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NSE characterization of projective special linear group $L_5(2)$

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ABSTRACT - Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k \mid k \in \omega(G)\}$. In Khatami et al. and Liu, $L_3(2)$ and $L_3(4)$ are uniquely determined by nse(G). In this paper, we prove that if G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_5(2))$, then $G \cong L_5(2)$.

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1. Introduction

Here we introduce some notations which will be used. If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k \mid k \in \omega(G)\}$. Let $\pi(G) = \pi(|G|)$. Let n_p or $n_p(G)$ denote the number of the Sylow $p-$subgroups P_p of G. Other notations are standard (see [1]).

A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [2]).

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Thompson's Problem. Let $T(G) = \{(n, s_n) \big| n \in \omega(G)$ and $s_n \in \mathrm{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$ for some (finite?) group H. If G is a finite solvable group, is it true that H is also necessarily solvable?

It was proved that: Let G be a group and M some K_i-group, $i = 3, 4$, then $G \cong M$ if and only if $|G| = |M|$ and $\mathrm{nse}(G) = \mathrm{nse}(M)$ (see [3, 4]). And also the groups A_{12}, A_{13} and $L_5(2)$ are characterizable by order and nse (see [5, 6, 7]). Recently, all sporadic simple groups are characterizable by nse and order (see [8]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the **Thompson's Problem**, in other words, it remains only nse(G), whether can it characterize finite simple groups? Up to now, some groups especially for $L_2(q)$, where $q = 7, 8, 9, 11, 13$, can be characterized by only the set nse(G) (see [9, 10]). The author has proved that the groups $L_3(4)$ and $L_2(16)$ are characterizable by nse (see [11, 12] respectively).

In this paper, it is shown that the group $L_5(2)$, which the number of the set of the same order is 13, also can be characterized by nse($L_5(2)$).

2. Some Lemmas

In this section we will give some lemmas used in the proof of the main theorem.

LEMMA 2.1 [14]. *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

LEMMA 2.2 [13]. *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic, then the number of elements of order n is always a multiple of p^s.*

LEMMA 2.3 [10]. *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

LEMMA 2.4 [15, Theorem 9.3.1]. *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \cdots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \cdots, s\}$:*

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ *for some* p_j.
(2) *The order of some chief factor of* G *is divided by* $q_i^{\beta_i}$.

To prove $G \cong L_5(2)$, we need the structure of simple K_n-groups with $n = 4, 5$.

LEMMA 2.5 [16]. *Let* G *be a simple* K_4*-group. Then* G *is isomoorphic to one of the following groups:*

(1) A_7, A_8, A_9 *or* A_{10}.
(2) M_{11}, M_{12} *or* J_2.
(3) *One of the following:*

(a) $L_2(r)$, *where* r *is a prime and* $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ *with* $a \geq 1$, $b \geq 1$, $c \geq 1$, *and* v *is a prime greater than* 3.
(b) $L_2(2^m)$, *where* $2^m - 1 = u$, $2^m + 1 = 3t^b$ *with* $m \geq 2$, u, t *are primes,* $t > 3$, $b \geq 1$.
(c) $L_2(3^m)$, *where* $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ *or* $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, *with* $m \geq 2$, u, t *are odd primes,* $b \geq 1$, $c \geq 1$.
(d) *One of the following* 28 *simple groups:* $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)$.

LEMMA 2.6 [17]. *Each simple* K_5*-group is isomorphic to one of the following simple groups:*

(1) $L_2(q)$ *with* $|\pi(q^2 - 1)| = 4$.
(2) $L_3(q)$ *with* $|\pi((q^2 - 1)(q^3 - 1))| = 4$.
(3) $U_3(q)$ *with* q *satisfies* $|\pi((q^2 - 1)(q^3 + 1))| = 4$.
(4) $O_5(q)$ *with* $|\pi(q^4 - 1)| = 4$.
(5) $Sz(2^{2m+1})$ *with* $|\pi((2^{2m+1} - 1)(2^{4m+2} + 1))| = 4$.
(6) $R(q)$ *where* q *is an odd power of* 3 *and* $|\pi(q^2 - 1)| = 3$ *and* $|\pi(q^2 - q + 1)| = 1$.
(7) *The following 30 simple groups:* A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSp_6(3)$, $PSp_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$ *or* $G_2(9)$.

LEMMA 2.7. *Let* G *be a simple* K_n*-group with* $n = 4, 5$ *and* $31 \mid |G| \mid 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. *Then* $G \cong L_2(31)$ *or* $L_5(2)$.

Proof. We prove the Lemma by the following two steps.

Step a. G is a simple K_4-group.

Order consideration rules out the cases (1) and (2) of Lemma 2.5.

So we consider Lemma 2.5(3). We will deal with this with the following cases.

- Case 1. $G \cong L_2(r)$, where $r \in \{5, 7, 31\}$.
  - Let $r = 5$ or 7, then $|\pi(q^2 - 1)| = 2$, which contradicts $|\pi(q^2 - 1)| = 3$.
  - Let $r = 31$, then $|\pi(q^2 - 1)| = 3$. So we have $G \cong L_2(31)$.

- Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 5, 7, 31\}$.
  - Let $u = 3$, then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbb{N}$, a contradiction.
  - Let $u = 5$ then the equation $2^m - 1 = 5$ has no solution in $\mathbb{N}$, a contradiction.
  - Let $u = 7$, then $m = 3$, and $2^3 + 1 = 3t^b$. Thus $t = 3$ and $b = 1$. But $t > 3$, a contradiction.
  - Let $u = 31$, then $m = 5$, $2^5 + 1 = 3 \cdot 11$. Thus we have $G \cong L_2(2^5)$. But $11 \mid |L_2(2^5)|$, a contradiction.

- Case 3. $G \cong L_2(3^m)$

We will consider the case by the following two subcases.

  - Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.

  We can suppose that $t \in \{3, 5, 7, 31\}$

  Let $t = 3, 5$ or 31, the equation $3^m + 1 = 4t$ has no solution. So we rule out the case.

  Let $t = 7$, then $m = 3$ and so $3^3 - 1 = 2 \cdot 11^1$, which means $11 \mid |G|$, a contradiction.

  - Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.

  We can suppose that $u \in \{3, 5, 7, 31\}$

  Let $u = 3, 7$ or 31, then the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$, a contradiction.

  Let $u = 5$, then $3^m = 11$, a contradiction.

- For the remaining case, considering the order of G we can rule out this case.

Step b. G is a simple K_5-group.

From Lemma 2.6, we will consider the following cases.

Let $G \cong L_2(q)$, we can suppose that $q = 2^m, 3^2, 5, 7, 31$.

- If $q = 2^m$, then we have $m = 6, 8, 9, 10$.
- If $m = 6$, then 13 divides the order of G.
- If $m = 8$, then $17 \mid |G|$. If $m = 9$, then 19 and 73 belong to $\pi(G)$.
- If $m = 10$, then 11 and 41 belong to $\pi(G)$.

Thus we rule out this case.
Similarly we can rule out the other groups except for $L_5(2)$.
From Steps a and b, we have that $G \cong L_2(31)$ or $L_5(2)$.
This completes the proof of the Lemma. □

3. Main result and its proof

Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_5(2))$, and s_n be the number of elements of order n of G. By Lemma 2.3 we have G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$(1) \qquad \begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d \end{cases}$$

THEOREM 3.1. *Let G be a group with* $\mathrm{nse}(G) = \mathrm{nse}(L_5(2)) = \{1, 6975, 75392, 416640, 476160, 624960, 666624, 833280, 952320, 1249920, 1333248, 1428480, 1935360\}$, *where $L_5(2)$ is the projective special linear group of degree 5 over the field of order 2. Then $G \cong L_5(2)$.*

PROOF. We prove the theorem by first proving that $\pi(G) \subseteqq \{2, 3, 5, 7, 31\}$, second showing that $|G| = |L_5(2)|$, and so $G \cong L_5(2)$.

By (1), $\pi(G) \subseteqq \{2, 3, 5, 7, 31, 41, 373, 624961, 833281, 952321, 1249921, 1333249\}$. If $m > 2$, then $\phi(m)$ is even, hence $s_2 = 6975, 2 \in \pi(G)$.

In the following, we prove that $41 \notin \pi(G)$. If $41 \in \pi(G)$, then by (1), $s_{41} = 1428480$. If $2 \cdot 41 \in \omega(G)$, then $s_{82} \notin \mathrm{nse}(G)$. Therefore $82 \notin \omega(G)$. Now we consider Sylow 41-subgroup P_{41} acts fixed point freely on the set of elements of order 2, then $|P_{41}| \mid s_2(= 46975)$, a contradiction. Similarly we can prove that the primes 373, 624961, 833281, 952321, 1249921, and 1333249 do not belong to $\pi(G)$. Hence we have $\pi(G) \subseteqq \{2, 3, 5, 7, 31\}$. Furthermore, by (1) $s_3 = 75392$, $s_5 = 666624$, $s_7 = 476160$ and $s_{31} = 1935360$.

If $7 \cdot 13 \in \omega(G)$, then by (1) $91 \mid 1 + s_7 + s_{13} + s_{91}$ and $\phi(91) \mid s_{91}$, and so we have $s_{91} \notin \mathrm{nse}(G)$, a contradiction. Hence $7 \cdot 13 \notin \omega(G)$.

If $2^{a_1} \in \omega(G)$, then $\phi(2^{a_1}) = 2^{a_1-1} \mid s_{2^{a_1}}$ and so $0 \leq a_1 \leq 12$.

If $5^{a_2} \in \omega(G)$, then $0 \leq a_2 \leq 2$. If $5^2 \in \omega(G)$, then $s_{25} \notin \mathrm{nse}(G)$, a contradiction. Hence $0 \leq a_2 \leq 1$.

If $7^{a_3} \in \omega(G)$, then $0 \leq a_3 \leq 2$. If $7^2 \in \omega(G)$, then $s_{49} \notin \mathrm{nse}(G)$, a contradiction. Therefore $0 \leq a_3 \leq 1$.

If $31^{a_4} \in \omega(G)$, then $0 \leq a_4 \leq 2$. Since $s_{31^2} \notin \mathrm{nse}(G)$, then $0 \leq a_4 \leq 1$.

If $3^{a_5} \in \omega(G)$, then $0 \leq a_5 \leq 4$.

In the following, we consider $\pi(G)$ as a subset of $\{2, 3, 5, 7, 31\}$ which contains the prime 2.

Case a. $\pi(G) = \{2\}$.

In this case, $\omega(G) \subset \{1, 2, 2^2, \cdots, 2^{12}\}$. But the number of the set of $\mathrm{nse}(G)$ is 13, it is easy to have a contradiction since the equation $9999360 + 75392k_1 + 416640k_2 + 476160k_3 + 624960k_4 + 666624k_5 + 833280k_6 + 952320k_7 + 1249920k_8 + 1333248k_9 + 1428480k_{10} + 1935360k_{11} = 2^m$, where $k_1, k_2, \cdots, k_{11}, m$ are nonnegative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + ... + k_{11} \leq 0$. Then the equation has no solution in $\mathbb{N}$.

Case b. $\pi(G) = \{2, 3\}$.

If $2^a \cdot 3^b \in \omega(G)$, then by Lemma 2.1, we have $0 \leq a \leq 11$ and $0 \leq b \leq 4$.

We have $\exp(P_3) = 3$, 9, 27 or 81.

Suppose that $\exp(P_3) = 3$, then by Lemma 2.1, we have that $|P_3| \mid 1 + s_3 (= 79393)$ and so $|P_3| \mid 9$.

- Let $|P_3| = 3$. Then $n_3 = s_3/\phi(3) = 75392/2 = 2^6 \cdot 19 \cdot 31$, which means $19, 31 \in \pi(G)$, a contradiction.
- Let $|P_3| = 9$. Then $9999360 + 75392k_1 + 416640k_2 + 476160k_3 + 624960k_4 + 666624k_5 + 833280k_6 + 952320k_7 + 1249920k_8 + 1333248k_9 + 1428480k_{10} + 1935360k_{11} = 2^m \cdot 3^2$, where $k_1, k_2, \cdots, k_{11}, m, n$ are nonnegative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + ... + k_{11} \leq 36$. So $156240 + 1178k_1 + 6510k_2 + 7440k_3 + 9765k_4 + 10416k_5 + 13020k_6 + 14880k_7 + 19530k_8 + 20832k_9 + 22320k_{10} + 30240k_{11} = 2^{m-6} \cdot 3^2$ and since $31(5040 + 38k_1 + 210k_2 + 240k_3 + 315k_4 + 336k_5 + 420k_6 + 480k_7 + 630k_8 + 672k_9 + 720k_{10}) + 30240k_{11} = 2^{m-6} \cdot 3^2$, the equation has no solution in $\mathbb{N}$.

Suppose that $\exp(P_3) = 9$, then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9$ and so $|P_3| \mid 27$.

- Let $|P_3| = 9$, then $s_9 = 1249920, 624960, 1428480, 1935360$ and so $n_3 = s_9/\phi(9)$. It follows that $5 \in \pi(G)$, a contradiction.
- Let $|P_3| = 27$, then by Lemma 2.2, we have $s_9 = 1935360$, and so $n_3 = s_9/\phi(9)$. We also have that $5 \in \pi(G)$, a contradiction.

Suppose that $\exp(P_3) = 27$, then if $|P_3| = 27$ and by Lemma 2.1, $s_{27} = 1244920, 1935360$ and so $n_3 = s_{27}/\phi(27)$. It follows that $5 \in \pi(G)$, a contradiction. If $|P_3| \geq 3^4$, then similarly as the Case "$\exp(P_3) = 3$ and $|P_3| = 9$", we also can rule out these cases.

Suppose that $\exp(P_3) = 81$, then $s_{81} = 1935360$ and so $n_3 = s_{81}/\phi(81) = 7680$, it follows that $5 \in \pi(G)$. If $|P_3| \geq 243$, then by Lemma 2.2, $s_{81} = 81t$ for some integer t, but the equation has no solution in $\mathbb{N}$.

Case c. $\pi(G) = \{2, 5\}$

Since $5^2 \notin \omega(G)$, then by Lemma 2.1, $|P_5| \mid 1 + s_5(= 5333 \cdot 125)$ and so $|P_5| \mid 125$.

- Let $|P_5| = 5$, then $n_5 = s_5/\phi(5) = 2^8 \cdot 3 \cdot 7 \cdot 31$. It follows that $3, 7, 31 \in \pi(G)$, a contradiction.
- Let $|P_5| \geq 5^2$. Then by Lemma 2.2, $s_5 = 25t$ for some integer t, but the equation has no solution in $\mathbb{N}$.

Case d. $\pi(G) = \{2, 7\}$.

Since $7^2 \notin \omega(G)$, then $|P_7| \mid 1 + s_7(= 12481)$ and so $|P_7| \mid 7$. So $n_7 = s_7/\phi(7) = 79360 = 2^9 \cdot 3 \cdot 31$, which means $3, 31 \in \pi(G)$, a contradiction.

Case e. $\pi(G) = \{2, 31\}$

Since $31^2 \notin \omega(G)$, then $\exp(P_{31}) = 31$. Then $|P_{31}| \mid 1 + s_{31}$ and so $|P_{31}| = 31$. Since $n_{31} = s_{31}/\phi(31) = 2^{10} \cdot 3^2 \cdot 7$, $3, 7 \in \pi(G)$, a contradiction.

Case f. $\pi(G)$ is equal to $\{2, 3, 5\}$, $\{2, 5, 7\}$, $\{2, 5, 31\}$, $\{2, 3, 5, 7\}$, $\{2, 3, 5, 31\}$ or $\{2, 5, 7, 31\}$.

Similarly as the Case c, we also get a contradiction.

Case g. $\pi(G) = \{2, 3, 7\}$, or $\pi(G) = \{2, 7, 31\}$

Similarly as the Case d or Case e, we get a contradiction.

Case h. $\pi(G) = \{2, 3, 5, 7, 31\}$.

We have known, as for Cases c and d, that $|P_7| = 7$ and $|P_{31}| = 31$.

Step 1. $|G| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 31$, $|G| = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, and $|G| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 31$.

We show that $5 \cdot 7 \notin \omega(G)$.

If $5 \cdot 7 \in \omega(G)$, set P and Q are Sylow 7-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{35} = \phi(35) \cdot n_7 \cdot k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_7)$. As $n_7 = s_7/\phi(7) = 476160/6 = 2080$, $79360 \cdot 24 \mid s_{15}$. We have $s_{35} \notin \mathrm{nse}(G)$. We conclude that $35 \notin \omega(G)$. It follows that the Sylow 5-group P_5 of G acts fixed point freely on the set of elements of order 7, $|P_5| \mid s_7$, and so $|P_5| = 5$.

We show that $2 \cdot 31 \notin \omega(G)$.

If $2 \cdot 31 \in \omega(G)$, set P and Q are Sylow 31-subgroups of G, then P and Q are conjugate in G and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G. Therefore we have $s_{62} = \phi(62) \cdot n_{31} \cdot k$, where k is the number of cyclic subgroups of order 2 in $C_G(P_{31})$. As $n_{31} = s_{31}/\phi(31) = 1935360/30 = 64512$, $64512 \mid s_{15}$ and so $s_{62} = s_{31}$. On the other hand, $62 \mid 1 + s_2 + s_{31} + s_{62}(= 3870721)$, we also get a contradiction. We conclude that $2 \cdot 31 \notin \omega(G)$. It follows that the Sylow 2-group of G acts fixed point freely on the set of elements of order 31, and so $|P_2| \mid s_{31}(= 1935360)$. Thus $|P_2| \mid 2^{11}$.

We show that $5 \cdot 31 \notin \omega(G)$.

If $5 \cdot 31 \in \omega(G)$, then $s_{155} = \phi(155) \cdot n_{31} \cdot k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_{31})$, and so $7849440 \mid s_{155}$. So we have $s_{155} \notin \mathrm{nse}(G)$, a contradiction. Hence $5 \cdot 31 \notin \omega(G)$. Similarly we can prove that $3 \cdot 31 \notin \omega(G)$ and so $|P_3| \mid s_{31}$. Thus $|P_3| \mid 3^3$.

Therefore $|G| = 2^m \cdot 3^n \cdot 5 \cdot 7 \cdot 31$. But $\sum_{s_k \in \mathrm{nse}(G)} s_k = 9999360 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31 \leq 2^m \cdot 3^n \cdot 5 \cdot 7 \cdot 31$. Hence we have that $|G| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$, $|G| = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, and $|G| = 2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$.

Step 2. $G \cong L_5(2)$.

In this step, we first prove that there are no groups such that $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$, $|G| = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, and $|G| = 2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ with $\mathrm{nse}(G) = \mathrm{nse}(L_5(2))$, then $|G| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ and $\mathrm{nse}(G) = \mathrm{nse}(L_5(2))$, it follows, from [6] that $G \cong L_5(2)$.

Let $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ with $\mathrm{nse}(G) = \mathrm{nse}(L_5(2))$.

If G is soluble, then by Lemma 2.4, for every prime p, $p^{p_i} \equiv 1 (\mathrm{mod}\ q)$ for some q. Thus we have $5 \equiv 1 (\mathrm{mod}\ 13)$, a contradiction. So G is insoluble. Therefore we can suppose that G has a normal series $1 \lhd K \lhd L \lhd G$ such that L/K is isomorphic to a simple K_i-group with $i = 3, 4, 5$ as $5^2, 7^2$ and 31^2 do not divide the order of G.

- If L/K is isomorphic to a simple K_3-group, then by [18], L/K is isomorphic to A_5, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$ or $U_4(2)$.
  - If $L/K \cong A_5$, then by [1], $n_3(L/K) = n_3(A_5) = 10$, and so $n_3(G) = 10t$ for some integer t with $3 \nmid t$. Hence the number of elements of order 3 in G is: $s_3 = 20t = 75392$. But the equation has no solution in $\mathbb{N}$, a contradiction.
  - Similarly, for the groups A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$ and $U_4(2)$, we also can rule out these cases.

- If L/K is isomorphic to a simple K_n-group with $n = 4, 5$, then by Lemma 2.7, L/K is isomorphic to $L_2(31)$ or $L_5(2)$.
  - If $L/K \cong L_2(31)$, then from [1], $n_{31}(L/K) = n_3(L_2(31)) = 32$, and so $n_{31}(G) = 32t$ for some integer t with $31 \nmid t$. Hence the number of elements of order 31 in G is $s_{31} = 32t \cdot 30 = 960t$ and so $t = 2016$. So $2^5 \cdot 3^2 \cdot 7 \mid |K| \mid 2^5 \cdot 3^2 \cdot 7$, Hence $|K| = 2^5 \cdot 3^2 \cdot 7$, and $|N_K(P_{31})| = 1$. So $K \cap N_G(P_{31}) = K \cap C_G(P_{31})$, which means that $K \rtimes P_{31}$ is a Frobenius group and so $|P_{31}| \mid |\mathrm{Aut}(K)|$, a contradiction.
  - If $L/K \cong L_5(2)$, set $\overline{G} = G/K$ and $\overline{L} = L/K$. Then

    $$L_3(4) \le \overline{L} \cong \overline{L}C_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \le \overline{G}/C_{\overline{G}}(\overline{L}) = N_{\overline{G}}(\overline{L})/C_{\overline{G}}(\overline{L}) \le \mathrm{Aut}(\overline{L})$$

    Set $M = \{xK \mid xK \in C_{\overline{G}}(\overline{L})\}$, then $G/M \cong \overline{G}/C_{\overline{G}}(\overline{L})$ and so $L_5(2) \le G/M \le$Aut$(L_5(2))$. Therefore G/M is isomorphic to $L_5(2)$ or $2.L_5(2)$

    If $G/M \cong L_5(2)$, then $|M| = 3$ and so $M = Z(G)$. Hence G has an element of order $3 \cdot 31$, a contradiction.

    If $G/M \cong 2.L_5(2)$, order consideration rules out this case.

Similarly as the arguments of "$|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ with nse$(G) =$ nse$(L_5(2))$", we can rule out these cases: $|G| = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$, and $|G| = 2^{11} \cdot 3^3 \cdot 5 \cdot 7 \cdot 31$ with nse$(G) =$ nse$(L_5(2))$.

Therefore we have $|G| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31 = |L_5(2)|$ and by assumption nse$(G) =$ nse$(L_5(2))$, then by [6], we have $G \cong L_5(2)$.

This completes the proof. □

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