REND. SEM. MAT. UNIV. PADOVA, Vol. 133 (2015)
DOI 10.4171/RSMUP/133-8

α-isoptics of a triangle and their connection to α-isoptic of an oval

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ABSTRACT - For a fixed positive angle α, $\alpha < \pi$ we get an explicit formulas for an α-isoptic curve of a triangle and study some of its properties. We use obtained results to show that α-isoptic of an oval is an envelope of α-isoptics of properly chosen triangles.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 53A04; 52A10; 52A38.

KEYWORDS. Isoptic curve, convex curve, support function, envelope, oval.

1. Introduction

For two nontrivial vectors in the complex plane $u = u_1 + iu_2$, $v = v_1 + iv_2$ let $[u, v] = u_1v_2 - u_2v_1$. On the other hand we know that $[u, v] = |u| \cdot |v| \sin \angle(u, v)$. Thus, we have the useful formula

$$\sin \angle(u, v) = \frac{[u, v]}{|u| \cdot |v|}. \tag{1.1}$$

An α-isoptic curve C_α of a plane, closed, convex curve C is a set of those points in the complex plane from which the curve C is seen under a fixed angle $\pi - \alpha$, $\alpha \in (0, \pi)$. If C is strictly convex and the origin of the plane is

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chosen inside C, then there exists (cf. [2]) a differentiable function p such that $p(t)$, $t \in [0, 2\pi]$ is the distance from the origin to the support line. Function p is called a support function and in its terms we have the parametrization of C

$$z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}, \quad t \in [0, 2\pi], \tag{1.2}$$

and the parametrization of its α-isoptic

$$z_\alpha(t) = p(t)e^{it} + \left\{ -p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha) \right\} ie^{it}, \quad t \in [0, 2\pi]. \tag{1.3}$$

Properties of isoptics of strictly convex curves were studied in [4], [5], [9] and in [7] some results for not strictly convex curves can be found. Interesting extension of the notion of isoptic to non-euclidean spaces are given in [6].

In this paper we find an α-isoptic curve of a triangle for a fixed positive angle α, $\alpha < \pi$ and study some of its properties. As an application of our results we show that a family of α-isoptics of properly chosen triangles has envelope which is α-isoptic of an oval.

2. Properties of an α-isoptic curve of a triangle

Let α be fixed and let $z_k = x_k + iy_k$, $k = 1, 2, 3$, denote the vertices of a counter-clockwise oriented triangle T on the complex plane $\mathbb{C}$. If we ever use a subindex k grater then 3, we always mean it modulo 3. For $k = 1, 2, 3$ we use the following notations. Let $\overrightarrow{a_k} = \overrightarrow{z_{k+1}z_{k+2}} = x_{k+2} - x_{k+1} + i(y_{k+2} - y_{k+1})$ be an oriented side of the triangle T, then $a_k = \sqrt{(x_{k+2} - x_{k+1})^2 + (y_{k+2} - y_{k+1})^2}$ denotes its length and β_k is an angle of T corresponding to the vertex z_k and opposite to the side $\overrightarrow{a_k}$. Without loss of generality we can assume throughout this paper that $\beta_1 \geq \beta_2 \geq \beta_3$ or equivalently $a_1 \geq a_2 \geq a_3$.

From the inscribed angle theorem it is known that C_α, the α-isoptic curve of T is a union of at least 3 and at most 6 circular arcs, one arc over each side of T and if α is greater then $\pi - \beta_k$ one has an additional arc over the vertex z_k. We find the equations for each part of C_α.

Over the side $\overrightarrow{a_k}$ a part of C_α is an arc of a circle $C(s_k, r_k)$ centered at the point s_k and with radius r_k. Moreover, if l_k is the line containing the side $\overrightarrow{a_k}$, then the center s_k and the vertex z_k lie in the same half plane of

l_k if and only if α is less then $\pi/2$. To find the equation of the circle $C(s_k, r_k)$ we observe that the point $z = x + iy$ belongs to the part C_α over the side $\overrightarrow{a_k}$ if the angle between the vectors $\overrightarrow{u} = \overrightarrow{zz_{k+1}}$ and $\overrightarrow{v} = \overrightarrow{zz_{k+2}}$ is equal to $\pi - \alpha$. Using formula (1.1) we obtain the equation

$$|u| \cdot |v| \sin(\pi - \alpha) = (x - x_{k+1})(y - y_{k+2}) - (y - y_{k+1})(x - x_{k+2}).$$

Taking square of both sides of the above equation we get after some calculations

$$\left| z - \frac{(z_{k+1} + z_{k+2}) \pm i \cdot \cot\alpha \cdot (z_{k+1} - z_{k+2})}{2} \right|^2 = \frac{a_k^2}{4\sin^2\alpha}. \tag{2.1}$$

Now, the property of the vertex z_k and the center of the circle s_k lying in the same half plane allow us to determine the sign in the above formula and finally we obtain

$$r_k = \frac{a_k}{2\sin\alpha}, \tag{2.2}$$

and

$$s_k = \frac{(z_{k+1} + z_{k+2})}{2} - i\cot\alpha \frac{(z_{k+1} - z_{k+2})}{2}. \tag{2.3}$$

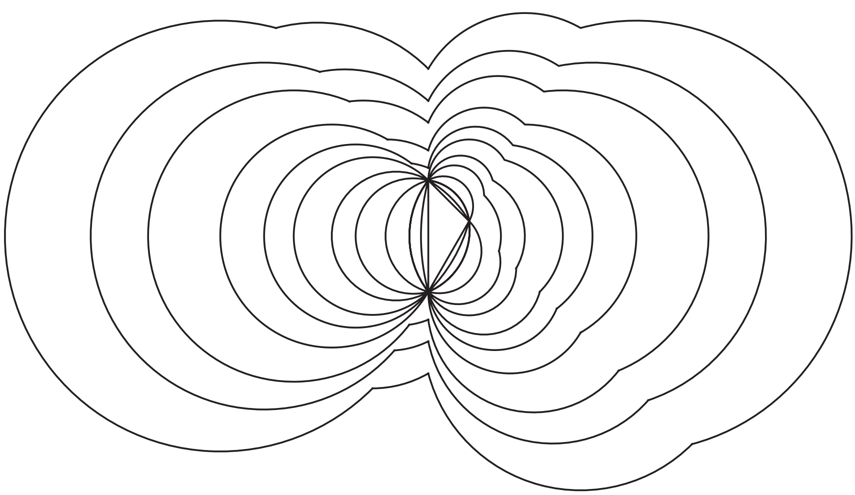

Figure 1. Isoptics of T for $\alpha \in \{\pi/12,\ 5\pi/24,\ 5\pi/12,\ 7\pi/12,\ 2\pi/3,\ 3\pi/4,\ 19\pi/24,\ 5\pi/6,\ 21\pi/24,\ 43\pi/48,\ 11\pi/12\}$.

It is worth to notice that the arc of the circle $C(p_k, r_k)$ which is the part of C_α over the vertex z_k, for α less then $\pi - \beta_k$, is obtained in the same way. The only deference is that the center of the circle p_k and vertex z_k lie in the same half plane of l_k if and only if α is greater then $\pi/2$. Thus for α less then $\pi - \beta_k$ from (2.1) we have the following

$$p_k = \frac{(z_{k+1} + z_{k+2})}{2} + i \cot \alpha \frac{(z_{k+1} - z_{k+2})}{2}. \tag{2.4}$$

Example 2.1. Let T be a triangle with vertices $z_1 = 1$, $z_2 = i$, $z_3 = -i\sqrt{3}$. Then from formulas (2.1), (2.2), (2.3) and (2.4) we get the isoptics of T.

The most interesting question is for which α the α-isoptic of T is a convex curve. To study this problem we need to find the points of intersection of two circular arcs of C_α.

If α is less or equal to $\pi - \beta_k$ then two circular arcs intersect at z_k. Let α be greater then $\pi - \beta_k$. Let ζ_k denote the intersection point of the circles $C(s_{k+1}, r_{k+1})$, $C(p_k, r_k)$ and the line l_{k+2}, and let η_k denote the intersection point of the circles $C(s_{k+2}, r_{k+2})$, $C(p_k, r_k)$ and the line l_{k+1}. We can find the coordinates of ζ_k and η_k by straightforward calculations. To simplify notations we put $\zeta_k = \eta_k = z_k$ for $\alpha \leq \pi - \beta_k$ and together we have

$$\zeta_k = z_k + (z_{k+1} - z_k) \frac{a_{k+1} \cdot \min\{0, \sin(\alpha + \beta_k)\}}{a_{k+2} \sin \alpha}, \tag{2.5}$$

$$\eta_k = z_k + (z_{k+2} - z_k) \frac{a_{k+2} \cdot \min\{0, \sin(\alpha + \beta_k)\}}{a_{k+1} \sin \alpha}. \tag{2.6}$$

Now we state a useful property of C_α which helps us to study its convexity.

Proposition 2.2. *Let $\alpha > \pi - \max\{\beta_1, \beta_2, \beta_3\}$ be fixed. Let T be a given triangle in the complex plane and let C_α be its α-isoptic curve. Then the triangle with the vertices s_{k+1}, p_k, ζ_k and the triangle with the vertices p_k, s_{k+2}, η_k are geometrically congruent to each other and both are similar to T for each $k \in \{1,2,3\}$ for which the arc of the circle $C(p_k, r_k)$ is a part of C_α.*

Proof. Let $k \in \{1,2,3\}$ be arbitrarily chosen. If $\alpha \leq \pi - \beta_k$ then none arc of the circle $C(p_k, r_k)$ is a part C_α. Let $\alpha > \pi - \beta_k$. Then we have $|\overrightarrow{s_{k+1}\zeta_k}| = r_{k+1}$, $|\overrightarrow{s_{k+2}\eta_k}| = r_{k+2}$ and $|\overrightarrow{p_k\zeta_k}| = |\overrightarrow{p_k\eta_k}| = r_k$. We can obtain lengths $|\overrightarrow{s_{k+2}p_k}|$ and $|\overrightarrow{s_{k+1}p_k}|$ using formulas (2.3) and (2.4), and we get the following equality

$$|\overrightarrow{s_{k+2}p_k}| = \left|\frac{(z_{k+2} - z_k)}{2} + i\cot\alpha\frac{(z_{k+2} - z_k)}{2}\right| = r_{k+1},$$

and analogously $|\overrightarrow{s_{k+1}p_k}| = r_{k+2}$. Thus the triangle with the vertices s_{k+1}, p_k, ζ_k and the triangle with the vertices p_k, s_{k+2}, η_k have the required properties. Since k was chosen arbitrarily we get the proof. □

Proposition 2.2 allows us to reduce the domain of α to the interval $(0, \beta_1]$ while studying the convexity of C_α. Namely we have

Corollary 2.3. *Let α be fixed. If $\alpha > \pi - \max\{\beta_1, \beta_2, \beta_3\}$ then the α-isoptic curve of T is not convex.*

Proof. Due to our assumption $\beta_1 = \max\{\beta_1, \beta_2, \beta_3\}$. Let $\alpha > \pi - \beta_1$ be fixed. Then, at least an arc of $C(p_1, r_1)$ is a part of C_α. The vector $\overrightarrow{\zeta_1 p_1}$ is normal to the tangent line to the circle $C(p_1, r_1)$ at the point ζ_1 and the vector $\overrightarrow{\zeta_1 s_2}$ is normal to the tangent line to the circle $C(s_2, r_2)$ at the point ζ_1. By Proposition 2.2 and formula (1.1) we get that the circles $C(p_1, r_1)$ and $C(s_2, r_2)$ intersects at ζ_1 under the angle $\pi - \beta_3$, thus C_α is not convex. □

In fact, the domain of convexity of the curve C_α is a proper subset of $(0, \beta_1]$. We prove the following

Theorem 2.4. *Let T be a given triangle in the complex plane and let C_α be its α-isoptic curve. Then C_α is a convex curve if $\alpha \leq (\pi - \max\{\beta_1, \beta_2, \beta_3\})/2$.*

Proof. By our assumption we have $\beta_1 = \max\{\beta_1, \beta_2, \beta_3\}$. Let $\alpha < \pi - \beta_1$ be fixed. Then C_α consists of 3 circular arcs $C(s_k, r_k)$, $k = 1, 2, 3$. Let γ_k denote the angle under which the circles $C(s_{k+1}, r_{k+1})$ and $C(s_{k+2}, r_{k+2})$ intersect at the point z_k. Similarly as in the proof of Corollary 2.3 the vector $\overrightarrow{z_k s_{k+2}}$ is normal to the tangent line to the circle $C(s_{k+2}, r_{k+2})$ at the point z_k and the vector $\overrightarrow{z_k s_{k+1}}$ is normal to the tangent line to the circle $C(s_{k+1}, r_{k+1})$ at the point z_k. Using formula (1.1) we find

$\angle(\overrightarrow{z_k s_{k+2}}, \overrightarrow{z_k s_{k+1}}) = -2\alpha - \beta_k$ and consequently $\gamma_k = \pi + 2\alpha + \beta_k$. Obviously $\gamma_k \leq 2\pi$ and hence we get the required result. □

Corollary 2.5. *The α-isoptic curve of an equilateral polygon with n sides is convex for $\alpha \leq \pi/n$ and the (π/n)-isoptic is a circle in which the polygon is inscribed.*

Proof. If a polygon is an equilateral triangle then by Theorem 2.4 its α-isoptic curve is convex for $\alpha \leq \pi/3$ and $C_{\pi/3}$ is a circle circumscribed on T.

Now let an equilateral polygon have n sides, $n > 3$. By inscribed angle theorem and Theorem 2.4 its α-isoptic curve is convex for $\alpha \leq \pi/n$. And again the (π/n)-isoptic is a circle. □

3. The length of an α-isoptic curve of a triangle

In this section we study some properties of the length function $L(\alpha)$ of C_α as a function of α. To find the length function we need some additional notations. For $k = 1, 2, 3$ let φ_k denote an angular measure in radians of the arc of the circle $C(s_k, r_k)$ which is a part of C_α and for a sufficiently large α let ψ_k denote an angular measure in radians of the arc of the circle $C(p_k, r_k)$ which is also a part of C_α. Then, by the inscribed angle theorem, we have for $k = 1, 2, 3$

$$\varphi_k = 2(\alpha - \max\{0, \beta_{k+1} + \alpha - \pi\} - \max\{0, \beta_{k+2} + \alpha - \pi\}), \tag{3.1}$$

$$\psi_k = 2\max\{0, \beta_k + \alpha - \pi\}. \tag{3.2}$$

Using the above formulas we obtain the length of C_α as a function of α

$$L(\alpha) = \sum_{k=1}^{3} (\varphi_k + \psi_k) r_k. \tag{3.3}$$

Theorem 3.1. *Let T be a triangle in the complex plane and let C_α be an α-isoptic curve of T for a given angle α. Then $L(\alpha)$ the length function of C_α defined by* (3.3) *is a continuous and strictly increasing function with respect to α. The function L is not convex.*

Proof. Assume that $\beta_1 \geq \beta_2 \geq \beta_3$. Form (3.3) and the sine theorem we have the explicit formula for L

$$L(\alpha) = \begin{cases} \frac{2\alpha R(\sin\beta_1+\sin\beta_2+\sin\beta_3)}{\sin\alpha} & \text{for } 0<\alpha\le\pi-\beta_1,\\ \frac{2R[2\alpha\sin\beta_1+(\pi-\beta_1)(-\sin\beta_1+\sin\beta_2+\sin\beta_3)]}{\sin\alpha} & \text{for } \pi-\beta_1<\alpha\le\pi-\beta_2,\\ \frac{2R[\alpha(\sin\beta_1+\sin\beta_2-\sin\beta_3)+(\beta_1-\beta_2)(\sin\beta_1-\sin\beta_2)+(\pi+\beta_3)\sin\beta_3]}{\sin\alpha} & \\ & \text{for } \pi-\beta_2<\alpha\le\pi-\beta_3,\\ \frac{4R(\beta_1\sin\beta_1+\beta_2\sin\beta_2+\beta_3\sin\beta_3)}{\sin\alpha} & \text{for } \pi-\beta_3<\alpha<\pi, \end{cases}$$

$$= \begin{cases} L_1(\alpha) & \text{for } 0<\alpha\le\pi-\beta_1,\\ L_2(\alpha) & \text{for } \pi-\beta_1<\alpha\le\pi-\beta_2,\\ L_3(\alpha) & \text{for } \pi-\beta_2<\alpha\le\pi-\beta_3,\\ L_4(\alpha) & \text{for } \pi-\beta_3<\alpha<\pi, \end{cases}$$

where R is the radius of the circle circumscribing T. The straightforward calculations show that L is continuous at the points $\pi-\beta_1,\ \pi-\beta_2,\ \pi-\beta_3$ and thus at all $\alpha\in(0,\pi)$. Each function $L_k(\alpha)$, $k=1,2,3,4$ is differentiable in an open subset of its domain and the derivatives are equal to

$$L_1'(\alpha)=\frac{2R}{\sin^2\alpha}(\sin\beta_1+\sin\beta_2+\sin\beta_3)(\sin\alpha-\alpha\cos\alpha),$$

$$L_2'(\alpha)=\frac{2R}{\sin^2\alpha}[2\sin\beta_1(\sin\alpha-\alpha\cos\alpha)-(\pi-\beta_1)(-\sin\beta_1+\sin\beta_2+\sin\beta_3)\cos\alpha],$$

$$L_3'(\alpha)=\frac{2R}{\sin^2\alpha}\{(\sin\beta_1+\sin\beta_2-\sin\beta_3)(\sin\alpha-\alpha\cos\alpha)$$
$$-[(\beta_1-\beta_2)(\sin\beta_1-\sin\beta_2)+(\pi+\beta_3)\sin\beta_3]\cos\alpha\},$$

$$L_4'(\alpha)=\frac{-4R}{\sin^2\alpha}(\beta_1\sin\beta_1+\beta_2\sin\beta_2+\beta_3\sin\beta_3)\cos\alpha.$$

Since they have property

$$(3.4)\quad L_{k+1}'(\pi-\beta_k)-L_k'(\pi-\beta_k)=\frac{a_k-a_{k+1}-a_{k+2}}{\sin\beta_k}<0,\quad \text{for } k=1,2,3,$$

thus the function L' is not defined at the points $\pi-\beta_1,\ \pi-\beta_2,\ \pi-\beta_3$.

We show that L' is a positive function in each interval of its domain and thus L is increasing.

First note that the function $f(\alpha)=\sin\alpha-\alpha\cos\alpha$ is positive and increasing for $\alpha\in(0,\pi)$ and the function $g(\alpha)=-\cos\alpha$ is positive for $\alpha\in(\pi/2,\pi)$. Consequently the functions $L_1',\ L_3',\ L_4'$ are positive in their domains.

The function L_2' is positive for $\alpha > \pi/2$ hence we have to show that it is positive in the interval $[\pi - \beta_1, \pi/2]$. Since the function

$$h(\alpha) = 2\sin\beta_1(\sin\alpha - \alpha\cos\alpha) - (\pi - \beta_1)(-\sin\beta_1 + \sin\beta_2 + \sin\beta_3)\cos\alpha$$

has nonnegative derivative for $\alpha \in [\pi - \beta_1, \pi/2]$ it is enough to show that $h(\pi - \beta_1)$ is positive. Let

$$\begin{aligned} &H(\beta_2, \beta_3) = h(\pi - \beta_1) \\ &= 2\sin^2(\beta_2 + \beta_3) - (\beta_2 + \beta_3)\cos(\beta_2 + \beta_3)[\sin(\beta_2 + \beta_3) + \sin\beta_2 + \sin\beta_3]. \end{aligned}$$

We need to show that the minimal value of H in

$$D = \{(\beta_2, \beta_3)|\ 0 \le \beta_3 \le \beta_2 \le \pi/2,\ 0 \le \beta_2 + \beta_3 \le \pi/2\}$$

is nonnegative. The function H has no critical points inside D. Moreover,

$$H(\beta_2, \pi/2 - \beta_2) = 2, \quad \text{for } \beta_2 \in [\pi/4, \pi/2],$$

$$H(\beta_2, 0) = 2\sin\beta_2(\sin\beta_2 - \beta_2\cos\beta_2) \ge 0, \quad \text{for } \beta_2 \in [0, \pi/2].$$

To complete this part of the proof we show that

$$\widetilde{H}(\beta_2) = H(\beta_2, \beta_2) = 2\sin^2 2 - 2\beta_2\cos 2\beta_2(\sin 2\beta_2 - 2\sin\beta_2)$$

is nonnegative for $\beta_2 \in [0, \pi/4]$. Once again, we use the fact that this function is nondecreasing and $\widetilde{H}(0) = 0$. Indeed, we have

$$\widetilde{H}'(\beta_2) = 8\cos\frac{\beta_2}{2}\left[(3\cos\beta_2 - 1)\sin\frac{\beta_2}{2}\cos 2\beta_2 + \beta_2(\cos 2\beta_2 - 2\cos 3\beta_2)\cos\frac{\beta_2}{2}\right]$$

$$= 8\cos\frac{\beta_2}{2}\left\{\left[\cos\frac{5\beta_2}{2}\sin^2\frac{\beta_2}{2}(\beta_2 - \sin\beta_2)\right] + \left[\sin\frac{5\beta_2}{2}(3\cos\beta_2 - 1)\sin^2\frac{\beta_2}{2}\right]\right.$$

$$\left. + \left[\cos\frac{5\beta_2}{2}(\sin\beta_2\cos\beta_2 - \beta_2) + \frac{3}{2}\beta_2\sin\beta_2\sin\frac{5\beta_2}{2}\right]\right\}.$$

From the first formula we obtain that $\widetilde{H}'(\beta_2)$ is positive for $\beta_2 \in (\pi/6, \pi/4]$. If $\beta_2 \in (0, \pi/6]$ each term in a square bracket in the second formula is positive. Thus, $\widetilde{H}'(\beta_2)$ is positive for $\beta_2 \in (0, \pi/4]$. Finally, we get that $H(\beta_2, \beta_3)$ is nonnegative and is equal to 0 only if $\beta_2 = \beta_3 = 0$. This completes the proof that L_2' is positive in its domain.

Moreover, the functions L_k', $k = 1, 2, 3, 4$ are positive and the property (3.4) implies that the graph of L is not convex. It is worth to notice that each L_k, $k = 1, 2, 3, 4$ is convex in its domain since $(f(\alpha)/\sin^2\alpha)' = \alpha(1 + \cos^2\alpha)/\sin^3\alpha > 0$ and $(-g(\alpha)/\sin^2\alpha)' = (1 + \cos^2\alpha)/\sin^3\alpha > 0$ for $\alpha \in (0, \pi)$. □

EXAMPLE 3.2. For the triangle T from Example 2.1 the length function $L(\alpha)$ given by (3.3) has the graph shown on Figure 2.

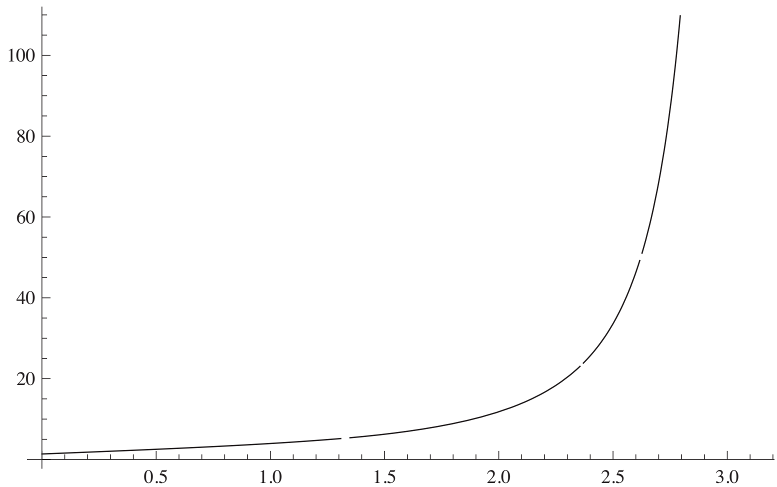

Figure 2. Graph of the length function $L(\alpha)$ of T.

4. The area of an α-isoptic curve of a triangle

Our aim in this section is to investigate some properties of the area function $A(\alpha)$ of the α-isoptic curve of T as a function of α. Let $\zeta_1\eta_1\zeta_2\eta_2, \zeta_3\eta_3$, where ζ_k and η_k are defined by (2.5) and (2.6), respectively, be a counter-clockwise oriented polygon. If $\eta_k = \zeta_k = z_k$ then the point z_k is counted only once in the polygon. Using formulas (3.1) and (3.2) we get the area function of C_α

$$\begin{aligned} A(\alpha) &= \text{area of } \zeta_1\eta_1\zeta_2\eta_2\zeta_3\eta_3 \\ &+ \sum_{k=1}^{3}\left(\varphi_k r_k^2 - \frac{1}{2}r_k^2 \sin\varphi_k\right) + \sum_{k=1}^{3}\left(\psi_k r_k^2 - \frac{1}{2}r_k^2 \sin\psi_k\right). \end{aligned} \tag{4.1}$$

The behavior of the function $A(\alpha)$ is described in the following

THEOREM 4.1. *Let T be a triangle in the complex plane and let C_α be an α-isoptic curve of T for a given angle α. Then $A(\alpha)$, the area function of C_α defined by* (4.1) *is a differentiable, strictly increasing and convex function with respect to α.*

Proof. Assume that $\beta 1 \geq \beta_2 \geq \beta_3$ and let ζ_k and η_k be defined by (2.5) and (2.6), respectively. To compute the area function given by (4.1) we need to find the area of a polygon $\zeta_1\eta_1\zeta_2\eta_2\zeta_3\eta_3$. To this end we use theorem 3 given by Radić in [10]. He proved that

$$\text{area of } \zeta_1\eta_1\zeta_2\eta_2\zeta_3\eta_3 = \frac{1}{2}|\zeta_1+\eta_1, \eta_1+\zeta_2, \zeta_2+\eta_2, \eta_2+\zeta_3, \zeta_3+\eta_3, \eta_3+\zeta_1|,$$

where

$$|z_1+z_2, z_2+z_3, \ldots, z_n+z_1| = \sum_{1\leq i<j\leq n} (-1)^{3+i+j}[z_i, z_j].$$

Obviously, if $\eta_k = \zeta_k = z_k$ then the point z_k is counted only once in the polygon. Applying the sine theorem we get

$$A(\alpha) = \begin{cases} \frac{2R^2[(\alpha-\sin\alpha\cos\alpha)(1+\cos\beta_1\cos\beta_2\cos\beta_3)+\sin^2\alpha\sin\beta_1\sin\beta_2\sin\beta_3]}{\sin^2\alpha} & \text{for } 0<\alpha\leq\pi-\beta_1, \\ \frac{2R^2\{(\alpha-\sin\alpha\cos\alpha)\sin^2\beta_1+\sin\beta_2\sin\beta_3[(\pi-\beta_1)\cos\beta_1+\sin\beta_1]\}}{\sin^2\alpha} & \text{for } \pi-\beta_1<\alpha\leq\pi-\beta_2, \\ \frac{2R^2\{[\alpha\cos\beta_3+\cos\alpha\sin(\beta_3-\alpha)]\sin\beta_1\sin\beta_2+[(\beta_2-\beta_1)\cos\beta_1+\sin\beta_1]\sin\beta_2\sin\beta_3\}}{\sin^2\alpha} \\ \quad +\frac{2R^2(\pi-\beta_2)\sin^2\beta_3}{\sin^2\alpha} & \text{for } \pi-\beta_2<\alpha\leq\pi-\beta_3, \\ \frac{2R^2[(3-2\sin^2\alpha)\sin\beta_1\sin\beta_2\sin\beta_3+\beta_1\sin^2\beta_1+\beta_2\sin^2\beta_2+\beta_3\sin^2\beta_3]}{\sin^2\alpha} & \text{for } \pi-\beta_3<\alpha<\pi, \end{cases}$$

$$= \begin{cases} A_1(\alpha) & \text{for } 0<\alpha\leq\pi-\beta_1, \\ A_2(\alpha) & \text{for } \pi-\beta_1<\alpha\leq\pi-\beta_2, \\ A_3(\alpha) & \text{for } \pi-\beta_2<\alpha\leq\pi-\beta_3, \\ A_4(\alpha) & \text{for } \pi-\beta_3<\alpha<\pi, \end{cases}$$

where R is the radius of the circle circumscribed on T. The function $A(\alpha)$ is continuous at the points $\pi-\beta_1,\ \pi-\beta_2,\ \pi-\beta_3$ and thus at all $\alpha \in (0,\pi)$. The same is true for its derivative and we have

$$A_1'(\alpha) = \frac{4R^2}{\sin^3\alpha}(\sin\alpha-\alpha\cos\alpha)(1+\cos\beta_1\cos\beta_2\cos\beta_3),$$

$$A_2'(\alpha) = \frac{4R^2}{\sin^3\alpha}\{(\sin\alpha-\alpha\cos\alpha)\sin^2\beta_1$$
$$-\ \cos\alpha\sin\beta_2\sin\beta_3[(\pi-\beta_1)\cos\beta_1+\sin\beta_1]\},$$

$$A_3'(\alpha) = \frac{4R^2}{\sin^3\alpha}\{[\sin(\alpha-\beta_3) - \alpha\cos\alpha\cos\beta_3]\sin\beta_1\sin\beta_2$$
$$-\cos\alpha[(\pi-\beta_2)\sin\beta_1\cos\beta_2\sin\beta_3 + [(\pi-\beta_1)\cos\beta_1 + \sin\beta_1]\sin\beta_2\sin\beta_3]\},$$

$$A_4'(\alpha) = \frac{-4R^2}{\sin^3\alpha}\cos\alpha[3\sin\beta_1\sin\beta_2\sin\beta_3 + \beta_1\sin^2\beta_1 + \beta_2\sin^2\beta_2 + \beta_3\sin^2\beta_3].$$

where the derivative of the function $A_k(\alpha)$, $k = 1, 2, 3, 4$, is defined in its domain. Using functions f and g defined in the proof of Theorem 3.1 we immediately obtain that A_1', A_3', A_4' are positive in their domains. Moreover, since $f(\alpha) \geq f(\pi - \beta_1)$ we have

$$A_2'(\alpha) \geq \frac{2f(\alpha)(\sin^2\beta_1 - \cos\alpha\sin\beta_2\sin\beta_3)}{\sin^3\alpha} \geq \frac{2f(\alpha)(\sin^2\beta_1 - \sin\beta_2\sin\beta_3)}{\sin^3\alpha} > 0.$$

Since $A'(\alpha)$ is positive then the function $A(\alpha)$ is strictly increasing.

The second derivative of the function $A_k(\alpha)$, $k = 1, 2, 3, 4$, is defined in each open subset of its domain and it is equal to

$$A_1''(\alpha) = \frac{4R^2}{\sin^4\alpha}[\alpha(1+2\cos^2\alpha) - 3\sin\alpha\cos\alpha](1+\cos\beta_1\cos\beta_2\cos\beta_3),$$

$$A_2''(\alpha) = \frac{4R^2}{\sin^4\alpha}\{[\alpha(1+2\cos^2\alpha) - 3\sin\alpha\cos\alpha]\sin^2\beta_1$$

$$+(1+2\cos^2\alpha)[(\pi-\beta_1)\cos\beta_1 + \sin\beta_1]\sin\beta_2\sin\beta_3\},$$

$$A_3''(\alpha) = \frac{4R^2}{\sin^4\alpha}\{[\alpha(1+2\cos^2\alpha) - 3\sin\alpha\cos\alpha]\sin\beta_1\sin\beta_2\cos\beta_3$$

$$+(1+2\cos^2\alpha)[(\pi-\beta_2)\sin\beta_1\cos\beta_2\sin\beta_3$$

$$+((\pi-\beta_1)\cos\beta_1 + \sin\beta_1)\sin\beta_2\sin\beta_3 + \sin\beta_1\sin\beta_2\sin\beta_3]\},$$

$$A_4''(\alpha) = \frac{4R^2}{\sin^4\alpha}(1+2\cos^2\alpha)[3\sin\beta_1\sin\beta_2\sin\beta_3 + \beta_1\sin^2\beta_1$$

$$+\beta_2\sin^2\beta_2 + \beta_3\sin^2\beta_3].$$

Thus the function $A''(\alpha)$ is not continuous at the points $\pi - \beta_1$, $\pi - \beta_2$, $\pi - \beta_3$, but still, it is positive in its domain. This completes the proof.

EXAMPLE 4.2. Once again, let us take the triangle T from Example 2.1. Then the graph of the area function $A(\alpha)$ given by (4.1) is shown on Figure 3.

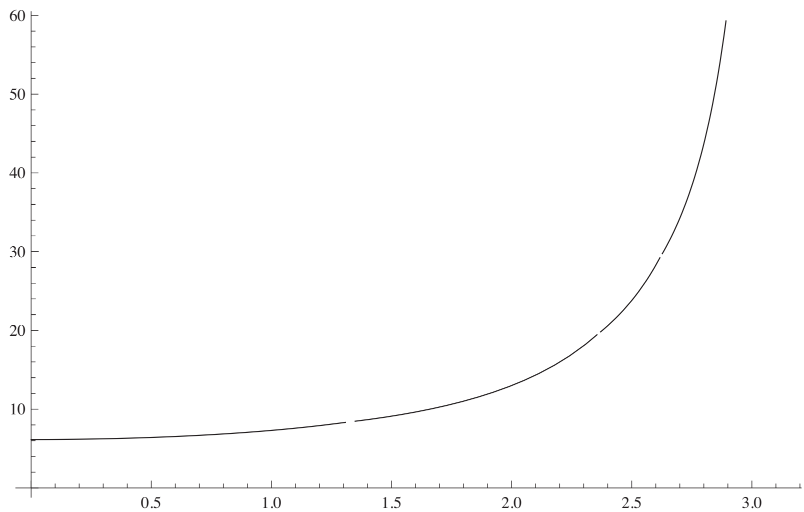

Figure 3. Graph of the area function $A(\alpha)$ of T.

5. Application of α-isoptic curves of a triangle

Let $\alpha \in (0, \pi)$ be fixed. In this section we use results obtained in Section 2 to study the α-isoptic C_α of an oval C. By an oval we understand C^2, plane closed simple curve with positive curvature.

Let $p(t) \in C^2([0, 2\pi])$, $t \in [0, 2\pi]$ be the support function of C. Then the point $z_\alpha(t)$ satisfying (1.3) belongs to C_α and it is an intersection of two lines tangent to C at points $z(t)$ and $z(t + \alpha)$. Let ξ be an arbitrary point from an open angle $\angle(\overrightarrow{z_\alpha(t)z(t+\alpha)}, \overrightarrow{z_\alpha(t)z(t)})$ and let T be a counter-clockwise oriented triangle with vertices $z(t + \alpha), z(t), \xi$. By $C_{\alpha,t}$ we denote an arc of a circle

$$(5.1) \qquad \left| z - \frac{(z(t+\alpha) + z(t)) + i \cot \alpha (z(t+\alpha) - z(t))}{2} \right| = \frac{|z(t+\alpha) - z(t)|}{2 \sin \alpha}$$

which is also a part of α-isoptic of T. We should mention that $C_{\alpha,t}$ does not depend on ξ. If $[\overrightarrow{\xi z(t+\alpha)}, \overrightarrow{\xi z(t)}] > 0$ then the center of the circle in (5.1) is obtained from equation (2.4), overwise it is obtained from equation (2.3). Finally we define the family of arcs as follows

$$\mathcal{F}_\alpha = \{C_{\alpha,t}, t \in [0, 2\pi]\}. \tag{5.2}$$

Using the above notations we have

THEOREM 5.1. *C_α is the envelope of the family $\mathcal{F}_\alpha$ defined by* (5.2).

PROOF. Let $F(x, y, t) = 0$ denote an equation for the family $\mathcal{F}_\alpha$ given by (5.2). Then, applying formula (1.2) to (5.1) with $z = x + iy$, we get

$$\begin{aligned} F(x, y, t) &= (x^2 + y^2)\sin\alpha \\ &+ x[p(t+\alpha)\sin t + \dot{p}(t+\alpha)\cos t - p(t)\sin(t+\alpha) - \dot{p}(t)\cos(t+\alpha)] \\ &+ y[-p(t+\alpha)\cos t + \dot{p}(t+\alpha)\sin t + p(t)\cos(t+\alpha) - \dot{p}(t)\sin(t+\alpha)] \\ &+ p(t+\alpha)\dot{p}(t) - p(t)\dot{p}(t+\alpha) = 0, \end{aligned} \tag{5.3}$$

thus $\mathcal{F}_\alpha$ is indeed a one parameter family of arcs. Moreover, the point

$$\begin{aligned} z_\alpha(t) &= x_\alpha(t) + iy_\alpha(t) \\ &= \frac{p(t)\sin(t+\alpha) - p(t+\alpha)\sin t + i(-p(t)\cos(t+\alpha) + p(t+\alpha)\cos t)}{\sin\alpha} \end{aligned}$$

defined by (1.3) satisfies the equation (5.3) and thus $z_\alpha(t) \in C_{\alpha,t}$ since ξ is an interior point of the angle $\angle(\overrightarrow{z_\alpha(t)z(t+\alpha)}, \overrightarrow{z_\alpha(t)z(t)})$.

Now Theorem 4 in [1] asserts that $C_{\alpha,t}$ and C_α are tangent at the point $z_\alpha(t)$. To show that C_α is the envelope of $\mathcal{F}_\alpha$ it is enough to check that $F'_t(x, y, t) = 0$ at $z_\alpha(t)$ (see, e.g., [3] or [11]). Indeed, we have

$$\begin{aligned} F'_t(x, y, t) &= x[R(t+\alpha)\cos t - R(t)\cos(t+\alpha)] \\ &+ y[R(t+\alpha)\sin t + R(t)\sin(t+\alpha)] + p(t+\alpha)R(t) - p(t)R(t+\alpha), \end{aligned}$$

where $R(t) = p(t) + \ddot{p}(t)$ is a radius of curvature of C, and finally,

$$F'_t(x_\alpha(t), y_\alpha(t), t) = 0,$$

which completes the proof. □

Theorem 5.1 remains true in special case when T is inscribed in oval C.

The above considerations can be related to those in paper of Martini [8] on the classical light field theory in $\mathbb{R}^d$, $d \geq 2$.

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Manoscritto pervenuto in redazione il 4 novembre 2013.