

Groups having complete bipartite divisor graphs for their conjugacy class sizes

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ABSTRACT - Given a finite group G , the *bipartite divisor graph* for its conjugacy class sizes is the bipartite graph with bipartition consisting of the set of conjugacy class sizes of $G \setminus \mathbf{Z}(G)$ (where $\mathbf{Z}(G)$ denotes the centre of G) and the set of prime numbers that divide these conjugacy class sizes, and with $\{p, n\}$ being an edge if $\gcd(p, n) \neq 1$.

In this paper we construct infinitely many groups whose bipartite divisor graph for their conjugacy class sizes is the complete bipartite graph $K_{2,5}$, giving a solution to a question of Taeri [15].

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1. Introduction

Given a finite group G , there are several sets of invariants that convey nontrivial information about the structure of G . Some classical examples include the set consisting of the orders of the elements of G , or the set of conjugacy class sizes of G , or the set of degrees of the irreducible complex characters of G .

For every set X as above, it is natural to ask to what extent the group structure of G is reflected and influenced by X , and a useful tool in this kind of investigation is the so-called *bipartite divisor graph* $B(X)$ of X . The

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bipartition of the vertex set of $B(X)$ consists of $X \setminus \{1\}$ and of the set of prime numbers dividing x , for some $x \in X$, and the edge set of $B(X)$ consists of the pairs $\{p, x\}$ with $\gcd(p, x) \neq 1$.

In this paper we are interested in the case that X consists of the conjugacy class sizes of G , that is, $X = \{|g^G| \mid g \in G\}$, and hence we study the bipartite divisor graph for the conjugacy class sizes. We will denote this graph simply by $B(G)$. We refer the reader to the beautiful survey [5] for the influence of the conjugacy class sizes on the structure of finite groups.

Historically, various graphs associated with the algebraic structure of a finite group have been extensively studied by a large number of authors, see for example [1, 2, 3, 4, 6, 9, 13, 16]. Recently, Lewis [14] discussed many remarkable connections among these graphs by analysing analogous of these graphs for arbitrary positive integer subsets. Then, inspired by the survey of Lewis, Praeger and Iranmanesh [12] introduced the bipartite divisor graph $B(X)$ for a finite set X of positive integers and studied some basic invariants of this graph (such as the diameter, girth, number of connected components and clique number).

One of the main questions that naturally arises in this area is classifying the groups whose bipartite divisor graphs have special graphical shapes. For instance, in [10], the first author of this paper and Iranmanesh have classified the groups whose bipartite divisor graphs are paths. Similarly, Taeri [15] considered the case that the bipartite divisor graph is a cycle, and in the course of his investigation posed the following question:

Question ([15, Question 1]). *Is there any finite group G such that $B(G)$ is isomorphic to a complete bipartite graph $K_{m,n}$, for some positive integers $m, n \geq 2$?*

The main theorem of this paper gives a family of examples which shows that the answer to this question is positive.

THEOREM 1.1. *For every odd prime p , there exists a $\{2, p\}$ -group G with $B(G) = K_{2,5}$.*

In light of Theorem 1.1, we pose the following question.

QUESTION 1.2. *For which positive integers $m, n \geq 2$, is there a finite group G with $B(G) = K_{m,n}$?*

In this paper we have tried to produce infinitely many groups G with $B(G) = K_{m,n}$, $m, n \geq 2$ and with m, n as small as possible. We were unable

to construct groups G with $B(G) = K_{2,3}$. Observe that Dolfi and Jabara [8] have classified the finite groups having only two non-trivial conjugacy class sizes, and from this remarkable work it follows [8, Corollary C] that there is no group G with $B(G) = K_{2,2}$.

REMARK 1.3. Shortly after the submission of our arXiv preprint [11] and during the refereeing process of an earlier draft of this paper, Casolo has provided us of a conjecture addressing Question 1.2. For the sake of completeness we include it here:

CONJECTURE 1.4 [7]. *If G is a finite group with $B(G) = K_{m,n}$ (where n is the number of non-identity conjugacy class sizes of G), then $n \geq 2^m$.*

In particular, Casolo conjectures that there exists no finite group G with $B(G) = K_{2,3}$. (Casolo has proved [7] that there are groups G with $B(G) = K_{2,4}$, we include a sketch of his construction in Section 3.)

Clearly a “working” conjecture can be very useful for the developing of new research, and hence we acknowledge our deepest gratitude to our friend Carlo Casolo for his contribution.

2. Groups with complete bipartite divisor graphs for their conjugacy class sizes

Given a finite group G , we denote by $\mathbf{Z}(G)$ the centre of G , and given $g \in G$, we denote by g^G the conjugacy class of g under G . Furthermore, $|g^G|$ denotes the cardinality of g^G .

NOTATION 2.1. We let p be an odd prime number and E be the extraspecial 2-group of plus-type

$$E = \langle x_1, \dots, x_{p-1}, y_1, \dots, y_{p-1}, z \mid \begin{aligned} x_i^2 &= y_i^2 = z^2 = [x_i, z] = [y_i, z] = 1 \ \forall i, \\ [x_i, x_j] &= [y_i, y_j] = [x_i, y_j] = 1 \ \forall i \neq j, \\ [x_i, y_i] &= z \ \forall i. \end{aligned} \rangle$$

Observe that

$$[x_1^{\varepsilon_1} \cdots x_{p-1}^{\varepsilon_{p-1}}, y_1^{\eta_1} \cdots y_{p-1}^{\eta_{p-1}}] = z^{\varepsilon_1 \eta_1 + \cdots + \varepsilon_{p-1} \eta_{p-1}},$$

and that every element $g \in E$ can be written uniquely as

$$g = x_1^{\varepsilon_1} \cdots x_{p-1}^{\varepsilon_{p-1}} y_1^{\eta_1} \cdots y_{p-1}^{\eta_{p-1}} z^v$$

with $\varepsilon_i, \eta_i \in \{0, 1\}$ for each $i \in \{1, \dots, p-1\}$ and $v \in \{0, 1\}$.

Finally, we let A and B be the maps from $\{x_1, \dots, x_{p-1}, y_1, \dots, y_{p-1}, z\}$ to E given (with exponential notation) by

$$\begin{aligned} z^A &= z, & z^B &= z, \\ x_i^A &= x_{i+1}, & x_i^B &= x_{p-i}, & \text{for each } i \in \{1, \dots, p-2\}, \\ x_{p-1}^A &= x_1 \cdots x_{p-1}, & x_{p-1}^B &= x_1, \\ y_i^A &= y_1 y_{i+1}, & y_i^B &= y_{p-i}, & \text{for each } i \in \{1, \dots, p-2\}, \\ y_{p-1}^A &= y_1, & y_{p-1}^B &= y_1. \end{aligned}$$

LEMMA 2.2. *Let E, A and B be as in Notation 2.1. Then A and B extend to two automorphisms a and b (respectively) of E . Moreover, $a^p = b^2 = (ab)^2 = 1$ and $\langle a, b \rangle$ is a dihedral group of order $2p$.*

PROOF. To show that A and B extend to two automorphisms, say a and b respectively, of E it suffices to prove that they preserve the defining relations of E . For instance, for every $i \in \{1, \dots, p-2\}$, we have $[x_i^A, y_i^A] = [x_{i+1}, y_1 y_{i+1}] = [x_{i+1}, y_{i+1}] = z = z^A = [x_i, y_i]^A$. Similarly, $[x_{p-1}^A, y_{p-1}^A] = [x_1 \cdots x_{p-1}, y_1] = [x_1, y_1] = z = z^A = [x_{p-1}, y_{p-1}]^A$. For $i, j \in \{1, \dots, p-2\}$ with $i \neq j$, we have $[x_i^A, y_j^A] = [x_{i+1}, y_1 y_{j+1}] = 1 = [x_i, y_j]^A$. Moreover, $[x_i^A, y_{p-1}^A] = [x_{i+1}, y_1] = 1 = [x_i, y_{p-1}]^A$ and $[x_{p-1}^A, y_j^A] = [x_1 \cdots x_{p-1}, y_1 y_{j+1}] = [x_1, y_1][x_{j+1}, y_{j+1}] = z \cdot z = 1 = [x_{p-1}, y_j]^A$. All the other computations are similar and are left to the conscientious reader.

It is readily seen that b^2 fixes each generator $x_1, \dots, x_{p-1}, y_1, \dots, y_{p-1}, z$ of E , and hence $b^2 = 1$. Observe that $x_{p-1}^{a^2} = (x_1 \cdots x_{p-2} x_{p-1})^a = x_1^a \cdots x_{p-2}^a x_{p-1}^a = x_2 \cdots x_{p-1} (x_1 \cdots x_{p-1}) = x_1$. Thus $x_i^{a^p} = (x_1^{a^{p-2}})^{a^2} = x_{p-1}^{a^2} = x_1$, and hence $x_i^{a^p} = (x_1^{a^{i-1}})^{a^p} = (x_1^{a^p})^{a^{i-1}} = x_1^{a^{i-1}} = x_i$ for every $i \in \{1, \dots, p-1\}$. Arguing inductively on i , we get $y_i^{a^{p-1-i}} = y_{p-1-i} y_{p-1}$ for each $i \in \{1, \dots, p-1\}$. It follows that $y_i^{a^{p-i}} = (y_{p-i-1} y_{p-1})^a = y_1 y_{p-i} y_1 = y_{p-i}$. Given $i \in \{1, \dots, p-1\}$, applying this equality first to the index i and then to the index $p-i$, we get $y_i^{a^p} = (y_i^{a^{p-i}})^{a^i} = y_{p-i}^{a^i} = y_{p-(p-i)} = y_i$. Therefore a^p fixes $x_1, \dots, x_{p-1}, y_1, \dots, y_{p-1}, z$ and hence $a^p = 1$. Finally, we have

$$\begin{aligned} z^{ab} &= z, \\ x_i^{ab} &= x_{i+1}^b = x_{p-i-1}, & \forall i \in \{1, \dots, p-2\}, \\ x_{p-1}^{ab} &= (x_1 \cdots x_{p-1})^b = x_{p-1} \cdots x_1 = x_1 \cdots x_{p-1}, \\ y_i^{ab} &= (y_1 y_{i+1})^b = y_{p-1} y_{p-i-1}, & \forall i \in \{1, \dots, p-2\}, \\ y_{p-1}^{ab} &= y_1^b = y_{p-1}, \end{aligned}$$

from which it easily follows that $(ab)^2 = 1$. □

PROOF OF THEOREM 1.1. Let p, E, A and B be as in Notation 2.1, let a and b be the automorphisms extending A and B as in Lemma 2.2, and set $M = \langle a, b \rangle$. Given two elements n_1 and n_2 of order p , define $N = \langle n_1 \rangle \times \langle n_2 \rangle \times E$ and $G = N \rtimes M$ where

$$n_1^a = n_1, \quad n_2^a = n_1 n_2, \quad n_1^b = n_1, \quad n_2^b = n_2^{-1}$$

(note that this is well-defined because the action of $M = \langle a, b \rangle$ on $\langle n_1, n_2 \rangle$ determines a dihedral group of automorphisms of $\langle n_1, n_2 \rangle$ order $2p$). We show that $B(G) = K_{2,5}$.

We start by determining the conjugacy class sizes of $G \setminus \mathbf{Z}(G)$. Clearly, $\mathbf{Z}(G) = \langle n_1, z \rangle$.

We claim that the conjugacy classes of G in $N \setminus \mathbf{Z}(G)$ have cardinality $2p$ or $4p$. Let $e \in \{h \in E \mid [a, h] \in \mathbf{Z}(E)\}$ and write $e = x_1^{\varepsilon_1} \cdots x_{p-1}^{\varepsilon_{p-1}} y_1^{\eta_1} \cdots y_{p-1}^{\eta_{p-1}} z^v$, for some $\varepsilon_1, \dots, \varepsilon_{p-1}, \eta_1, \dots, \eta_{p-1}, v \in \{0, 1\}$. We have

$$\begin{aligned} e^a &= (x_1^{\varepsilon_1})^a \cdots (x_{p-1}^{\varepsilon_{p-1}})^a (y_1^{\eta_1})^a \cdots (y_{p-1}^{\eta_{p-1}})^a (z^v)^a \\ &= x_2^{\varepsilon_1} x_3^{\varepsilon_2} \cdots x_{p-1}^{\varepsilon_{p-2}} (x_1 \cdots x_{p-1})^{\varepsilon_{p-1}} (y_1 y_2)^{\eta_1} (y_1 y_3)^{\eta_2} \cdots (y_1 y_{p-1})^{\eta_{p-2}} y_1^{\eta_{p-1}} z^v \\ &= x_1^{\varepsilon_{p-1}} x_2^{\varepsilon_1 + \varepsilon_{p-1}} \cdots x_{p-1}^{\varepsilon_{p-2} + \varepsilon_{p-1}} y_1^{\eta_1 + \cdots + \eta_{p-1}} y_2^{\eta_1} \cdots y_{p-1}^{\eta_{p-2}} z^v. \end{aligned}$$

As $e^{-1}e^a = [e, a] \in \mathbf{Z}(E) = \langle z \rangle$, by comparing e with e^a , we get $\varepsilon_i = \eta_i = 0$ for every $i \in \{1, \dots, p-1\}$, and hence $e \in \mathbf{Z}(E)$. Thus $\{h \in E \mid [a, h] \in \mathbf{Z}(E)\} = \mathbf{Z}(E)$. In particular, $C_E(a) = \mathbf{Z}(E)$ and a acts fixed point freely on $E \setminus \mathbf{Z}(E)$. It follows that the orbits of M acting by conjugation on $E \setminus \mathbf{Z}(E)$ have size p or $2p$. Since $|E'| = 2$, for every $e \in E \setminus \mathbf{Z}(E)$ we have $|E : C_E(e)| = 2$. Thus e^G has size $2p$ or $4p$. Now let $g \in N \setminus \mathbf{Z}(N)$. Write $g = ne$, for some $n \in \langle n_1, n_2 \rangle \setminus \langle n_1 \rangle$ and $e \in E \setminus \langle z \rangle$. Now $C_N(g) = \langle n_1, n_2 \rangle C_E(e)$ has index 2 in N and so (as a does not centralise n) $C_G(g)$ has index divisible by $2p$. It follows (with an elementary computation) that the conjugacy classes of G in $N \setminus \mathbf{Z}(G)$ have size $2p$ or $4p$. Observe that both possibilities can occur: $|x_1^G| = 4p$ and $|(x_1 x_{p-1})^G| = 2p$.

Let $g \in \langle N, a \rangle \setminus N$ and write $g = na^i$, for some $n \in N$ and $i \in \{1, \dots, p-1\}$. Now, given $e \in E$, we have $e^n = ez^v$, for some $v \in \{0, 1\}$. In particular, if $e \in C_E(g)$, then $e = e^g = (e^n)^{a^i} = (ez^v)^{a^i} = e^{a^i} z^v$, and hence $[a, e] \in \mathbf{Z}(E)$. Thus $e \in \mathbf{Z}(E)$. It follows that $C_N(g) = \mathbf{Z}(G)$ and $|C_G(g)| = p|C_N(g)| = 2p^2$. Thus $|g^G| = 2^{2p-1}p$.

It remains to compute the size of the conjugacy classes of G in $G \setminus \langle a, N \rangle$. Observe that every element of $G \setminus \langle a, N \rangle$ is conjugate to an element of $\langle b, N \rangle \setminus N$. In particular, since we are interested only on the conjugacy class sizes, we may assume that $g \in \langle b, N \rangle \setminus N$. Write $g = nb$, for

some $n \in N$. Observe that

$$\{h \in E \mid [h, b] \in \mathbf{Z}(E)\} = \langle x_1 x_{p-1}, x_2 x_{p-2}, \dots, x_{(p-1)/2} x_{(p+1)/2}, \\ y_1 y_{p-1}, y_2 y_{p-2}, \dots, y_{(p-1)/2} y_{(p+1)/2}, z \rangle = \mathbf{C}_E(b),$$

and $|\mathbf{C}_E(b)| = 2^p$. Now, arguing as above, it is easy to verify that $\mathbf{C}_N(g) \leq \langle n_1, \mathbf{C}_E(b) \rangle$ and $|\langle n_1, \mathbf{C}_E(b) \rangle : \mathbf{C}_N(g)| \in \{1, 2\}$. Thus $|\mathbf{C}_G(g)| \in \{p^2 2^p, p^2 2^{p+1}\}$ and hence $|g^G|$ equals $2^{p-1} p^2$ or $2^p p^2$. Observe that both of these cases can occur: in fact $|b^G| = |G : \mathbf{C}_G(b)| = 2^{p-1} p^2$ and $|(x_1 b)^G| = |G : \mathbf{C}_G(x_1 b)| = 2^p p^2$.

Summing up, the conjugacy class sizes $|g^G|$ (as g runs through $G \setminus \mathbf{Z}(G)$) are $2p, 4p, 2^{2p-1} p, 2^{p-1} p^2, 2^p p^2$, from which it follows that $B(G) = K_{2,5}$. \square

3. The case $B(G) = K_{2,4}$

Here we describe a group G with $B(G) = K_{2,4}$: the construction is due to Casolo [7]. Let Q be the quaternion group of order 8, and let $Q_1 = \langle q_1 \rangle, Q_2 = \langle q_2 \rangle$ and $Q_3 = \langle q_3 \rangle$ be the three maximal subgroups of Q . Let p be an odd prime and let E be the extra-special p -group

$$\langle x_1, x_2, x_3, y_1, y_2, y_3, z \mid x_i^p = y_i^p = z^p = [x_i, z] = [y_i, z] = 1 \forall i, \\ [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \forall i \neq j, [x_i, y_i] = z \forall i \rangle.$$

Arguing as in Section 2, it is easy to verify that by assigning

$$x_1^{q_1} = x_1, \quad y_1^{q_1} = y_1, \quad x_2^{q_1} = x_2^{-1}, \quad y_2^{q_1} = y_2^{-1}, \quad x_3^{q_1} = x_3^{-1}, \quad y_3^{q_1} = y_3^{-1}, \quad z^{q_1} = z, \\ x_1^{q_2} = x_1^{-1}, \quad y_1^{q_2} = y_1^{-1}, \quad x_2^{q_2} = x_2, \quad y_2^{q_2} = y_2, \quad x_3^{q_2} = x_3^{-1}, \quad y_3^{q_2} = y_3^{-1}, \quad z^{q_2} = z, \\ x_1^{q_3} = x_1^{-1}, \quad y_1^{q_3} = y_1^{-1}, \quad x_2^{q_3} = x_2^{-1}, \quad y_2^{q_3} = y_2^{-1}, \quad x_3^{q_3} = x_3, \quad y_3^{q_3} = y_3, \quad z^{q_3} = z,$$

we define a group-action of Q on E . Set $G = E \rtimes Q$.

Now, following the proof of Theorem 1.1 we see that the conjugacy class sizes $|g^G|$, as g runs through $G \setminus \mathbf{Z}(G)$, are $2p, 4p, 2p^4$ and $2p^5$. In particular, $B(G) = K_{2,4}$.

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