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Evaluations of a continued fraction of Ramanujan

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Abstract - We study the properties of a general continued fraction of Ramanujan. In certain cases we evaluate it completely.

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1. Introduction

Let

$$(a;q)_k := \prod_{n=0}^{k-1}(1-aq^n) \tag{1}$$

Then the Ramanujan eta function is

$$f(-q) := (q;q)_\infty \tag{2}$$

and the Weber function is

$$\Phi(-q) := (-q;q)_\infty. \tag{3}$$

Also we denote by

$$K(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1-x^2\sin^2(t)}}dt \tag{4}$$

the elliptic integral of the first kind.

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The function k_r is defined as the solution to the equation (see [2], [7])

$$\frac{K(k_r')}{K(k_r)} = \sqrt{r} \tag{5}$$

where r is positive, $q = e^{-\pi\sqrt{r}}$ and $k' = \sqrt{1-k^2}$. It is known that whenever r is positive rational, k_r is an algebraic number.

In Berndt's book [5] pg. 21 and in [6], one can find the following expansion

THEOREM 1.1. *Suppose that q, a and b are complex numbers with $|q| < 1$, or that q, a, and b are complex numbers with $a = bq^m$ for some integer m. Then*

$$U = U(a,b;q) = \frac{(-a;q)_\infty(b;q)_\infty - (a;q)_\infty(-b;q)_\infty}{(-a;q)_\infty(b;q)_\infty + (a;q)_\infty(-b;q)_\infty} = \tag{6}$$

$$= \frac{a-b}{1-q+}\frac{(a-bq)(aq-b)}{1-q^3+}\frac{q(a-bq^2)(aq^2-b)}{1-q^5+}\frac{q^2(a-bq^3)(aq^3-b)}{1-q^7+}\cdots$$

Our main concern in the present article is to simplify and evaluate U for some specific cases of a,b and q. For this purpose we define

$$X = \frac{(-a;q)_\infty(b;q)_\infty}{(a;q)_\infty(-b;q)_\infty}, \tag{7}$$

then

$$\frac{X-1}{X+1} = U \tag{8}$$

2. Propositions

PROPOSITION 2.1. *Set*

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{9}$$

then

$$\frac{\phi(q)-1}{\phi(q)+1} = \frac{q}{1+q+}\frac{-q^3}{1+q^3+}\frac{-q^5}{1+q^5+}\frac{-q^7}{1+q^7+}\cdots \tag{10}$$

PROOF. Take $q \to q^2$ in (6) and then set $a \to q$ and $b \to q^2$. Using [5] chapter 16 Entry 22, we get the result (10).

The expansion (10) can also be found independently and directly by using Euler's general continued fraction expansion [8]. $\square$

PROPOSITION 2.2.

$$\frac{\Phi(-q) - f(-q)}{\Phi(-q) + f(-q)} = \frac{q}{1-q+}\,\frac{q^3}{1-q^3+}\,\frac{q^5}{1-q^5+}\,\frac{q^7}{1-q^7+}\cdots \tag{11}$$

PROOF. Set $b = 0$ in (6) and then $a = q$. The result follows imediately from the definitions of f and Φ by relations (2) and (3). $\square$

PROPOSITION 2.3.

$$\frac{f(-q) - \Phi(-q)}{f(-q) + \Phi(-q)} = \frac{\phi(-q) - 1}{\phi(-q) + 1} \tag{12}$$

PROOF. This follows from Propositions 2.1 and 2.2 $\square$

We continue by setting

$$u_0(a,q) := \frac{2a}{1-q+}\,\frac{a^2(1+q)^2}{1-q^3+}\,\frac{a^2q(1+q^2)^2}{1-q^5+}\,\frac{a^2q^2(1+q^3)^2}{1-q^7+}\cdots \tag{13}$$

and

$$P = \left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2. \tag{14}$$

Then

$$\frac{P-1}{P+1} = u_0(a,q), \tag{15}$$

On solving (15) with respect to P, we get

COROLLARY 2.4. *If* $|q| < 1$, *then*

$$\left(\frac{(-a;q)_\infty}{(a;q)_\infty}\right)^2 = -1 + \frac{2}{1-}\,\frac{2a}{1-q+}\,\frac{a^2(1+q)^2}{1-q^3+}\,\frac{a^2q(1+q^2)^2}{1-q^5+}\,\frac{a^2q^2(1+q^3)^2}{1-q^7+}\cdots \tag{16}$$

Corollary 2.5. *If $|q|<1$ and $|a/q|<1$ then*

$$4\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-u_0(a,q)}\right) \tag{17}$$

Proof. Take the logarithm on both sides of (16) and expand the two products in Taylor series. Then the double sums are easily rearranged to get the desired result. □

From Corollaries 2.4 and 2.5, we proceed to the more general formula:

Proposition 2.6.

$$2\sum_{n=0}^{\infty}\frac{a^{2n+1}-b^{2n+1}}{(2n+1)(1-q^{2n+1})}=\log\left(-1+\frac{2}{1-U(a,b;q)}\right) \tag{18}$$

and hence

$$\left(-1+\frac{2}{1-U(a,b;q)}\right)^2=\frac{\left(-1+\frac{2}{1-u_0(a,q)}\right)}{\left(-1+\frac{2}{1-u_0(b,q)}\right)} \tag{19}$$

From relation (19) it is clear that to study U we have only to examine $u_0(a,q)$.

One can find many useful results on internet sites, such as

http://pi.physik.uni-bonn.de/ dieckman/InfProd/InfProd.html

In some cases the fraction $u_0(a,q)$ can calculated in terms of elliptic functions. For example

$$-1+\frac{2}{1-u_0(q,q)}=\frac{\pi}{2k_r'K(k_r)}. \tag{20}$$

More generally,

$$4\sum_{n=0}^{\infty}\frac{q^{\nu(2n+1)}}{(2n+1)(1-q^{2n+1})}=-4\sum_{j=1}^{\nu-1}\operatorname{arctanh}(q^j)-\log\left(\frac{2k_r'K(k_r)}{\pi}\right) \tag{21}$$

from which we obtain:

PROPOSITION 2.7. *Let v be positive integer, then*

$$(22)\quad -1+\frac{2}{1-u_0(q^v,q)}=-$$

$$-1+\frac{2}{1-}\,\frac{2q^v}{1-q+}\,\frac{q^{2v}(1+q)^2}{1-q^3+}\,\frac{q^{2v+1}(1+q^2)^2}{1-q^5+}\,\frac{q^{2v+2}(1+q^3)^2}{1-q^7+}\cdots=$$

$$=\frac{\pi}{2k_r'K(k_r)}\exp\left[-4\sum_{j=1}^{v-1}\operatorname{arctanh}(q^j)\right]$$

LEMMA 2.8. *Let v_1, v_2 be positive integers, then*

$$(23)\quad -1+\frac{2}{1-U(q^{v_1},q^{v_2},q)}=\exp\left[-4\left(\sum_{j_1=1}^{v_1-1}\operatorname{arctanh}(q^{j_1})-\sum_{j_2=1}^{v_2-1}\operatorname{arctanh}(q^{j_2})\right)\right]$$

PROOF. The proof follows easily from Propositions 2.6 and 2.7. □

PROPOSITION 2.9.

$$(24)\quad \sum_{n=0}^{\infty}\frac{q^n}{1-a^2q^{2n}}=\frac{1}{1-q+}\,\frac{-a^2(1-q)^2}{1-q^3+}\,\frac{-qa^2(1-q^2)^2}{1-q^5+}\,\frac{-q^2a^2(1-q^3)^2}{1-q^7+}\cdots$$

PROOF. Solve relation (18) for $U(a,b,q)$, divide by $a-b$ and then take the limit $b\to a$. □

PROPOSITION 2.10. *If $q=e^{-\pi\sqrt{r}}$, then*

$$(25)\quad \frac{K(k_r)}{2\pi}+\frac{1}{4}=\frac{1}{1-q+}\,\frac{(1-q)^2}{1-q^3+}\,\frac{q(1-q^2)^2}{1-q^5+}\,\frac{q^2(1-q^3)^2}{1-q^7+}\cdots$$

PROOF. Set in (24) $a=i$, $q=e^{-\pi\sqrt{r}}$ and use the identity

$$\sum_{n=0}^{\infty}\frac{q^n}{1+q^{2n}}=\frac{K(k_r)}{2\pi}+\frac{1}{4}$$

□

It is also known that when v is an integer,

$$(26)\quad -1+\frac{2}{1-u_0(q^{v+1/2},q)}=\exp\left[-4\sum_{n=0}^{\infty}\frac{q^{(2n+1)(v+1/2)}}{(2n+1)(1-q^{2n+1})}\right]$$

$$=\exp\left[-4\sum_{j=0}^{v-1}\operatorname{arctanh}(q^{j+1/2})+\operatorname{arctanh}(k_r)\right]$$

Hence

$$k_r=\tanh\left[4\sum_{j=0}^{v-1}\operatorname{arctanh}(q^{j+1/2})+\log\left(-1+\frac{2}{1-u_0(q^{v+1/2},q)}\right)\right]$$

For every positive integer v. By this means we get a continued fraction expansion for the singular modulus k_r

(27) $$\frac{k'_r}{1-k_r}=-1+\frac{2}{1-u_0(q^{1/2},q)}$$

From Propositions 2.6 and 2.7 and Lemma 2.8, we have

Proposition 2.11. *If c is positive real and v_1, v_2 are positive integers then:*

(28) $$-1+\frac{2}{1-U(q^{v_1+c},q^{v_2+c},q)}=$$

$$=\exp\left[-2\left(\sum_{j_1=1}^{v_1-1}\operatorname{arctanh}(q^{j_1+c})-\sum_{j_2=1}^{v_2-1}\operatorname{arctanh}(q^{j_2+c})\right)\right].$$

Moreover:
If $U=U(a,b,q)$, where $q=e^{-\pi\sqrt{r}}$ and

(29) $$c=\left\{\frac{-\log(a)}{\pi\sqrt{r}}\right\}=\left\{\frac{-\log(b)}{\pi\sqrt{r}}\right\}$$

then

$$-1+\frac{2}{1-U(a,b,q)}=$$

$$=\exp\left[-2\left(\sum_{j_1=1}^{\left[\frac{-\log(a)}{\pi\sqrt{r}}\right]-1}\operatorname{arctanh}(q^{j_1+c})-\sum_{j_2=1}^{\left[\frac{-\log(b)}{\pi\sqrt{r}}\right]-1}\operatorname{arctanh}(q^{j_2+c})\right)\right]$$

where $\{x\}$ is the fractional part of x and $[x]$ is the largest integer not exceeding x.

Remark 2.12. Observe that for $v_1=v_2=v$

$$-1+\frac{2}{1-U(q^{v+c},q^{v+c},q)}=1$$

Also, we observe that

$$\left(-1+\frac{2}{1-U(a,-b;q)}\right)=\left(-1+\frac{2}{1-u_0(a,q)}\right)\left(-1+\frac{2}{1-u_0(b,q)}\right) \tag{30}$$

This relation is similar to (19).

PROPOSITION 2.13.

$$\log\left(-1+\frac{2}{1-U(aq,b,q)}\right)-\log\left(-1+\frac{2}{1-U(a,b,q)}\right)=-2\,\mathrm{arctanh}\,(a) \tag{31}$$

PROOF. From (20)

$$\log\left(-1+\frac{2}{1-U(aq,b,q)}\right)+2\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)(1-q^{2n+1})}=2\sum_{n=0}^{\infty}\frac{q^{2n+1}a^{2n+1}}{(2n+1)(1-q^{2n+1})}$$

or

$$\log\left(-1+\frac{2}{1-U(aq,b,q)}\right)+2\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)(1-q^{2n+1})}-2\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)(1-q^{2n+1})}=$$

$$=2\sum_{n=0}^{\infty}\frac{q^{2n+1}a^{2n+1}}{(2n+1)(1-q^{2n+1})}-2\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)(1-q^{2n+1})}=-2\sum_{n=0}^{\infty}\frac{a^{2n+1}}{2n+1}$$

□

PROPOSITION 2.14.

$$\log\left(-1+\frac{2}{1-U(a,b,q)}\right)=2\sum_{j_1=1}^{\infty}\mathrm{arctanh}\,(aq^{j_1})-2\sum_{j_2=1}^{\infty}\mathrm{arctanh}\,(bq^{j_2}) \tag{32}$$

PROOF. In (31) replace a with aq and add the result to (31) to arrive at

$$\log\left(-1+\frac{2}{1-U(0,b,q)}\right)-\log\left(-1+\frac{2}{1-U(a,b,q)}\right)=-2\sum_{n=0}^{\infty}\mathrm{arctanh}\,(aq^{n})$$

By doing the same with

$$\log\left(-1+\frac{2}{1-U(0,b,q)}\right)=-2\sum_{n=0}^{\infty}\frac{b^{2n+1}}{(2n+1)(1-q^{2n+1})},$$

using the fact that

$$\log\left(-1+\frac{2}{1-U(0,0,q)}\right)=0$$

we obtain (32). □

REMARK 2.15. From (32) one can show that when α, β, s are positive reals and

$$\frac{\beta-\alpha}{s}=\nu=\nu_1-\nu_2\in\mathbf{N} \tag{33}$$

then for

$$x=\alpha \text{ and } y=\frac{\alpha(\beta+\nu_2)-b\nu_1-\alpha^2}{\alpha-\beta} \tag{34}$$

and $\nu_1=[y]$, $\nu_2=[x]$, we can evaluate

$$\log\left(-1+\frac{2}{1-U(q^\alpha,q^\beta,q^s)}\right)=\log\left(-1+\frac{2}{1-U(q^{s(x+y-[y])},q^{s(x+y-[x])},q^s)}\right)=$$

in the closed form

$$-2\sum_{j=1}^{[x]-1}\operatorname{arctanh}\left[q^{s(j+\{x\}+\{y\})}\right]+2\sum_{j=1}^{[y]-1}\operatorname{arctanh}\left[q^{s(j+\{x\}+\{y\})}\right] \tag{35}$$

From the relations (see [2])

$$\operatorname{sn}(u)=\operatorname{sn}(q,u)=\frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}\sin\left(\frac{2K}{\pi}(2n+1)u\right)}{1-q^{2n+1}} \tag{36}$$

$$\operatorname{cn}(u)=\operatorname{cn}(q,u)=\frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}\cos\left(\frac{2K}{\pi}(2n+1)u\right)}{1+q^{2n-1}} \tag{37}$$

where $u=2Kz/\pi$ and

$$\int\operatorname{sn}(u)du=\frac{k_r^{-1}}{2}\log\left(\frac{1-k_r\operatorname{cd}(u)}{1+k_r\operatorname{cd}(u)}\right)=\frac{1}{2k}\log\left(\frac{\vartheta_3-\vartheta_2\sqrt{k}}{\vartheta_3+\vartheta_2\sqrt{k}}\right) \tag{38}$$

and from

$$\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{(2n+1)(1-q^{2n+1})}=\frac{-i}{4}\arctan\left(\sqrt{\frac{m(-q)}{1-m(-q)}}\right) \tag{39}$$

with $m(q)=k_r^2$ and (17) we get next

PROPOSITION 2.16. *If $q = e^{-\pi\sqrt{r}}$, $r > 0$ and $\theta \in \mathbf{R}$, then*

$$Re\left[\log\left(-1+\frac{2}{1-u_0(q^{1/2}e^{i\theta},q)}\right)\right] = \tag{40}$$

$$= -\frac{1}{2}\left[\log\left(\frac{\vartheta_3-\vartheta_2\sqrt{k_r}}{\vartheta_3+\vartheta_2\sqrt{k_r}}\right)\right]_0^{\theta} - i\arctan\left(\sqrt{\frac{m(-q)}{1-m(-q)}}\right)$$

where $\vartheta_{2,3}$ are the Jacobi theta functions:

$$\vartheta_2(q,z) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos\left((2n+1)z\right) \tag{41}$$

$$\vartheta_3(q,z) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \tag{42}$$

PROPOSITION 2.17. *If θ_1, θ_2 are real numbers and $|q| < 1$, then*

$$Re\left[\log\left(-1+\frac{2}{1-U(q^{1/2}e^{i\theta_1},q^{1/2}e^{i\theta_2},q)}\right)\right] = \frac{1}{4}\left[\log\left(\frac{\vartheta_3-\vartheta_2\sqrt{k_r}}{\vartheta_3+\vartheta_2\sqrt{k_r}}\right)\right]_{\theta_1}^{\theta_2} \tag{43}$$

More generally,

$$\mathrm{cc}(u) = \mathrm{cc}(q,u) := \frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}\cos\left((2n+1)\frac{2Ku}{\pi}\right)}{1+q^{2n+1}} = \tag{44}$$

$$= \tan\left(\frac{4Ku}{\pi}\right)\mathrm{ss}(u) + q^{-1}\sec\left(\frac{4Ku}{\pi}\right)\mathrm{cn}(u) - \frac{2\pi\sqrt{q}\cos\left(\frac{2Ku}{\pi}\right)\sec\left(\frac{4Ku}{\pi}\right)}{Kk(1+q)}$$

where

$$\mathrm{ss}(u) = \mathrm{ss}(q,u) := \frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+1/2}\sin\left((2n+1)\frac{2Ku}{\pi}\right)}{1+q^{2n+1}} \tag{45}$$

But $\mathrm{cc}(q,u) = -i\cdot\mathrm{sign}(q)\,\mathrm{cd}(-q,u)$, so by solving (44) for the function ss we get

$$\mathrm{ss}(q,u) = -\frac{\mathrm{cn}(q,u)\csc\left(\frac{4Ku}{\pi}\right)}{q} + \frac{\pi\sqrt{q}\csc\left(\frac{2Ku}{\pi}\right)}{Kk_r(1+q)} - i\cdot\mathrm{cd}(-q,u)\cot\left(\frac{4Ku}{\pi}\right) \tag{46}$$

which leads to

PROPOSITION 2.18. *If θ is in $\mathbf{C}$ such that $|q^{1/2}e^{2i\theta K/\pi}| < 1$ and $0 < q < 1$, then*

$$\text{(47)} \qquad \frac{-\pi^2}{4K^2k}\frac{d}{dt}\left[\log\left(-1+\frac{2}{1-u_0(iq^{1/2}e^{2itK/\pi},q)}\right)\right]_{t=\theta} =$$

$$= \frac{i\pi\sqrt{q}\csc\left(\frac{2K\theta}{\pi}\right)}{kK(1-q)} + \left[1+i\cot\left(\frac{4K\theta}{\pi}\right)\right]\mathrm{cd}(q,\theta) + \frac{\csc\left(\frac{4K\theta}{\pi}\right)\mathrm{cn}(-q,\theta)}{q}$$

PROOF. From (17), the definitions of cc, ss and $e^{ix} = \cos(x) + i\cdot\sin(x)$, we have

$$\frac{-\pi^2}{4K^2k}\frac{d}{d\theta}\log\left(-1+\frac{2}{1-u_0(iq^{1/2}e^{2i\theta K/\pi},q)}\right) = \mathrm{ss}(-q,\theta) - i\cdot\mathrm{cc}(-q,\theta) =$$

$$= \mathrm{ss}(-q,\theta) + \mathrm{cd}(q,\theta)$$

Using (46), we get (47). □

REMARK 2.19. The function $\mathrm{ss}(q,u)$ plays very important role here. If one finds its generalized integral, then the problem of the evaluation of U will be solved.

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