact sequence

$$(\dagger) \qquad \cdots \to W_0^{-2} \oplus W_1^{-1} \oplus W_2^0 \to W_0^{-1} \oplus W_1^0 \to W_0^0 \to K^1 \to 0$$

of complexes of R-modules such that it is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact by Corollary 3.11.

On the other hand, note that there is a CE-exact sequence

$$0 \to G^i \to W_{-1}^i \to W_{-2}^i \to \cdots$$

of complexes of R-modules for each $i \geq 1$ with $W_j^i \in \mathcal{V}$, such that it is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact, then there is a CE-exact sequence

$$(\ddagger) \qquad 0 \to K^1 \to W_{-1}^1 \to W_{-2}^1 \oplus W_{-1}^2 \to W_{-3}^1 \oplus W_{-2}^2 \oplus W_{-1}^3 \to \cdots$$

of complexes of R-modules such that it is $\mathrm{Hom}(\mathcal{V}, -)$ and $\mathrm{Hom}(-, \mathcal{V})$-exact by the dual version of Corollary 3.11.

Now assembling the sequences (†) and (‡) we get $K^1 \in \mathcal{G}_{\mathrm{CE}}(\mathcal{V})$ since $\mathcal{V}$ is closed under finite direct sums. □

Lemma 3.13. *The following statements hold.*

(1) $\mathcal{W} \perp \mathcal{X}$ *if and only if* $\overline{\mathrm{Ext}}^i(C, D) = 0$ *for any* $C \in \mathrm{CE}(\mathcal{W})$, $D \in \mathrm{CE}(\mathcal{X})$ *and* $i \geq 1$.
(2) $\mathcal{X} \perp \mathcal{W}$ *if and only if* $\overline{\mathrm{Ext}}^i(D, C) = 0$ *for any* $C \in \mathrm{CE}(\mathcal{W})$, $D \in \mathrm{CE}(\mathcal{X})$ *and* $i \geq 1$.

Proof. We prove part (1); the proof of part (2) is dual. Let $W \in \mathcal{W}$ and $k \in \mathbb{Z}$. Then $\overline{\mathrm{Ext}}^i(\Sigma^k \overline{W}, D) \cong \overline{\mathrm{Ext}}^1(\Sigma^k \overline{T}, D) \cong \mathrm{Ext}^1(T, D_k) \cong \mathrm{Ext}^i(W, D_k) = 0$ by Lemma 2.6, where T is an R-module. On the other hand, $\overline{\mathrm{Ext}}^i(\Sigma^k \underline{W}, D) \cong \overline{\mathrm{Ext}}^1(\Sigma^k \underline{T}, D) \cong \mathrm{Ext}^1(T, \mathrm{Z}_k(D)) \cong \mathrm{Ext}^i(W, \mathrm{Z}_k(D)) = 0$ by Lemma 2.6. Note that $\mathcal{W} \perp \mathcal{W}$, then C can be written as the form of $(\oplus_{i \in \mathbb{Z}} \Sigma^i \overline{W_i}) \times (\oplus_{i \in \mathbb{Z}} \Sigma^i \underline{W_i'})$ with W_i and W_i' in $\mathcal{W}$ by Lemma 3.1, and so $\overline{\mathrm{Ext}}^i(C, D) = 0$ for each $i \geq 1$.

Conversely, fix $W \in \mathcal{W}$ and $X \in \mathcal{X}$, then $\mathrm{Ext}^i(W, X) \cong \mathrm{Ext}^1(T, X) \cong \overline{\mathrm{Ext}}^1(\underline{T}, \underline{X}) \cong \overline{\mathrm{Ext}}^i(\underline{W}, \underline{X})$ by Lemma 2.6, where T is an R-module. Note that $\underline{W} \in \mathrm{CE}(\mathcal{W})$ and $\underline{X} \in \mathrm{CE}(\mathcal{X})$, then $\overline{\mathrm{Ext}}^i(\underline{W}, \underline{X}) = 0$, and so $\mathrm{Ext}^i(W, X) = 0$ for each $i \geq 1$. □

The next result can be checked easily.

Lemma 3.14. *Let $\mathcal{W}$ be closed under extensions, and let*

$$0 \to X' \to X \to X'' \to 0$$

be an CE*-exact sequence of complexes of R-modules. If X' and X'' are in* $\mathrm{CE}(\mathcal{W})$*, then so is X.*

We close the paper with the following result that contains a partial converse of Theorem 3.12.

PROPOSITION 3.15. *Let* $\mathcal{W} \perp \mathcal{W}$, *and let* $G \equiv \cdots \to G^{-1} \to G^0 \to G^1 \to \cdots$ *be a* CE-*exact sequence of complexes of* R-*modules. Then* $G^i \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$ *for each* $i \in \mathbb{Z}$ *and* G *is* $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ *and* $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-*exact if and only if* $K^i = \mathrm{Ker}(G^i \to G^{i+1}) \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$ *for each* $i \in \mathbb{Z}$.

PROOF. By Theorem 3.12, we only need to prove the "if" part. Let $K^i \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$ for each $i \in \mathbb{Z}$. Since $\mathrm{CE}(\mathcal{G}(\mathcal{W})) = \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$ by Proposition 3.7, we get that $G^i \in \mathrm{CE}(\mathcal{G}_{\mathcal{M}}(\mathcal{W}))$ for each $i \in \mathbb{Z}$ by Lemma 3.14 since $\mathcal{G}_{\mathcal{M}}(\mathcal{W})$ is closed under extensions by [7, Theorem B], and so $G^i \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. Now we only need to prove that $\overline{\mathrm{Ext}}^1(T, K) = 0$ and $\overline{\mathrm{Ext}}^1(K, T) = 0$ for any $T \in \mathrm{CE}(\mathcal{W})$ and $K \in \mathcal{G}_{\mathrm{CE}}(\mathrm{CE}(\mathcal{W}))$. Let $\cdots \to C^{-1} \to C^0 \to C^1 \to \cdots$ be a CE-exact sequence of complexes of R-modules with $C^i \in \mathrm{CE}(\mathcal{W})$ such that it is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact and $K = \mathrm{Ker}(C^0 \to C^1)$. Since $\overline{\mathrm{Ext}}^1(T, C^0) = 0$ by Lemma 3.13, we get an exact sequence

$$0 \to \mathrm{Hom}(T, K) \to \mathrm{Hom}(T, C^0) \to \mathrm{Hom}(T, L^1) \to \overline{\mathrm{Ext}}^1(T, K) \to \overline{\mathrm{Ext}}^1(T, C^0) = 0,$$

where $L^1 = \mathrm{Ker}(C^1 \to C^2)$. On the other hand, the sequence

$$0 \to \mathrm{Hom}(T, K) \to \mathrm{Hom}(T, C^0) \to \mathrm{Hom}(T, L^1) \to 0$$

is exact, and so $\overline{\mathrm{Ext}}^1(T, K) = 0$. Similarly, one can check that $\overline{\mathrm{Ext}}^1(K, T) = 0$. Thus G is $\mathrm{Hom}(\mathrm{CE}(\mathcal{W}), -)$ and $\mathrm{Hom}(-, \mathrm{CE}(\mathcal{W}))$-exact. □

Acknowledgments. This work was partly supported by NSF of China (Grant Nos. 11301240, 11226059, 11101197), the Young Scholars Science Foundation of Lanzhou Jiaotong University (Grant No. 2012020), and NSF of Gansu Province of China (No. 145RJZA079). The authors would like to thank the referee for a careful reading of the paper and for many useful comments and suggestions.

REFERENCES

[1] M. AUSLANDER - M. BRIDGER, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969.

[2] H. CARTAN - S. EILENBERG, *Homological algebra*, Princeton University Press, Princeton, N.J., 1956.

[3] E. E. ENOCHS, *Cartan-Eilenberg complexes and resolutions*, J. Algebra **342** (2011), pp. 16–39.

[4] E. E. ENOCHS - O. M. G. JENDA, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), pp. 611–633.
[5] E. E. ENOCHS - O. M. G. JENDA, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000.
[6] Z. HUANG, *Proper resolutions and Gorenstein categories*, J. Algebra **393** (2013), pp. 142–169.
[7] S. SATHER-WAGSTAFF - T. SHARIF - D. WHITE, *Stability of Gorenstein categories*, J. London Math. Soc. **77** (2008), pp. 481–502.
[8] J.-L. VERDIER, *Des catégories dérivées des catégories abéliennes*, Astérisque 239, Soc. Math. France, Paris, 1996.

Manoscritto pervenuto in redazione il 2 Ottobre 2012.