

## Simple sufficient conditions for bounded turning

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ABSTRACT - Let  $f$  be an analytic function in the open unit disk normalized such that  $f(0) = f'(0) - 1 = 0$ . In this paper the modulus and the real part of the linear combination of  $f'(z)$  and  $f(z)/z$  is studied and conditions when  $f$  is of bounded turning are obtained.

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### 1. Introduction and preliminaries

Let  $\mathcal{A}$  denote the class of analytic functions in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$  that are normalized such that  $f(0) = f'(0) - 1 = 0$ , i.e.  $f(z) = z + a_2z^2 + a_3z^3 + \dots$ .

A function  $f \in \mathcal{A}$  is in the class of *starlike functions*,  $S^*$ , if and only if

$$\operatorname{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in \mathbb{D}.$$

Such functions are univalent and their geometric characterization (which motivates the name of the class) is that they map the unit disk onto a starlike region, i.e. if  $\omega \in f(\mathbb{D})$  then  $t\omega \in f(\mathbb{D})$  for all  $t \in [0, 1]$ .

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Another well-known class of univalent functions is the *class of functions with bounded turning*,

$$R = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > 0, z \in \mathbb{D}\}.$$

More details on these classes can be found in [2] (section 2.5) and [3] (section 5.5). One of the main results concerning them is due to Krzyz ([7]), who proved that  $S^*$  does not contain  $R$  and  $R$  does not contain  $S^*$ . This makes the class  $R$  interesting and lots of research is dedicated to it. Some references in that direction are [6] – [9].

In this paper we will study the linear combination of two simple expressions,  $f'(z)$  and  $f(z)/z$ , i.e. we will study the modulus and the real part of

$$(1) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z}$$

and obtain criteria for a function  $f \in \mathcal{A}$  to be of bounded turning. For that purpose we will use a method from the theory of differential subordinations. Valuable references on this topic are [1] and [3].

First we introduce subordination. Let  $f, g \in \mathcal{A}$ . Then we say that  $f(z)$  is *subordinate* to  $g(z)$ , and write  $f(z) \prec g(z)$ , if there exists a function  $\omega(z)$ , analytic in the unit disc  $\mathbb{D}$ , such that  $\omega(0) = 0$ ,  $|\omega(z)| < 1$  and  $f(z) = g(\omega(z))$  for all  $z \in \mathbb{D}$ . In particular, if  $g(z)$  is univalent in  $\mathbb{D}$  then  $f(z) \prec g(z)$  if and only if  $f(0) = g(0)$  and  $f(\mathbb{D}) \subseteq g(\mathbb{D})$ .

For obtaining our main result we will use the method of differential subordinations. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [4] and [5]. Namely, if  $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$  (where  $\mathbb{C}$  is the complex plane) is analytic in a domain  $D$ , if  $h(z)$  is univalent in  $\mathbb{D}$ , and if  $p(z)$  is analytic in  $\mathbb{D}$  with  $(p(z), zp'(z)) \in D$  when  $z \in \mathbb{D}$ , then  $p(z)$  is said to satisfy a first-order differential subordination if

$$(2) \quad \phi(p(z), zp'(z)) \prec h(z).$$

A univalent function  $q(z)$  is said to be a *dominant* of the differential subordination (2) if  $p(z) \prec q(z)$  for all  $p(z)$  satisfying (2). If  $\tilde{q}(z)$  is a dominant of (2) and  $\tilde{q}(z) \prec q(z)$  for all dominants of (2), then we say that  $\tilde{q}(z)$  is the *best dominant* of the differential subordination (2).

From the theory of first-order differential subordinations we will make use of the following lemma.

LEMMA 1.1 [[5]]. Let  $q(z)$  be univalent in the unit disk  $\mathbb{D}$ , and let  $\theta(\omega)$  and  $\phi(\omega)$  be analytic in a domain  $D$  containing  $q(\mathbb{D})$ , with  $\phi(\omega) \neq 0$  when  $\omega \in q(\mathbb{D})$ . Set  $Q(z) = zq'(z)\phi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$ , and suppose that

- i)  $Q(z)$  is starlike in the unit disk  $\mathbb{D}$ ; and
- ii)  $\operatorname{Re} \frac{zh'(z)}{Q(z)} \left(= \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} \right) > 0, z \in \mathbb{D}.$

If  $p(z)$  is analytic in  $\mathbb{D}$ , with  $p(0) = q(0)$ ,  $p(\mathbb{D}) \subseteq D$  and

$$(3) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then  $p(z) \prec q(z)$ , and  $q(z)$  is the best dominant of (3).

Now, using Lemma 1.1 we will prove the following result.

LEMMA 1.2. Let  $f \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  be such that  $\alpha + \beta = 0$  or  $\alpha + \beta = 1$ . Also, let  $q(z)$  be univalent in the unit disk  $\mathbb{D}$  with  $q(0) = 0$  and

$$(4) \quad \operatorname{Re} \left[ 1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D}.$$

Additionally, in the case when  $\alpha + \beta = 1$ , let  $\operatorname{Re} \frac{1}{\alpha} > -1$  and

$$(5) \quad \operatorname{Re} \left[ 1 + \frac{zq''(z)}{q'(z)} \right] > -\operatorname{Re} \frac{1}{\alpha}, \quad z \in \mathbb{D}.$$

If

$$(6) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta) \cdot [q(z) + 1] + \alpha z q'(z) (\equiv h(z))$$

then  $\frac{f(z)}{z} - 1 \prec q(z)$ , and  $q(z)$  is the best dominant of (6).

PROOF. The functions  $\theta(\omega) = (\alpha + \beta) \cdot (\omega + 1)$  and  $\phi(\omega) = \alpha$  are analytic in the domain  $D = \mathbb{C}$  which contains  $q(\mathbb{D})$  and  $\phi(\omega) \neq 0$  when  $\omega \in q(\mathbb{D})$ . Further,  $Q(z) = zq'(z)\phi(q(z)) = \alpha z q'(z)$  is starlike since

$$\operatorname{Re} \frac{zQ'(z)}{Q(z)} = \operatorname{Re} \left[ 1 + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D};$$

furthermore for the function  $h(z) = \theta(q(z)) + Q(z)$  we have

$$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left[ 1 + \frac{\alpha + \beta}{\alpha} + \frac{zq''(z)}{q'(z)} \right] > 0, \quad z \in \mathbb{D},$$

for  $\alpha + \beta = 0$  due to (4) and for  $\alpha + \beta = 1$  due to (5).

Now, let choose  $p(z) = \frac{f(z)}{z} - 1$  which is analytic in  $\mathbb{D}$ ,  $p(0) = q(0) = 0$  and  $p(\mathbb{D}) \subseteq D = \mathbb{C}$ . Finally, bearing in mind that subordinations (3) and (6) are equivalent, from Lemma 1.1 we deduce the conclusions of Lemma 1.2.  $\square$

## 2. Results on the modulus of (1)

In this section we will study the modulus of (1) and obtain conclusions that lead to criteria for a function  $f$  to be in the class  $R$ .

**THEOREM 2.1.** *Let  $f \in \mathcal{A}$ ,  $\mu > 0$  and  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  be such that  $\alpha + \beta = 0$  or  $\alpha + \beta = 1$ . Also, when  $\alpha + \beta = 1$  let  $\operatorname{Re} \frac{1}{\alpha} > -1$ . If*

$$(7) \quad \left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| < \delta \equiv \begin{cases} \mu \cdot |\alpha|, & \alpha + \beta = 0 \\ \mu \cdot |1 + \alpha|, & \alpha + \beta = 1 \end{cases},$$

for all  $z \in \mathbb{D}$ , then

$$(8) \quad \left| \frac{f(z)}{z} - 1 \right| < \mu, \quad z \in \mathbb{D}.$$

This implication is sharp, i.e., in the inequality (8)  $\mu$  cannot be replaced by a smaller number for the implication to hold. Also,

$$|f'(z) - 1| < \lambda \equiv \begin{cases} 2\mu, & \alpha + \beta = 0 \\ \mu \cdot \left( \left| 1 + \frac{1}{\alpha} \right| + \left| 1 - \frac{1}{\alpha} \right| \right), & \alpha + \beta = 1 \end{cases}, \quad z \in \mathbb{D}.$$

This implication is also sharp, i.e.,  $\lambda$  cannot be replaced by a smaller number for the implication to hold, if

- (i)  $\alpha + \beta = 0$ ; or
- (ii)  $\alpha + \beta = 1$  and  $\left| 1 + \frac{1}{\alpha} \right| + \left| 1 - \frac{1}{\alpha} \right| = 2$ .

Additionally, if  $\mu \leq \frac{1}{2}$  for  $\alpha + \beta = 0$  or  $\left| 1 + \frac{1}{\alpha} \right| + \left| 1 - \frac{1}{\alpha} \right| \leq \frac{1}{\mu}$  for  $\alpha + \beta = 1$  then  $f \in R$ .

**PROOF.** Choosing  $q(z) = \mu z$  we have  $1 + \frac{zq''(z)}{q'(z)} = 1$ , meaning that (4) and (5) from Lemma 1.2 both hold. Further, for the function  $h(z)$  defined

in (6) we have

$$h(z) = \alpha + \beta + \mu z(2\alpha + \beta),$$

meaning that the subordination (6) is equivalent to

$$\left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| < \mu \cdot |2\alpha + \beta| = \delta, \quad z \in \mathbb{D},$$

and so equivalent to (7). Therefore, (8) follows directly from Lemma 1.2 and the definition of subordination.

Further, for all  $z \in \mathbb{D}$ ,

$$\left| \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} - (\alpha + \beta) \right| = \left| \alpha \cdot [f'(z) - 1] + \beta \cdot \left[ \frac{f(z)}{z} - 1 \right] \right|$$

and

$$\begin{aligned} |\alpha| \cdot |f'(z) - 1| &\leq \left| \alpha \cdot [f'(z) - 1] + \beta \cdot \left[ \frac{f(z)}{z} - 1 \right] \right| + \left| \beta \cdot \left[ \frac{f(z)}{z} - 1 \right] \right| \\ &< \delta + |\beta| \cdot \mu = |\alpha| \cdot \lambda, \end{aligned}$$

since  $|w_1| \leq |w_1 + w_2| + |w_2|$ . Therefore, the implications of this corollary holds.

Both implications are sharp as the function  $f_*(z) = z + \mu z^2$  shows, since

$$\left| \alpha \cdot f'_*(z) + \beta \cdot \frac{f_*(z)}{z} - (\alpha + \beta) \right| = \mu \cdot |2\alpha + \beta| \cdot |z| = \delta \cdot |z|, \quad z \in \mathbb{D},$$

$$\left| \frac{f_*(z)}{z} - 1 \right| = \mu \cdot |z|, \quad z \in \mathbb{D},$$

$$|f'_*(z) - 1| = 2 \cdot \mu \cdot |z|, \quad z \in \mathbb{D},$$

and  $2\mu = \lambda$  if (i) or (ii) hold.  $\square$

### 3. Results on the real part of (1)

In this section we study the real part of the expression (1) and obtain criteria that ensure that a function  $f \in \mathcal{A}$  is in the class  $R$ .

**THEOREM 3.1.** *Let  $f \in \mathcal{A}$ ,  $\mu > 0$  and  $\alpha, \beta \in \mathbb{C} \setminus \{0\}$  be such that  $\alpha + \beta = 0$  or  $\alpha + \beta = 1$ . Also, when  $\alpha + \beta = 1$  let  $\operatorname{Re} \frac{1}{\alpha} > 0$ . If*

$$(9) \quad \alpha \cdot f'(z) + \beta \cdot \frac{f(z)}{z} \prec (\alpha + \beta) \left( 1 + \frac{2\mu z}{1-z} \right) + \frac{2\alpha\mu z}{(1-z)^2} \equiv h_2(z)$$

then

$$(10) \quad \operatorname{Re} \left[ \frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

This implication is sharp, i.e., in the inequality (10)  $\mu$  cannot be replaced by a bigger number so that the implication still holds.

PROOF. The implication of this theorem follows directly from Lemma 1.2 for  $q(z) = \frac{2\mu z}{1-z}$ . Condition  $\operatorname{Re} \frac{1}{\alpha} > 0$  stands in place of  $\operatorname{Re} \frac{1}{\alpha} > -1$  in order for (5) to hold. The result is sharp, as can be seen from the function  $f_*(z) = z + z \cdot q(z)$  for which

$$\alpha \cdot f'_*(z) + \beta \cdot \frac{f_*(z)}{z} = (\alpha + \beta) \left( 1 + \frac{2\mu z}{1-z} \right) + \frac{2\alpha\mu z}{(1-z)^2}$$

and  $\operatorname{Re} \frac{f(z)}{z} = 1 - \mu$  for  $z = -1$ .

In the case when  $\alpha + \beta = 1$  we obtain the following corollary.

COROLLARY 3.2. Let  $f \in \mathcal{A}$ ,  $\alpha > 0$  and  $\mu > 0$ . If

$$(11) \quad \operatorname{Re} \left[ \alpha \cdot f'(z) + (1 - \alpha) \cdot \frac{f(z)}{z} \right] > 1 - \mu \cdot \left( 1 + \frac{\alpha}{2} \right), \quad z \in \mathbb{D},$$

then

$$\operatorname{Re} \left[ \frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

If, additionally,

- (i)  $\alpha > 1$  and  $\mu \leq 1$ ; or
- (ii)  $\alpha < 1$  and  $\mu \geq 1$ ;

then

$$(12) \quad \operatorname{Re} f'(z) > 1 - \frac{3}{2} \cdot \mu, \quad z \in \mathbb{D}.$$

These results are sharp.

PROOF. Let  $\alpha + \beta = 1$ . Then, for the function  $h_2$  defined in (9) we have

$$h_2(z) = 1 + \frac{2\mu z}{1-z} + \frac{2\alpha\mu z}{(1-z)^2},$$

$h_2(0) = 1$  and

$$h_2(e^{i\theta}) = 1 - \frac{\mu\alpha}{2}(1 + t^2) - \mu + \mu t i,$$

where  $t = \operatorname{ctg}(\theta/2)$ . Therefore,

$$X = \operatorname{Re} h(e^{i\theta}) = 1 - \mu\left(\frac{\alpha}{2} + 1\right) - \frac{\alpha}{2\mu} \cdot Y^2,$$

where

$$Y = \operatorname{Im} h(e^{i\theta}) = \mu t$$

attains all real numbers. This leads to

$$h_2(e^{i\theta}) = \left\{ x + iy : x = 1 - \mu\left(1 + \frac{\alpha}{2}\right) - \frac{\alpha}{2\mu} \cdot y^2, y \in \mathbb{R} \right\}.$$

From here, bearing in mind the definition of subordination, the inequality (11) and the fact that

$$\left\{ x + iy : x > 1 - \mu\left(1 + \frac{\alpha}{2}\right), y \in \mathbb{R} \right\} \subseteq h_2(\mathbb{D}),$$

we obtain the subordination (9). Therefore, from Theorem 3.1 it follows that

$$\operatorname{Re} \left[ \frac{f(z)}{z} \right] > 1 - \mu, \quad z \in \mathbb{D}.$$

Further, in the case when (i) or (ii) holds we have

$$\begin{aligned} \operatorname{Re} f'(z) &= \frac{1}{\alpha} \cdot \left\{ \operatorname{Re} \left[ \alpha \cdot f'(z) + (1 - \alpha) \cdot \frac{f(z)}{z} \right] - (1 - \alpha) \cdot \operatorname{Re} \left[ \frac{f(z)}{z} \right] \right\} \\ &> \frac{1}{\alpha} \cdot \left[ 1 - \mu\left(1 + \frac{\alpha}{2}\right) - (1 - \alpha)(1 - \mu) \right] = 1 - \frac{3}{2} \cdot \mu, \end{aligned}$$

for all  $z \in \mathbb{D}$ .

The results are sharp in view of the function  $f_*(z) = z + \frac{2\mu z^2}{1-z}$  for which  $f_*(z)/z = 1 + \frac{2\mu z}{1-z} \equiv g(z)$ ,  $g(\mathbb{D}) = \{x + iy : x > 1 - \mu, y \in \mathbb{R}\}$ ,

$$\alpha \cdot f'_*(z) + (1 - \alpha) \cdot \frac{f_*(z)}{z} = h_2(z)$$

and

$$\operatorname{Re} f'_*(z) = \operatorname{Re} h_2(z) = 1 - \frac{3}{2} \cdot \mu \quad \text{for } z = -1.$$

□

In a similar way to Corollary 3.2, for the case  $\alpha = -\beta = 1$  we obtain

COROLLARY 3.3. *Let  $f \in \mathcal{A}$  and  $\mu > 0$ . If*

$$(13) \quad \operatorname{Re} \left[ f'(z) - \frac{f(z)}{z} \right] > -\frac{\mu}{2}, \quad z \in \mathbb{D},$$

*then  $\operatorname{Re} \left[ \frac{f(z)}{z} \right] > 1 - \mu$ ,  $z \in \mathbb{D}$ , and  $\operatorname{Re} f'(z) > 1 - \frac{3}{2} \cdot \mu$ ,  $z \in \mathbb{D}$ . If additionally,  $\mu \leq \frac{2}{3}$ , then  $\operatorname{Re} f'(z) > 0$ ,  $z \in \mathbb{D}$ , i.e.  $f \in R$ . Both implications are sharp.*

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