Rend. Sem. Mat. Univ. Padova, Vol. 133 (2015)
DOI 10.4171/RSMUP/133-3

Finite p-supersoluble groups with some E-S-supplemented subgroups

Changwen Li (*) - Xiaolong Yu (**) - Na Tang (‡)

Abstract - Let H be a subgroup of a group G and H_{eG} denote the subgroup of H generated by all those subgroups of H which are S-permutably embedded in G. H is said to be E-S-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$. In this paper, we investigate the influence of some E-S-supplemented subgroups on the structure of finite groups. Some new characterizations of p-supersoluble groups are obtained.

Mathematics Subject Classification (2010). 20D10, 20D20.

Keywords. S-permutable, E-S-supplemented, p-supersoluble.

1. Introduction

Throughout this paper, all groups are finite. Most of the notation is standard and can be found in [4] and [17]. G always denotes a finite group and p denotes a prime. G is said to be p-nilpotent if it has a normal p-complement and G is called a p-supersoluble group if its every p-chief factor is cyclic. It is

(*) Indirizzo dell'A.: School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China.
E-mail: lcw2000@126.com
Supported by NNSF of China (Grant No. 11401264) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.
(**) Indirizzo dell'A.: School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China.
E-mail: yuxiaolong0710@sina.com
Supported by NNSF of China (Grant No. 11071229) and Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20113402110036).
(‡) Indirizzo dell'A.: School of Mathematical Science, Huaiyin Normal University, Huaiyin, 223300, China.
E-mail: hytn999@126.com
Supported by Tianyuan foundation (Grant No. 11226285).

easy to see that if G is p-nilpotent then G is also p-supersoluble. In the sequel, $\mathcal{U}_p$ will denote the class of all p-supersoluble groups and $G^{\mathcal{U}_p}$ will denote the $\mathcal{U}_p$-residual of G, i.e., the intersection of all normal subgroups N of G with $G/N \in \mathcal{U}_p$.

Recall that a subgroup H of G permutes with a subgroup K of G if HK is a subgroup of G; H is said to be S-permutable in G if H permutes with all Sylow subgroups of G. This property extends normality and permutability and was introduced by Kegel [10] in 1962. It turns out to be useful in establishing results concerning the group structure. On the other hand, the study of the generalizations of S-permutability is also one of the most fruitful research areas. For instance, Ballester-Bolinches and Pedraza-Aguilera [2] said that H is S-permutably embedded in G, if for each prime p dividing the order of $|H|$, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-permutable subgroup of G. Skiba [20] called that H is weakly S-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G. Using these generalized permutable subgroups, one has given a series of necessary and sufficient conditions for nilpotency, supersolubility, existence of Sylow towers and, more generally, for belonging to saturated formations containing some of these classes (see [1, 11, 14, 18, 19]). In order to unify above mentioned subgroups, Li, Zhang and Yi in [12] introduced the following concept:

DEFINITION 1.1. Let H be a subgroup of G and H_{eG} denote the subgroup of H generated by all those subgroups of H which are S-permutably embedded in G. If there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$, then H is called E-S-supplemented in G.

In [12], the authors improved a nice result of Skiba which gives some conditions under which every G-chief factor of a normal subgroup E of G is cyclic. In present paper, we continue the work of [12] to study the influence of E-S-supplemented on the structure of finite groups. More precisely, we study p-supersolubility of finite groups under the assumption that some special subgroups are E-S-supplemented.

2. Preliminaries

LEMMA 2.1 ([12, Lemma 2.2]). *Suppose that H is E-S-supplemented in G.*

(1) *If $H \leq L \leq G$, then H is E-S-supplemented in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is E-S-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is E-S-supplemented in G/N.*

LEMMA 2.2 ([14, Lemma 2.3]). *Suppose that H is S-permutable in G, and let P be a Sylow p-subgroup of H. If $H_G = 1$, then P is S-permutable in G.*

LEMMA 2.3 ([13, Lemma 2.6]). *Assume that L is a nontrivial normal subgroup of G and $L \cap \Phi(G) = 1$. Then the Fitting subgroup $F(L)$ of L lies in Soc(G) and therefore $F(L)$ is the direct product of the minimal normal subgroups of G contained in $F(L)$.*

LEMMA 2.4 ([3, Theorem 2.1.6]). *If G is p-supersoluble and $O_{p'}(G) = 1$, then the Sylow p-subgroup of G is normal in G.*

3. Main Results

THEOREM 3.1. *Let p be a fixed prime dividing the order of G and L a p-soluble normal subgroup of G such that G/L is p-supersoluble. If there exists a Sylow p-subgroup P of L such that every maximal subgroup of P is E-S-supplemented in G, then G is p-supersoluble.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G is p-soluble.

This follows directly from the p-solubility of L and the p-supersolubility of G/L.

(2) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/T$. Clearly, $\overline{G}/\overline{L} \cong G/LT \cong (G/L)/(TL/L)$ is p-supersoluble by the p-supersolubility of G/L, where $\overline{L} = LT/T$. Let $\overline{P_1} = P_1T/T$ be a maximal subgroup of PT/T. We may assume that P_1 is a maximal subgroup of P. Since P_1 is E-S-supplemented in G, the subgroup P_1T/T is E-S-supplemented in G/T by Lemma 2.1(3). The minimal choice of G yields that $\overline{G}$ is p-supersoluble, and so G is also p-supersoluble, a contradiction.

(3) If N is a minimal normal subgroup of G, then N is an elementary abelian p-group.

This follows from Steps (1) and (2).

(4) G has a unique minimal normal subgroup N contained in L such that G/N is p-supersoluble.

Let N be a minimal normal subgroup of G contained in L. Obviously, $N \leq P$ and P/N is a Sylow p-subgroup of L/N. Let P_1/N be a maximal subgroup of P/N. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 is E-S-supplemented in G and so P_1/N is E-S-supplemented in G/N by Lemma 2.1(2). Since $(G/N)/(L/N) \cong G/L$ is p-supersoluble, G/N satisfies all the hypotheses of our theorem. It follows that G/N is p-supersoluble by the minimality of G. Noticing that the class of all p-supersoluble groups is a saturated formation, we have N is the unique minimal normal subgroup of G contained in L.

(5) $|N| > p$.

It is clear by Step (4).

(6) The final contradiction.

If N is contained in all maximal subgroups of G, then $N \leq \Phi(G)$ and so G is p-supersoluble. This contradiction shows that there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Let G_p be a Sylow p-subgroup of G containing P. Then $G_p = N(G_p \cap M)$ and $G_p \cap M < G_p$. Take a maximal subgroup G_1 of G_p containing $G_p \cap M$ and set $P_1 = G_1 \cap P$. Then $|P : P_1| = |P : G_1 \cap P| = |PG_1 : G_1| = |G_p : G_1| = p$ and so P_1 is a maximal subgroup of P. First, we have $N \nsubseteq P_1$. If not, $P = P \cap G_p = P \cap NG_1 = N(G_1 \cap P) = NP_1 = P_1$, a contradiction. Secondly, we have $N \cap P_1 \neq 1$. If not, $|N : P_1 \cap N| = |NP_1 : P_1| = |P : P_1| = p$ and so $P_1 \cap N$ is a maximal of N. Therefore $|N| = p$, which contradicts Step (5).

By hypothesis, P_1 is E-S-supplemented in G. Then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{eG}$. Since $|G : T|$ is a power of p and $T \lhd\lhd G$, $O^p(G) \leq T$. We know $G/O^p(G)$ is a p-subgroup, so $G/O^p(G)$ is p-supersoluble and $G/(N \cap O^p(G)) \lesssim G/N \times G/O^p(G)$ is p-supersoluble. Then $N \cap O^p(G) \neq 1$. Since N is the minimal subgroup of G, $N \cap O^p(G) = N$ and so $N \leq O^p(G)$. It follows that $N \cap P_1 = N \cap (P_1)_{eG}$. Obviously, $(P_1)_{eG} \neq 1$. Let $U_1, U_2, ..., U_s$ be all the nontrivial subgroups of P_1 which are S-permutably embedded in G. For every $i \in \{1, 2, ..., s\}$, then there is an S-permutable subgroup K_i of G such that U_i is a Sylow p-subgroup of K_i. Suppose that for some $i \in \{1, 2, ..., s\}$, we have $(K_i)_G \neq 1$. Then we can take a minimal normal subgroup D of G such that $D \leq (K_i)_G$. In view of Step (3), D is a p-group. Then $D \leq (P_1)_{eG} \leq P_1 \leq L$. By virtue of Step (4), $D = N$. Consequently, $N \leq P_1$. This contradiction shows that for all $i \in \{1, 2, ..., s\}$, we have $(K_i)_G = 1$. By Lemma 2.2, U_i ($i \in \{1, 2, ..., s\}$) are S-permutable in G. It follows that $(P_1)_{eG}$ is S-permutable in G. Let G_q be any Sylow q-subgroup of G with $p \neq q$. Then $(P_1)_{eG} G_q = G_q (P_1)_{eG}$. Since $N \cap P_1 = N \cap (P_1)_{eG} = N \cap (P_1)_{eG} G_q$, we have $N \cap P_1$ is normalized by G_q. Since $P = G_p \cap L$, we have $P \trianglelefteq G_p$ and so $P_1 \trianglelefteq G_p$. It follows that $N \cap P_1$ is normalized by G_p. Therefore, $N \cap P_1$ is normal in G. The minimality of N implies that $N \cap P_1 = 1$ or $N \subseteq P_1$, a contradiction. □

We can choose L to get some results of special interest. For example, if we choose $L = G$ or $L = G^{\mathcal{U}_p}$, then we obtain the following criteria for p-supersolubility of groups.

COROLLARY 3.2. *Let P be a Sylow p-subgroup of a p-soluble G, where p is a fixed prime divisor of $|G|$. If all maximal subgroups of P are E-S-supplemented in G, then G is p-supersoluble.*

COROLLARY 3.3. *A p-soluble G is p-supersoluble if and only if all maximal subgroups of any Sylow p-subgroup of $G^{\mathcal{U}_p}$ are E-S-supplemented in G.*

In the following, we shall denote by $F_p(G)$ the p-Fitting subgroup of G. In fact, $F_p(G) = O_{p'p}(G)$.

THEOREM 3.4. *Let p be a fixed prime dividing the order of G and L a p-soluble normal subgroup of G such that G/L is p-supersoluble. If all maximal subgroups of $F_p(L)$ containing $O_{p'}(L)$ are E-S-supplemented in G, then G is p-supersoluble.*

PROOF. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(L) = 1$.

If $O_{p'}(L) \neq 1$, we consider the factor group $G/O_{p'}(L)$. First,

$$(G/O_{p'}(L))/(L/O_{p'}(L)) \cong G/L$$

is p-supersoluble. Now $O_{p'}(L/O_{p'}(L)) = 1$ and $F_p(L/O_{p'}(L)) = F_p(L)/O_{p'}(L)$. Let $M/O_{p'}(L)$ be a maximal subgroup of $F_p(L/O_{p'}(L))$. Then M is a maximal subgroup of $F_p(L)$ containing $O_{p'}(L)$. Since M is E-S-supplemented in G, in view of Lemma 2.1(2), $M/O_{p'}(L)$ is E-S-supplemented in $G/O_{p'}(L)$. Thus

$G/O_{p'}(L)$ satisfies the hypotheses of the theorem. The minimality of G implies that $G/O_{p'}(L)$ is p-supersoluble and so G is p-supersoluble, contrary to the choice of G.

(2) $L \cap \Phi(G) = 1$.

Write $H = L \cap \Phi(G)$. If $H \neq 1$, we consider the factor group G/H. By virtue of [9, III, 3.5], we have $F(L/H) = F(L)/H$ and so $F(L/H) = O_p(L)/H$. On the other hand, writing $K/H = O_{p'}(L/H)$ and letting S be a Hall p'-subgroup of K we have $K = SH$, and by the Frattini argument $G = KN_G(S) = HN_G(S) = N_G(S)$ and $S \lhd G$. Therefore $S = 1$ and $O_{p'}(L/H) = 1$. This implies that $F_p(L/H) = O_p(L/H) = O_p(L)/H = F_p(L)/H$. If P_1/H is a maximal subgroup of $F_p(L/H)$, then P_1 is maximal in $F_p(L)$ and, by our hypothesis, it is an E-S-supplemented subgroup of G. Hence P_1/H is E-S-supplemented in G/H by Lemma 2.1(2). Now the minimality of G implies that G/H is p-supersoluble and then so is G, which contradicts the choice of G.

(3) The final contradiction.

Since L is p-soluble and $O_{p'}(L) = 1$, we have $C_L(O_p(L)) \leq O_p(L)$ by [5, Theorem 6.3.2]. Now $\Phi(L) = 1$ implies that $F(L) = O_p(L)$ is a nontrivial elementary abelian p-group by [9, III, 4.5]. Thus $C_L(F(L)) = F(L)$. By hypothesis, all maximal subgroups of $F(L)$ are E-S-supplemented in G. Applying [12, Theorem 1.4], all G-chief factors of $F(L)$ are cyclic. In view of Lemma 2.3, we have $F(L) = N_1 \times N_2 \times \cdots \times N_r$, where N_i is a minimal normal of G with order p. Since for each i the factor group $G/C_G(N_i)$ is isomorphic to some subgroup of $\mathrm{Aut}(N_i)$, we have $G/C_G(N_i)$ is cyclic. Consequently, $G/C_G(N_i)$ is p-supersoluble. From the p-supersolubility of G/L, it follows that $G/(L \cap C_G(N_i)) = G/C_L(N_i)$ is p-supersoluble. Therefore $G/\bigcap_{i=1}^{r} C_L(N_i)$ is p-supersoluble. In fact, what we really have is $G/F(L)$ is p-supersoluble because

$$\bigcap_{i=1}^{r} C_L(N_i) = C_L(F(L)) = F(L).$$

However all G-chief factors of $F(L)$ are cyclic of order p and hence G is p-supersoluble, a contradiction. □

Corollary 3.5. *Let G be a p-soluble group, where p is a fixed prime divisor of $|G|$. Then G is p-supersoluble if and only if all maximal subgroups of $F_p(G^{\mathcal{U}_p})$ containing $O_{p'}(G^{\mathcal{U}_p})$ are E-S-supplemented in G.*

THEOREM 3.6. *Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and all maximal subgroups of P are E-S-supplemented in G, then G is p-nilpotent. In particular, G is p-supersoluble.*

PROOF. It is easy to see that the theorem holds when $p = 2$ by [12, Theorem 5.1], so it suffices to prove the theorem for the case of odd prime.

Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) If H is a proper subgroup of G with $P \leq H < G$, then H is p-nilpotent.

It is easy to see that $N_H(P) \leq N_G(P)$ and hence $N_H(P)$ is p-nilpotent. By Lemma 2.1(1), all maximal subgroups of P are E-S-supplemented in H. Hence H satisfies the hypothesis of our theorem. The minimal choice of G implies that H is p-nilpotent.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. Obviously, $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. In view of Lemma 2.1(3), it is easy to see that all maximal subgroups of $PO_{p'}(G)/O_{p'}(G)$ are E-S-supplemented in $G/O_{p'}(G)$. Since

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent, $G/O_{p'}(G)$ satisfies the hypothesis of our theorem. The minimal choice of G yields that $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

Let $J(P)$ be the Thompson subgroup of P. Then $N_G(P) \leq N_G(Z(J(P))) \leq G$. If $N_G(Z(J(P))) < G$, then, in view of Step (1), $N_G(Z(J(P)))$ is p-nilpotent and so G is p-nilpotent by [5, Theorem 8.3.1], a contradiction. Hence $N_G(Z(J(P))) = G$, which shows that $Z(J(P))$ is a normal p-subgroup of G and $1 < O_p(G) < P$.

(4) G is p-soluble.

It is easy to see that the factor group $G/O_p(G)$ satisfies the hypothesis of our theorem. Now, by the minimality of G, we see that $G/O_p(G)$ is p-nilpotent. Consequently, $G/O_p(G)$ is p-soluble and so G is p-soluble.

(5) The final contradiction.

Applying Corollary 3.2, G is p-supersoluble. In view of Lemma 2.4, P is normal in G. Therefore, $G = N_G(P)$ is p-nilpotent, a contradiction. □

Corollary 3.7. *Let p be a prime dividing the order of G and L a normal subgroup of G such that G/L is p-nilpotent. Suppose that there exists a Sylow p-subgroup P of L such that all maximal subgroups of P are E-S-supplemented in G and $N_G(P)$ is p-nilpotent. Then G is p-nilpotent.*

Proof. It is obvious that $N_L(P)$ is p-nilpotent and all maximal subgroups of P are E-S-supplemented in L. Applying Theorem 3.6, L is p-nilpotent. Now we have $L_{p'}$ is the normal Hall p'-subgroup of L. Furthermore, $L_{p'}$ is a normal subgroup of G. If $L_{p'} \neq 1$, we consider the factor group $G/L_{p'}$. First, $(G/L_{p'})/(L/L_{p'}) \cong G/L$ is p-nilpotent and all maximal subgroups of $PL_{p'}/L_{p'}$ are E-S-supplemented in $G/L_{p'}$ by Lemma 2.1(3). Secondly, $N_{G/L_{p'}}(PL_{p'}/L_{p'}) = N_G(P)L_{p'}/L_{p'}$ is p-nilpotent. Hence $G/L_{p'}$ satisfies the hypothesis of our corollary. By induction, $G/L_{p'}$ is p-nilpotent and so G is p-nilpotent, as desired. Hence we may assume $L_{p'} = 1$, i.e., $L = P$. By hypothesis, $N_G(P) = G$ is p-nilpotent. □

Theorem 3.8. *Let L be a normal subgroup of G such that G/L is p-supersoluble, where p is a prime divisor of $|L|$ with $(p-1, |L|) = 1$. Suppose that for a Sylow p-subgroup P of L, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) is E-S-supplemented in G. Then G is p-supersoluble.*

Proof. Suppose that this theorem is false and consider a counterexample (G, L) for which $|G||L|$ is minimal.

(1) L is p-nilpotent.

By Lemma 2.1, it is easy to see that every subgroup H of P with order $|H| = |D|$ and every cyclic subgroup of P with order 4 (if $|D| = 2$ and P is a non-abelian 2-group) are E-S-supplemented in L. Applying [12, Theorem 1.5], L is p-nilpotent.

(2) $P = L$.

From Step 1, we know $O_{p'}(L)$ is the normal Hall p'-subgroup of L. Assume that $O_{p'}(L) \neq 1$. In view of Lemma 2.1, the hypothesis holds for $(G/O_{p'}(L), L/O_{p'}(L))$. Hence, by the minimal choice of (G, L), the theorem is true for $(G/O_{p'}(L), L/O_{p'}(L))$ and so $G/O_{p'}(L)$ is p-supersoluble. Consequently, G is p-supersoluble. This contradiction shows that $O_{p'}(L) = 1$. Hence L is a normal p-subgroup of G.

(3) The final contradiction.

Applying [12, Theorem 1.4], all G-chief factors of L are cyclic. From the p-supersolubility of G/L, we have G is p-supersoluble. □

COROLLARY 3.9. *Let p be a prime divisor of $|G^{\mathcal{U}_p}|$ with $(p-1,|G^{\mathcal{U}_p}|)=1$. Suppose that for a Sylow p-subgroup P of $G^{\mathcal{U}_p}$, there exists a subgroup D of P such that $1<|D|<|P|$ and every subgroup H of P with order $|H|=|D|$ and every cyclic subgroup of P with order 4 (if $|D|=2$ and P is a non-abelian 2-group) is E-S-supplemented in G. Then G is p-supersoluble.*

COROLLARY 3.10. *Let p be the smallest prime divisor of $|G|$. Suppose that for a Sylow p-subgroup P of $G^{\mathcal{U}_p}$, there exists a subgroup D of P such that $1<|D|<|P|$ and every subgroup H of P with order $|H|=|D|$ and every cyclic subgroup of P with order 4 (if $|D|=2$ and P is a non-abelian 2-group) is E-S-supplemented in G. Then G is p-nilpotent.*

PROOF. By virtue of Corollary 3.9, G is p-supersoluble. Since p is the smallest prime divisor of $|G|$, we have G is p-nilpotent by [11, Lemma 2.8]. □

4. Some Applications

It is clear that all the subgroups, whether they are c-normal subgroups [21], c^*-normal subgroups [22], S-permutably embedded subgroups or weakly S-permutable subgroups, are all E-S-supplemented subgroups. Hence the following results are all special cases of our Theorems.

COROLLARY 4.1 ([22, Theorem 3.7]). *Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. If all maximal subgroups of $F_p(H)$ containing $O_{p'}(H)$ are c^*-normal in G, then G is p-supersoluble.*

COROLLARY 4.2 ([22, Theorem 3.5]). *Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c^*-normal in G, then G is p-supersoluble.*

COROLLARY 4.3 ([16, Theorem 3.1]). *Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is c-normal in G, then G is p-supersoluble.*

Corollary 4.4 ([8, Theorem 3.10]). *Let p be a prime, G a p-soluble group and H a normal subgroup of G such that G/H is p-supersoluble. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is S-permutably embedded in G, then G is p-supersoluble.*

Corollary 4.5 ([15, Theorem 3.3]). *Let p be a prime and G a p-soluble group. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly S-permutable in G, then G is p-supersoluble.*

Corollary 4.6 ([6, Theorem 3.1]). *Let p be an odd prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 4.7 ([14, Theorem 3.2]). *Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is S-permutably embedded in G, then G is p-nilpotent.*

5. Some Remarks

Remark 5.1. The hypothesis that L is p-soluble in Theorems 3.1 and 3.4 cannot be removed. Consider for example the group $G = A_5$, the alternating group of degree 5. Clearly 1 is the maximal subgroup of any Sylow 5-subgroup of G and $F_5(G) = 1$. However, G is not 5-supersoluble.

Remark 5.2. In proving our Theorem 3.6, the assumption that $N_G(P)$ is p-nilpotent is essential. To illustrate the situation, we may also consider $G = A_5$ and $p = 5$.

Remark 5.3. From [7] or [11], we know if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{eG}$, then H is called SE-supplemented or E-supplemented in G. Obviously, the set of all SE-supplemented subgroups of a group is wider than the set of all its E-S-supplemented subgroups. However, the following example shows that we cannot generalize our Theorems 3.1 and 3.6 using SE-supplemented subgroups: Let $A = Z_3 \rtimes Z_2 \cong S_3$, where Z_3 is a cyclic subgroup of order 3, Z_2 is a cyclic subgroup of order 2 and S_3 is the symmetric group of degree 3. Let $B = A \wr Z_3$, the regular wreath product of A by Z_3. Put $G = O^2(B) = \langle x \mid \circ(x) = 3\rangle$. Then $G \cong (Z_3 \times Z_3 \times Z_3) \rtimes A_4$ and G is soluble, where A_4 is the alternating group of degree 4. Let P be a Sylow 3-subgroup of G. Then $N_G(P) = P$ is 3-nilpotent. It is easy to see that every maximal subgroup of P is SE-supplemented in G. But G is not 3-supersoluble.

Acknowledgments. The authors would like to thank the referee who read the manuscript carefully and contributed a lot of valuable suggestions and useful comments.

REFERENCES

[1] M. Asaad and A. A. Heliel, *On S-quasinormally embedded subgroups of finite groups*, J. Pure Appl. Algebra, **165** (2001), pp. 129–135.

[2] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, *Sufficient conditions for supersolubility of finite groups*, J. Pure Appl. Algebra, **127** (1998), pp. 113–118.

[3] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of finite groups*, Walter de Gruyter, Berlin, 2010.

[4] K. Doerk and T. Hawkes, *Finite solvable Groups*, Walter de Gruyter, Berlin-New York, 1992.

[5] D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.

[6] X. Guo and K. P. Shum, *On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups*, Arch. Math., **80** (2003), pp. 561–569.

[7] W. Guo, A. N. Skiba and N. Yang, *SE-supplemented subgroups of finite groups*, Rend. Sem. Mat. Univ. Padova, **129** (2013), pp. 245–263.

[8] A. A. Heliel and S. M. Alharbia, *The influence of certain permutable subgroups on the structure of finite groups*, Int. J. Algebra, **4** (2010), pp. 1209–1218.

[9] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin-New York, 1967.

[10] O. H. Kegel, *Sylow Gruppen und subnormalteiler endlicher Gruppen*, Math. Z., **78** (1962), pp. 205–221.

[11] C. Li, X. Zhang and Y. Wang, *On E-supplemented subgroups and the structure of finite groups*, J. Algebra Appl., **12** (2013), 1350019.

[12] C. Li, X. Zhang and X. Yi, *On E-S-supplemented subgroups of finite groups*, Colloq. Math., **131** (2013), pp. 41–51.

[13] Y. Li, Y. Wang and H. Wei, *The influence of π-quasinormality of some subgroups of a finite group*, Arch. Math., **81** (2003), pp. 245–252.

[14] Y. Li, Y. Wang and H. Wei, *On p-nilpotency of finite groups with some subgroups π-quasinormally embedded*, Acta Math. Hungar., **108** (2005), pp. 283–298.

[15] L. Miao, *On weakly S-permutable subgroups*, Bull. Braz. Math. Soc, New Series, **41** (2010), pp. 223–235.

[16] M. Ramadan, M. Ezzat-Mohamed and A. A. Heliel, *On c-normality of certain subgroups of prime power order of finite groups*, Arch. Math., **85** (2005), pp. 203–210.
[17] D. J. S. Robinson, *A Course in Theory of Group*, Springer-Verlag, 1982.
[18] L. A. Shemetkov and A. N. Skiba, *On the $\mathcal{X}\Phi$-hypercentre of finite groups*, J. Algebra, **322** (2009), pp. 2106–2117.
[19] A. N. Skiba, *Cyclicity conditions for G-chief factors of normal subgroups of a group G*, Siberian Math. J., **52** (2011), pp. 127–130.
[20] A. N. Skiba, *On weakly S-permutable subgroups of finite groups*, J. Algebra, **315** (2007), pp. 192–209.
[21] Y. Wang, *c-Normality of groups and its properties*, J. Algebra, **180** (1996), pp. 954–965.
[22] H. Wei and Y. Wang, *On c^*-normality and its properties*, J. Group Theory, **10** (2007), pp. 211–223.

Manoscritto pervenuto in redazione il 12 luglio 2013.