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A characterization of automorphism groups of simple K_3-groups

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Abstract - In this paper, a new characterization of automorphism groups of simple K_3-groups is presented.

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1. Introduction

All groups considered in this paper are finite.

It is known that the set $cs(G)$ of sizes of the conjugacy classes of a finite group G encodes much information about the structure of G. Many authors have studied the connection between arithmetical properties of $cs(G)$ and structural properties of G. The present paper is a contribution along this line, which is related to Thompson's conjecture (see [11, Problem 12.38] and [2, 3, 4, 5, 12, 1] for detail).

CONJECTURE 1.1. *Let G be a group with $Z(G) = 1$ and M is a non-abelian simple group satisfying that $cs(G) = cs(M)$, then $G \simeq M$.*

In this conjecture, M is a non-abelian simple group. Hence it seems interesting to consider the following question.

Let G be a group with $Z(G) = 1$ and M is an almost simple group satisfying that $cs(G) = cs(M)$. Then, what can we say about the structure of G?

A group G is almost simple if there exists a non-abelian simple group S such that $S \leqslant G \leqslant Aut(S)$. In [9, 10], the almost sporadic groups, $Aut(PSL(2, q))$ and $PGL(2, p)$ for some special cases are discussed. In this paper, we investigate the almost simple K_3-groups. A group G is called a K_3-group if $\pi(G)$ consists of exactly three distinct primes. Among the known simple groups, there are exactly eight simple K_3-groups (see [7, p. 12]):

$$A_5, \quad A_6, \quad L_2(7), \quad L_2(8), \quad L_2(17), \quad L_3(3), \quad U_3(3), \quad U_4(2).$$

Our main result is as follows.

THEOREM 1.2. *Let G be a group with $Z(G) = 1$ and S be one of the simple K_3-groups. Set $M = Aut(S)$.*

(1) *If $S \in \{A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)\}$ and $cs(G) = cs(M)$, then $G \simeq M$.*

(2) *If $S = U_4(2)$, $cs(G) = cs(M)$ and $|G|_p = |M|_p$, then $G \simeq M$, where $|G|_p$ denotes the order of the Sylow p-subgroups of G for $p \in \{2, 3, 5\}$.*

2. Preliminaries

Let G be a group and construct its prime graph $\Gamma(G)$ as follows: the vertices are the primes dividing the order of G, two vertices p and q are joined by an edge if and only if G contains an element of order pq (see [13]). We denote the set of all connected components of the graph $\Gamma(G)$ by $T(G) = \{\pi_i(G) | 1 \leqslant i \leqslant t(G)\}$, where $t(G)$ is the number of the connected components of $\Gamma(G)$. If the order of G is even, we always assume that $2 \in \pi_1(G)$. For $x \in G$, x^G denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. A group G is called a 2-*Frobenius group* if G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K and G/H are Frobenius groups with kernels H and K/H respectively (see, for example [9, Definition 2.1]).

The other notation and terminologies in this paper are standard and the reader is referred to ATLAS [6] and [8] if necessary.

The following lemma is well-known. It follows from [4, Lemma 1.1] or [12, Lemma 3].

Lemma 2.1. *Let G and M be groups satisfying $Z(G) = Z(M) = 1$ and $cs(G) = cs(M)$. Then $\pi(G) = \pi(M)$.*

Lemma 2.2 [4, Lemma 1.4]. *Let G be a group with $Z(G) = 1$ and M a group with $t(M) > 1$. Suppose that $cs(G) = cs(M)$. Then $|G| = |M|$.*

Lemma 2.3 [4, Lemma 1.5]. *Let G and M be groups satisfying that $|G| = |M|$ and $cs(G) = cs(M)$. Then $t(G) = t(M)$ and $T(G) = T(M)$.*

Lemma 2.4. *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$ and G has one of the following structures:*

(i) *$2 \in \pi(H)$ and all Sylow subgroups of K are cyclic;*
(ii) *$2 \in \pi(K)$, H is an abelian group, K is a solvable group, the Sylow subgroups of K of odd order are cyclic groups and the Sylow 2-subgroups of K are cyclic or generalized quaternion groups;*
(iii) *$2 \in \pi(K)$, H is abelian, and there exists a subgroup K_0 of K such that*

$$|K : K_0| \leqslant 2, K_0 = Z \times SL(2,5), (|Z|, 2 \times 3 \times 5) = 1,$$

and the Sylow subgroups of Z are cyclic.

Proof. This is Lemma 1.6 in [4]. □

LEMMA 2.5. *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, the order of G/K divides the order of the automorphism group of K/H, and both G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$ and G is solvable.*

PROOF. This is Lemma 1.7 in [4]. □

LEMMA 2.6. *Let G be a group with more than one prime graph component. Then G is one of the following:*

(i) *a Frobenius or 2-Frobenius group;*
(ii) *G has a normal series $1 \subseteq H \subseteq K \subseteq G$, where H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a π_1-group such that $|G/K|$ divides the order of the outer automorphism group of K/H. Besides, $\pi_i(K/H) = \pi_i(G)$ for $i \geqslant 2$.*

PROOF. It follows straight forward from Lemmas 1-3 in [13], Lemma 1.5 in [3] and Lemma 7 in [5]. □

LEMMA 2.7 [9, Corollary 5.1]. *Let G be a group with $Z(G) = 1$ and $M = Aut(L_2(q))$, where $\Gamma(M)$ is not connected. If $cs(G) = cs(M)$, then $G \simeq M$.*

LEMMA 2.8 [12, Lemma 4]. *Suppose that G is a group with $Z(G) = 1$ and p is a prime in $\pi(G)$ such that p^2 does not divide $|x^G|$ for all x in G. Then a Sylow p-subgroup of G is elementary abelian.*

3. Proof of Theorem 1.2

LEMMA 3.1. *Let G be a group with $Z(G) = 1$ and*

$$S \in \{A_5, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)\}.$$

Let $M = Aut(S)$ and $cs(G) = cs(M)$. Then $G \simeq M$.

PROOF. By [6], we have that $\Gamma(M)$ is not connected if

$$S \in \{A_5, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)\}.$$

Therefore, by Lemma 2.7, $G \simeq M$ provided that

$$S \in \{L_2(7), L_2(8), L_2(17)\}.$$

Now, it suffices to discuss the remaining three cases.

First, we suppose that $S = A_5$. Then $\Gamma(M)$ is not connected. By the hypothesis and lemma 2.2, we obtain that $|G| = |M|$. Since $\pi_2(M) = \{5\}$ by [6], it follows from Lemma 2.3 that $\pi_2(G) = \{5\}$. Since 5 does not divide 23, by Lemma 2.4, we have G is not a Frobenius group. Suppose that G is a 2-Frobenius group. Then, by Lemma 2.5, G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $|K/H| = 5$. Let P be a Sylow 3-subgroup of H. Then P is a normal subgroup of G of order 3. Let $x \in G$ such that $|x| = 5$. Then x acts trivially on P and so $\Gamma(G)$ is connected. This contradiction shows that G is not a 2-Frobenius group either. Now, by lemma 2.6, G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that K/H is a non-abelian simple group. It is easy to see that K/H is isomorphic to A_5. If $H \neq 1$, then H is of order 2 and so $H \leqslant Z(G)$, contrary to our assumption for G. Hence $H = 1$ and therefore $K \simeq A_5$. It follows that $G \simeq S_5$, as desired.

Next, we assume that $S = L_3(3)$. Since $\Gamma(M) = \Gamma(Aut(S))$ is not connected, Lemma 2.2 together with the hypothesis imply that $|G| = |M|$. By [6], we have that $\{13\}$ is a component of $\Gamma(M)$. Hence, by Lemma 2.3, $\{13\}$ is a component of $\Gamma(G)$. Similar to above discussion, we can show that G is not a Frobenius group. We assert that G is not a 2-Frobenius group either. If not, then, by Lemma 2.5, G has a normal series $1 \subseteq H \subseteq K \subseteq G$ such that $|K/H| = 13$. Let P be a Sylow 2-subgroup of H and $\Omega = \Omega_1(Z(P))$. Since $(13, |GL_n(2)|) = 1$, where $n = 2, 3, 4, 5$, an element x of G of order 13 act trivially on Ω. Therefore G has an element of order 26, a contradiction. Thus G is not a 2-Frobenius group. By Lemma 2.6, G has a chief factor K/H such that K/H is a simple K_3-group and $13 \in \pi(K/H)$. By [6], K/H must be isomorphic to $L_3(3)$. Similarly as above, we obtain that $H = 1$ and so $G \simeq M = Aut(L_3(3))$.

Finally, we prove that $G \simeq M = Aut(U_3(3))$. As above, one can show that G is neither a Frobenius group nor a 2-Frobenius group. By [6], $\pi_2(M) = \{7\}$ and so $\pi_2(G) = \{7\}$ by Lemma 2.3. It follows from Lemma 2.6 that G has a chief factor K/H such that K/H is a simple K_3-group and $7 \in \pi(K/H)$. Then K/H is isomorphic to $L_2(7)$, $L_2(8)$ or $U_3(3)$. Let P be a Sylow 2-subgroup of H, Q a Sylow 3-subgroup of H and x an element of G of order 7. If $K/H \simeq L_2(7)$, then $|Q| = 3^2$. Since $(7, |Aut(Q)|) = 1$, x acts trivially on Q and so 7 and 3 are joined, a contradiction. If $K/H \simeq L_2(8)$, then P is of order 2^3 and so G has an element x such that $|x^G| \leq 7$, a contradiction by [6]. Hence $K/H \simeq U_3(3)$. It is easy to see that $H = 1$ since otherwise, H is contained in $Z(G)$, a contradiction. Thus, $K \simeq U_3(3)$ and $G \simeq Aut(U_3(3))$. □

LEMMA 3.2. *Let G be a group with $Z(G) = 1$ and $M = Aut(A_6)$. If $cs(G) = cs(M)$, then $G \simeq M$.*

PROOF. We proceed the proof by several steps.

(1) By [6], $cs(G)$ consists of $n_1 = 1$, $n_2 = 3^2 \cdot 5$, $n_3 = 2^4 \cdot 5$, $n_4 = 2 \cdot 3^2 \cdot 5$, $n_5 = 2^4 \cdot 3^2$, $n_6 = 2 \cdot 3 \cdot 5$, $n_7 = 2^4 \cdot 3 \cdot 5$, $n_8 = 2^2 \cdot 3^2$, $n_9 = 2^2 \cdot 3^2 \cdot 5$.

(2) By Lemma 2.1, we have that $\pi(G) = \pi(M) = \{2, 3, 5\}$.

(3) The Sylow 5-subgroups of G are of order 5.

Let P be a Sylow 5-subgroup of G. Then, by Lemma 2.8, P is elementary abelian since 5^2 does not divide any element in $cs(G)$. We assert that P is of order 5. Assume that 5^2 divide the order of G. Then, the centralizer of every element of G contains an element of order 5. Let $y \in G$ such that $|y^G| = 2^4 \cdot 3 \cdot 5$ and x an element of $C_G(y)$ of order 5. Since the Sylow 5-subgroup of G is elementary abelian, we have that 5 does not divide $|x^G|$ and consequently $x^G = 2^4 \cdot 3^2$ or $2^2 \cdot 3^2$. If 5 does not divide $|y|$, then $C_G(xy) = C_G(x) \cap C_G(y)$, from which we can deduce that $2^6 \cdot 3^3 \cdot 5$ divide $(xy)^G$, which is impossible. If 5 divides $|y|$, then $y = y_1 y_2$ with $|y_1| = 5$ and $(|y_1|, |y_2|) = 1$. It follows that $|y_1^G|$ should divide $|y^G|$, a contradiction. Hence the Sylow 5-subgroup of G is of order 5.

(4) $O_{22'}(G) = O_2(G)$.

Write $K = O_2(G)$ and $\overline{G} = G/K$. Suppose that the statement is false. Then there is $r \in \{3, 5\}$ such that $O_r(\overline{G}) \neq 1$. If $P = O_5(\overline{G}) \neq 1$, then $|P| = 5$. Let Q be a Sylow 3-subgroup of G and x an element of $Z(Q)$ of order 3. Then $|x^G| = n_3 = 2^4 \cdot 5$ and so 5 does not divide $|C_G(x)|$ by Step 1. Since $(3, 5-1) = 1$, we see that $\overline{x}$ acts trivially on P. Thus 5 divides $C_{\overline{G}}(\overline{x})$. Since $(3, |K|) = 1$, we have that $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$ and so 5 divides $|C_G(x)|$, a contradiction. Now, we assume that $O_3(\overline{G}) \neq 1$. Put $V = \Omega_1(Z(O_3(\overline{G})))$. Then V is a nontrivial normal subgroup of $\overline{G}$. Let y be an element of G of order 5. Then $V = [V, \overline{y}] \times C_V(\overline{y})$. By Step 1, $|V : C_V(\overline{y})|$ is at most 3^2. It follows that $[[V, \overline{y}], \overline{y}] = 1$, which implies that $[V, \overline{y}] = 1$. Therefore $V = C_V(\overline{y})$. Let Q be a Sylow 3-subgroup of $\overline{G}$. Then $V \cap Z(Q) \neq 1$. Let $\overline{z}$ be an element of $V \cap Q$ of order 3. Then $\overline{y}\,\overline{z} = \overline{z}\,\overline{y}$. Since $(3, |K|) = 1$, there exists a preimage z of $\overline{z}$ in G such that z is contained in the center a Sylow 3-subgroup of G and so 5 divides $|C_G(z)|$. But, by Step 1, this is impossible.

(5) $G \simeq Aut(A_6)$.

Set $K = O_2(G)$ and $\overline{G} = G/K$. Then $\overline{G}$ is insoluble by Step 4. Furthermore, we have that $M \leqslant \overline{G} \leqslant Aut(M)$, where $M = S_1 \times S_2 \times \cdots \times S_k$ is a direct product of non-abelian simples $S_1, S_2, \ldots, S_k$. Since $\pi(G) = \{2, 3, 5\}$ and the order of the Sylow 5-subgroups of G is 5, we obtain that $k = 1$, that is, M is a non-abelian simple K_3-group. Hence M is isomorphic to A_5, A_6 or $U_4(2)$.

If $M \simeq A_5$, then $G/K \simeq A_5$ or $Aut(A_5)$ and so 3^2 does not divide $|G|$, a contradiction.

Suppose that $M \simeq A_6$. Assume first that $\overline{G} \simeq A_6$ or S_6. Let x be an element of G of order 5 such that $n_8 = |x^G| = 2^2 \cdot 3^2$. Then $|\overline{x}^{\overline{G}}|$ divides $2^2 \cdot 3^2$. By [6], we have that $|\overline{x}^{\overline{G}}| = 1$, which implies that $\overline{x} \leqslant Z(\overline{G})$. Therefore $\overline{x} = 1$ for that $Z(\overline{G})$ is trivial and so $x \in K$, a contradiction.

Assume that $\overline{G} \simeq PGL_2(9)$. Pick $x \in G$ with $|x| = 5$ such that $|x^G| = 2^2 \cdot 3^2$. Then, by [6], $|\overline{x}^{\overline{G}}| = 1$ or $2^2 \cdot 3^2$. As above, it is impossible that $|\overline{x}^{\overline{G}}| = 1$. If $|\overline{x}^{\overline{G}}| = 2^2 \cdot 3^2$, then x centralizes K. If $K \neq 1$, then x centralizes an element of G which lies in the center of a Sylow 2-subgroup of G, contrary to Step 1. If $K = 1$, then $G \simeq PGL_2(9)$, but $cs(PGL_2(9)) \neq cs(G)$ by [6], a contradiction.

Suppose that $\overline{G} \simeq M_{10}$. Let x be an element of G of order 5 such that $|x^G| = 2^4 \cdot 3^2$. Then $|\overline{x}^{\overline{G}}| = 1$ or $2^4 \cdot 3^2$ by [6]. Similar to the above case, we can derive a contradiction.

If $\overline{G} \simeq Aut(A_6)$, then, similar to the forgoing argument, one can show that $K = 1$ and so $G \simeq Aut(A_6)$, as desired.

If $\overline{G} \simeq U_4(2)$, then G has an element x such that $|\overline{x}^{\overline{G}}| = 2^7 \cdot 3^2 \cdot 5$. It follows that $|x^G|$ is divisible by $2^7 \cdot 3^2 \cdot 5$, which is impossible by Step 1.

Thus, the proof is complete. □

Lemma 3.3. *Let G be a group with $Z(G) = 1$ and $M = Aut(U_4(2))$. Suppose that $cs(G) = cs(M)$ and $|G|_p = |M|_p$, where $|G|_p$ denotes the order of the Sylow p-subgroups of G for $p \in \{2, 3, 5\}$. Then $G \simeq M$.*

Proof. We proceed the proof by the following steps.

(1) $cs(G)$ consists of $n_1 = 1$, $n_2 = 3^2 \cdot 5$, $n_3 = 2 \cdot 3^3 \cdot 5$, $n_4 = 2^4 \cdot 5$, $n_5 = 2^4 \cdot 3 \cdot 5$, $n_6 = 2^5 \cdot 3 \cdot 5$, $n_7 = 2^2 \cdot 3^3 \cdot 5$, $n_8 = 2^3 \cdot 3^4 \cdot 5$, $n_9 = 2^6 \cdot 3^4$, $n_{10} = 2^4 \cdot 3^2 \cdot 5$, $n_{11} = 2^5 \cdot 3^2 \cdot 5$, $n_{12} = 2^4 \cdot 3^3 \cdot 5$, $n_{13} = 2^7 \cdot 3^2 \cdot 5$, $n_{14} = 2^5 \cdot 3^3 \cdot 5$, $n_{15} = 2^2 \cdot 3^2$, $n_{16} = 2^2 \cdot 3^4 \cdot 5$, $n_{17} = 2^4 \cdot 3^4 \cdot 5$.

This follows from [6].

(2) $\pi(G) = \pi(M) = \{2, 3, 5\}$.

This follows from [6] and Lemma 2.1.

(3) $O_{22'}(G) = O_2(G)$.

Write $K = O_2(G)$ and $\overline{G} = G/K$. Suppose that the assertion is false. Then $O_r(\overline{G}) \neq 1$, where $r \in \{3, 5\}$. Assume first that $O_5(\overline{G}) \neq 1$. Then, $|O_5(\overline{G})| = 5$. Let $x \in Z(Q)$ with $|x| > 1$, where Q is a Sylow 3-subgroup of G. Then $|x^G| = 2^4 \cdot 5$ by (1). Since $\overline{x}$ acts trivially on $O_5(\overline{G})$, we have that 5 divides the order of $C_{\overline{G}}(\overline{x})$, by which we conclude that 5 divides the order of $C_G(x)$. This is impossible by (1). Now assume that $O_3(\overline{G}) \neq 1$ and set $V = \Omega_1(Z(O_3(\overline{G})))$. Let y be an element of G of order 5. Then $|y^G| = 2^6 \cdot 3^4$ or $2^2 \cdot 3^2$. Since $V = [V, \overline{y}] \times C_V(\overline{y})$, we have that $|V : C_V(\overline{y})| \leqslant 3^4$. If

$|V : C_V(\overline{y})| \leqslant 3^3$, then $[[V, \overline{y}], \overline{y}] = 1$ and so $[V, \overline{y}] = 1$. Discussing as in Lemma 3.2, we see that this is impossible. If $|V : C_V(\overline{y})| = 3^4$, then the Sylow 3-subgroups of G is elementary abelian, which contradicts that $|x^G| = n_{13} = 2^7 \cdot 3^2 \cdot 5$ for some 3-element $x \in G$.

(4) $G \simeq M$.

By (3), $\overline{G}$ is insoluble. Hence we have that $M \leqslant \overline{G} \leqslant Aut(M)$, where M is a direct product of non-abelian simple groups. Since, by (2) $\pi(G) = \pi(M) = \{2, 3, 5\}$, we know that M is simple K_3-groups by (3). Thus, M is isomorphic to A_5, A_6 or $U_4(2)$. If M is isomorphic to A_5 or A_6, then 3^3 does not divide the order of G, a contradiction by (1). Suppose that $M \simeq U_4(2)$. Then $\overline{G} \simeq U_4(2)$ or $U_4(2).2$. If $\overline{G} \simeq U_4(2)$, then $Z(G)$ is non-trivial, a contradiction. If $\overline{G} \simeq U_4(2).2$, then G is isomorphic to $Aut(U_4(2))$, as desired.

Proof of Theorem 1.2.

It follows from Lemmas 3.1-3.3. □

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