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Almost-periodic solution of linearized Hasegawa–Wakatani equations with vanishing resistivity

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Abstract - In this paper we consider the zero-resistivity limit for linearized Hasegawa–Wakatani equations in a cylindrical domain when the initial data are Stepanov-almost-periodic to the axial direction. We prove two results: one is the existence and uniqueness of a strong Stepanov-almost-periodic solution to the initial boundary value problem for linearized Hasegawa–Wakatani equations with zero resistivity; another is the convergence of the solution of linearized Hasegawa–Wakatani equations established in [24] to the solution of the problem studied at the first stage as the resistivity tends to zero. In the proof we obtain two useful lemmas for Stepanov-almost-periodic functions.

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1. Introduction

There are many kinds of instabilities in plasma phenomena, and drift wave instability is one of those. Drift wave instability is classed as micro-instabilities (which is a technical term of plasma physics, [36]). It has been well known that the spatial gradients in plasma lead to the drift waves and the drift wave turbulence is a natural cause of anomalous transport from which the dramatic reduction in confinement results in tokamak. Here tokamak is the most advanced magnetic confinement device, in which an

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axisymmetric plasma is confined by a strong magnetic field. Thereby the analysis of such drift wave turbulences is important.

In order to describe the resistive drift wave turbulence in tokamak, Hasegawa and Wakatani ([19]) proposed in 1983 the following equations for the perturbations of plasma density n and the electrostatic potential ϕ:

$$(1.1)\qquad \begin{cases} \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right) \Delta\phi = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) + c_2\Delta^2\phi, \\ \left(\dfrac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right)(n + \log n^*) = -\dfrac{c_1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n) \end{cases}$$

(Hasegawa–Wakatani equations) from the two fluids model in a homogeneous strong magnetic field $\mathbf{B} = B_0\vec{e}$ and an inhomogeneous plasma equilibrium density $n^* = n^*(|x'|)$ $(x = (x_1, x_2, x_3) = (x', x_3))$ (see, [16], [20], [29]). Here the total density N is divided into equilibrium and fluctuating parts, $N = n^* + n^1$, and the normalization $e\phi/T_e \equiv \phi$, $n^1/n^* \equiv n$, $\omega_{ci}t \equiv t$ and $x/\rho_s \equiv x$ are used. Here B_0 is the strength of a magnetic field assumed to be a constant, $\vec{e} = (0, 0, 1)$, $c_1 = T_e/(e^2\eta\omega_{ci})$, $c_2 = \mu/(\rho_s^2\omega_{ci})$, T_e is the electron temperature, e is the elementary charge, μ is the kinematic ion-viscosity coefficient, η is the resistivity, m_i is the ion mass, $\omega_{ci} = eB_0/m_i$ is the cyclotoron frequency and $\rho_s = \sqrt{T_e}/(\omega_{ci}\sqrt{m_i})$ is the ion Larmor radius. For simplicity we assume that c_1 and c_2 are positive constants.

Concerning the mathematical issue for (1.1) we have a few results. In [21] we established the existence and uniqueness of a strong solution on some time interval to the initial boundary value problems for (1.1) in a cylindrical domain when the initial data are periodic to the axial direction. In [22], [23] we proved that the solution of Hasegawa–Wakatani equations established in [21] converges strongly to that of the model equations of drift wave turbulence with zero resistivity as the resistivity tends to zero. In [24] we established the existence and uniqueness of a strong Stepanov-almost-periodic solution on some time interval to the initial boundary value problems for (1.1) in a cylindrical domain when the initial data are Stepanov-almost-periodic to the axial direction.

In advance of Hasegawa–Wakatani equations Hasegawa and Mima in 1977 ([17], [18]) proposed the equation

$$(1.2)\qquad \left(\frac{\partial}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla\right)(\Delta\phi - \phi - \log n^*) = 0$$

(Hasegawa–Mima equation) from the one fluid model under the same magnetic field and plasma equilibrium state as Hasegawa–Wakatani equations. Concerning the mathematical results for (1.2) we refer to [21], [22] and references therein.

By differencing the first and the second equations of (1.1) and linearizing it around $(\phi, n) = (0, 0)$ and by denoting $\varepsilon = 1/c_1$, we have

$$
\begin{cases} \dfrac{\partial}{\partial t}(\Delta\phi - n) + (\nabla\phi \times \vec{e}) \cdot \nabla \log n^* = c_2 \Delta^2 \phi, \\ \varepsilon\left(\dfrac{\partial n}{\partial t} - (\nabla\phi \times \vec{e}) \cdot \nabla \log n^*\right) = -\dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}(\phi - n). \end{cases} \tag{1.3}
$$

Notice that $\partial^2 n/\partial x_1^2$ and $\partial^2 n/\partial x_2^2$ don't appear in the right side of $(1.3)_2$. In this paper it will be found that when studying the zero-resistivity limit for (1.3) with almost-periodic initial data, this anisotropy causes the unexpected difficulties. Generally when looking for almost-periodic solutions, one looks for almost-periodicity in the time variable. However in this paper we consider another problem as follows:

For given an initial electrostatic potential ϕ_0^ε, an initial plasma density n_0^ε and the background density $n^* = n^*(|x'|)$, let $(\phi^\varepsilon, n^\varepsilon) = (\phi^\varepsilon, n^\varepsilon)(x, t)$ be a solution of the initial boundary value problem for (1.3) with $\varepsilon > 0$ in $\omega \times \mathbf{R} \times (0, \infty) \equiv \Omega \times (0, \infty)$ under the initial and the boundary conditions

$$
\begin{cases} \phi^\varepsilon(x, 0) = \phi_0^\varepsilon(x), \quad n^\varepsilon(x, 0) = n_0^\varepsilon(x) \quad \text{for } x \in \Omega, \\ \phi^\varepsilon(x, t) = \Delta\phi^\varepsilon(x, t) = n^\varepsilon(x, t) = 0 \quad \text{for } x \in \Gamma,\ t > 0, \end{cases} \tag{1.4}
$$

when the initial data are Stepanov-almost-periodic in the direction $\vec{e}$. Here $\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \,\big|\, |x'| < R\}$, $\partial\omega = \{x' = (x_1, x_2) \in \mathbf{R}^2 \,\big|\, |x'| = R\}$, $\Gamma = \{x \in \mathbf{R}^3 \,\big|\, x' \in \partial\omega\}$, and R is a positive real number.

For convenience, we introduce

$$
\bar{f}(x') = \mathcal{M}\{f(x)\} \equiv \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} f(x)\, dx_3,
$$
$$
\tilde{f}(x) = f(x) - \mathcal{M}f(x) \equiv (\mathcal{I} - \mathcal{M})f(x).
$$

Then it is easily seen that problem (1.3), (1.4) is equivalent to the problem

$$
\begin{cases} \dfrac{\partial}{\partial t}(\Delta\phi^\varepsilon - n^\varepsilon) + (\nabla\phi^\varepsilon \times \vec{e}) \cdot \nabla \log n^* = c_2 \Delta^2 \phi^\varepsilon, \\ \varepsilon\left(\dfrac{\partial \widetilde{n^\varepsilon}}{\partial t} - (\nabla\widetilde{\phi^\varepsilon} \times \vec{e}) \cdot \nabla \log n^*\right) = -\dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}\left(\widetilde{\phi^\varepsilon} - \widetilde{n^\varepsilon}\right), \\ \dfrac{\partial \overline{n^\varepsilon}}{\partial t} - (\nabla\overline{\phi^\varepsilon} \times \vec{e}) \cdot \nabla \log n^* = 0 \quad \text{for } x \in \Omega,\ t > 0, \end{cases}
$$

and (1.4).

Putting $\varepsilon = 0$ in this problem, we have

$$(1.5)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left(\Delta\phi^0 - n^0\right) + \left(\nabla\phi^0 \times \vec{e}\right)\cdot\nabla\log n^* = c_2\Delta^2\phi^0, \\ \dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}\left(\widetilde{\phi^0} - \widetilde{n^0}\right) = 0, \\ \dfrac{\partial\overline{n^0}}{\partial t} - (\nabla\overline{\phi^0}\times\vec{e})\cdot\nabla\log n^* = 0 \quad \text{for } x\in\Omega,\ t>0, \end{cases}$$

and (1.4) with $\varepsilon = 0$.

The aim of this paper is to establish the unique existence of a strong Stepanov-almost-periodic solution to the problem (1.5), (1.4) with $\varepsilon = 0$ when the initial data are Stepanov-almost-periodic to the magnetic field direction in the same way as in [24], and the convergence of $(\phi^\varepsilon, n^\varepsilon)$ to $\left(\phi^0, n^0\right)$ as ε tends to zero, which corresponds to the vanishing resistivity of linearized Hasegawa–Wakatani equations. The basic scheme of the proof in [24] essentially consists of the following steps: i) getting the approximate solutions in the form of the Bochner–Fejér sum; ii) proving that the Bochner–Fejér sum forms a sequence bounded and equi-almost-periodic; iii) overcoming the difficulty caused by the inapplicability of the Riesz–Fischer theorem with the help of [11], [14].

Concerning Stepanov-almost-periodic solutions of Navier–Stokes equations, we have had some results. When the external force fields are sufficiently small and Stepanov-almost-periodic in time variable, the existence and uniqueness of such solutions of the initial boundary value problem for incompressible Navier–Stokes equations were proved by Foias ([15]) in 1962 in three-dimensional case and by Prouse ([31]) in 1963 in two-dimensional case. For compressible Navier–Stokes equations, similar result was obtained by Marcati and Valli ([28]) in 1985 in three-dimensional case. The basic scheme of the proof essentially consists of the following steps ([32]): i) global existence on $[0, +\infty)$ with zero initial data; ii) global existence on $(-\infty, +\infty)$; iii) Stepanov-almost-periodicity by contradiction.

In this paper, we consider the following problems:

- Let f, g be almost periodic functions, and $s,\ \delta \in \mathbf{R},\ \delta > 1$ and $\eta_s(x) \in C^1(\mathbf{R})$ be a cut-off function such that $\eta_s \equiv 1$ on $[s, s+\delta]$, $\eta_s \equiv 0$ on $(-\infty, s-\delta] \cup [s+2\delta, +\infty)$, $0 \le \eta_s(x) \le 1$ and $\eta_s'(x+2\delta) = -\eta_s'(x)$ for $x \in [s-\delta, s]$. Then we have

$$\int_{s-\delta}^{s+2\delta} f'(x)g(x)\eta_s(x)\,\mathrm{d}x = -\int_{s-\delta}^{s+2\delta} \left\{f(x)g'(x)\eta_s(x) + f(x)g(x)\eta_s'(x)\right\}\mathrm{d}x.$$

When f, g are periodic functions with period L, the last term of above equation is zero for $2\delta = Ln$ ($n \in \mathbf{N}$), since

$$\int_{s-\delta}^{s+2\delta} f(x)g(x)\eta_s'(x)\,\mathrm{d}x = \int_{s-\delta}^{s} \{f(x)g(x) - f(x+2\delta)g(x+2\delta)\}\eta_s'(x)\,\mathrm{d}x.$$

In other cases it is not zero for any $\delta > 0$. In Lemma 2.1 we obtain some estimates for it.

- It is well known that if f is a periodic function with period L, and $\int_0^L f(x)\,\mathrm{d}x = 0$, then the Poincaré estimate holds: $\|f\| \le c\|\partial_x f\|$. In Lemma 2.2 we obtain the Poincaré estimate when f is an almost-periodic function and $\overline{f} = 0$.

Before describing the main theorem we introduce the function spaces and the almost periodic functions that we use in the sequel ([1], [2], [5], [7], [8], [9], [13]).

Let Ω be a domain in $\mathbf{R}^m$ ($m = 1, 2, 3, \dots$). By $W_2^l(\Omega)$ ($l \in \mathbf{R},\ l \ge 0$) we denote the space of functions $u(x),\ x \in \Omega$, equipped with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{|\alpha|<l} \left\|\mathrm{D}_x^\alpha u\right\|^2_{L^2(\Omega)} + \|u\|^2_{\dot W_2^l(\Omega)},$$

where

$$\|u\|^2_{\dot W_2^l(\Omega)} = \begin{cases} \sum\limits_{|\alpha|=l} \left\|\mathrm{D}_x^\alpha u\right\|^2_{L^2(\Omega)} & \text{for } l \in \mathbf{Z}, \\ \sum\limits_{|\alpha|=[l]} \int\limits_\Omega \int\limits_\Omega \dfrac{\left|\mathrm{D}_x^\alpha u(x) - \mathrm{D}_y^\alpha u(y)\right|^2}{|x-y|^{m+2(l-[l])}}\,\mathrm{d}x\,\mathrm{d}y & \text{for } l \notin \mathbf{Z} \end{cases}.$$

Here $[l]$ is the integral part of l, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a multi-index, and $\mathrm{D}_x^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_m^{\alpha_m}$ is the generalized derivative of order $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_m$. For $1 \le p \le \infty$, we denote by $\|\cdot\|_{L^p(\Omega)}$ the norm of the Lebesgue space $L^p(\Omega)$.

The anisotropic Sobolev–Slobodetskiĭ space $W_2^{l,l/2}(Q_T)$ ($Q_T \equiv \Omega \times (0,T)$) is defined as $L^2(0,T; W_2^l(\Omega)) \cap L^2(\Omega; W_2^{l/2}(0,T))$, equipped with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \|u\|^2_{W_2^{l,0}(Q_T)} + \|u\|^2_{W_2^{0,l/2}(Q_T)}$$
$$\equiv \int_0^T \|u(t)\|^2_{W_2^l(\Omega)}\,\mathrm{d}t + \int_\Omega \|u(x)\|^2_{W_2^{l/2}(0,T)}\,\mathrm{d}x.$$

Let X be a Banach space with the norm $\|\cdot\|_X$. By $C(X)$, we denote the space of all continuous functions on $\mathbf{R}$ with values in X. The function $f(x) \in C(X)$ is called almost-periodic (a.p.) if for any $\varepsilon > 0$ the set

$$E_\varepsilon(f) \equiv \left\{ \sigma \in \mathbf{R} \,\Big|\, \sup_{s\in\mathbf{R}} \|f(s+\sigma) - f(s)\|_X \leq \varepsilon \right\}$$

is relatively dense in $\mathbf{R}$, that is, there exists $L = L(\varepsilon) > 0$ (inclusion length) such that $E_\varepsilon(f) \cap (a, a+L) \neq \emptyset$ for any $a \in \mathbf{R}$. By $AP(X)$, we denote the space of all a.p. functions from $\mathbf{R}$ to X.

By $S^p(X)$ $(1 \leq p < \infty)$, we denote the subspace of $L^p_{loc}(\mathbf{R}; X)$ equipped with the finite norm

$$\|u\|^p_{S^p(X)} \equiv \sup_{s\in\mathbf{R}} \int_s^{s+1} \|u(x)\|^p_X \, \mathrm{d}x.$$

The function $f(x) \in L^p_{loc}(\mathbf{R}; X)$ is called Stepanov-almost-periodic (S^p-a.p.) ([34], [35], [37]) if for any $\varepsilon > 0$ the set

$$E_\varepsilon(f) \equiv \left\{ \sigma \in \mathbf{R} \,\Big|\, \sup_{s\in\mathbf{R}} \left(\int_s^{s+1} \|f(x+\sigma) - f(x)\|^p_X \, \mathrm{d}x \right)^{1/p} \leq \varepsilon \right\}$$

is relatively dense in $\mathbf{R}$, that is, there exists $L = L(\varepsilon) > 0$ (inclusion length) such that $E_\varepsilon(f) \cap (a, a+L) \neq \emptyset$ for any $a \in \mathbf{R}$. By $S^p_{ap}(X)$, we denote the space of all S^p-a.p. functions from $\mathbf{R}$ to X.

Let $\omega_T \equiv \omega \times (0, T)$ and $l \in \mathbf{Z}$, $l \geq 0$. We introduce the following spaces:

$$\widetilde{S}^l(X) = \left\{ u \in S^2(X) \,\Big|\, \|u\|^2_{\widetilde{S}^l(X)} \equiv \sum_{|\alpha|=0}^{l} \left\| \mathrm{D}^\alpha_x u \right\|^2_{S^2(X)} < \infty \right\},$$

$$\widetilde{S}^l_{ap}(X) = \left\{ u \in \widetilde{S}^l(X) \,\Big|\, \mathrm{D}^\alpha_x u \in S^2_{ap}(X),\ |\alpha| = 0, 1, \ldots, l \right\},$$

$$\widetilde{S}^{l,l/2}(\omega_T) = \widetilde{S}^l(L^2(\omega_T)) \cap \widetilde{S}^0(L^2(\omega; W_2^{l/2}(0,T))),$$

$$\widetilde{S}^{l,l/2}_{ap}(\omega_T) = \widetilde{S}^l_{ap}(L^2(\omega_T)) \cap \widetilde{S}^0_{ap}(L^2(\omega; W_2^{l/2}(0,T))).$$

Moreover we define the norm

$$\|u\|_\delta^2 \equiv \sup_{s\in\mathbf{R}} \frac{1}{\delta}\int_s^{s+\delta} \|u(x)\|_{L^2(\omega)}^2 \, \mathrm{d}x,$$

$$|||u|||_{t,\delta}^2 \equiv \sup_{s\in\mathbf{R}} \frac{1}{\delta}\int_s^{s+\delta} \|u(x)\|_{L^2(\omega\times(0,t))}^2 \, \mathrm{d}x,$$

$$\|u\|_{\widetilde{S}_\delta^l}^2 \equiv \sum_{|\alpha|=0}^{l} \|\mathrm{D}_x^\alpha u\|_\delta^2.$$

For simplicity, $\nabla' \equiv (\partial_1, \partial_2)$, $\varDelta' \equiv \partial_1^2 + \partial_2^2$, $\partial_k = \partial/\partial x_k$.

First we prove the following theorem on the global in time existence for problem (1.3), (1.4).

THEOREM 1.1. *Let ε and T be any positive numbers, $n^*(|x'|) \in W_2^2(\omega)$ and $n^*(|x'|) \geq n_*$ with n_* being a positive constant. Assume that $(\phi_0^\varepsilon, n_0^\varepsilon) \in \widetilde{S}_{ap}^4(\omega) \times \widetilde{S}_{ap}^2(\omega)$ satisfies the compatibility conditions*

$$\phi_0^\varepsilon(x) = \varDelta\phi_0^\varepsilon(x) = n_0^\varepsilon(x) = 0 \quad \text{for } x \in \varGamma. \tag{1.6}$$

Then there exists a unique solution $(\phi^\varepsilon, n^\varepsilon)$ to problem (1.3), (1.4) *on* $[0, T]$ *such that $(\phi^\varepsilon, n^\varepsilon) \in L^2(0, T; \widetilde{S}_{ap}^4(\omega)) \times \widetilde{S}_{ap}^{2,1}(\omega_T)$, $\partial\phi^\varepsilon/\partial t \in L^2(0, T; \widetilde{S}_{ap}^2(\omega))$. Here T is a constant independent of ε.*

Next, it is clear that the second equation of (1.5) implies $\widetilde{\phi^0} - \widetilde{n^0} = 0$ by virtue of the almost-periodicity condition in x_3 and $\mathcal{M}\widetilde{\phi^0} = \mathcal{M}\widetilde{n^0} = 0$. Inserting this into (1.5), (1.4) with $\varepsilon = 0$, we have

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\varDelta\phi^0 - \phi^0 + \overline{\phi^0}\right) + \left(\nabla\widetilde{\phi^0} \times \vec{e}\right) \cdot \nabla \log n^* = c_2\varDelta^2\phi^0, \\ \dfrac{\partial\overline{n^0}}{\partial t} - (\nabla\overline{\phi^0} \times \vec{e}) \cdot \nabla \log n^* = 0 \quad \text{for } x \in \varOmega,\ t > 0, \\ \phi^0(x, 0) = \phi_0^0(x) \quad \text{for } x \in \varOmega, \\ \overline{n^0}(x', 0) = \overline{n_0^0}(x') \quad \text{for } x' \in \omega, \\ \phi^0(x, t) = \varDelta\phi^0(x, t) = 0 \quad \text{for } x \in \varGamma,\ t > 0, \\ \overline{n^0}(x', t) = 0 \quad \text{for } x' \in \partial\omega,\ t > 0. \end{cases} \tag{1.7}$$

It is to be noted that under an additional conditions $\overline{n_0^0}(x') = 0$ and $\log n^*(x') = const.$, the equations of (1.7) is similar to the linearized equations of the Hasegawa–Mima equation (1.2) with an higher order correction term.

Second we prove the following theorem on the global in time existence for problem (1.7).

THEOREM 1.2. *Let T be any positive number. Assume that* $(\phi_0^0, \overline{n_0^0}) \in \widetilde{S}^4_{ap}(\omega) \times W_2^3(\omega)$ *satisfies the compatibility conditions* (1.6) *with* $\varepsilon = 0$. *Then there exists a unique solution* $(\phi^0, \overline{n^0})$ *to the problem* (1.7) *on* $[0, T]$ *such that* $(\phi^0, \overline{n^0}) \in L^2(0, T; \widetilde{S}^4_{ap}(\omega)) \times L^\infty(0, T; W_2^3(\omega))$, $\partial\phi^0/\partial t \in L^2(0, T; \widetilde{S}^2_{ap}(\omega))$, $\partial\overline{n^0}/\partial t \in L^2(0, T; W_2^2(\omega))$.

For this solution ϕ^0 let $n^0(x,t) = \phi^0(x,t) - \overline{\phi^0}(x', t)$ and $n_0^0(x) = \phi_0^0(x) - \overline{\phi^0}(x', 0)$. Then it is easily seen that (ϕ^0, n^0) satisfy (1.5) and (1.4) with $\varepsilon = 0$.

Finally we prove the following main result.

THEOREM 1.3. *Let T be any positive number,* $(\phi^\varepsilon, n^\varepsilon)$ *and* (ϕ^0, n^0) *be the solutions established in Theorems* 1.1 *and* 1.2, *respectively. If the initial data* $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ *as* $\varepsilon \to 0$ *in* $\widetilde{S}^3(\omega) \times \widetilde{S}^2(\omega)$, *then as* $\varepsilon \to 0$, $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ *in* $L^2\big(0, T; \widetilde{S}^4(\omega)\big) \times \widetilde{S}^{2,0}(\omega_T)$, $\Delta\phi^\varepsilon - n^\varepsilon \to \Delta\phi^0 - n^0$ *in* $\widetilde{S}^{0,1}(\omega_T)$ *and* $\overline{n^\varepsilon} \to \overline{n^0}$ *in* $\widetilde{S}^{0,1}(\omega_T)$ $(\omega_T \equiv \omega \times (0, T))$ *on* $[0, T]$.

This paper is organized as follows. In § 2 we prove Theorem 1.1 from our result in [24] and the *a priori* estimates for problem (1.3), (1.4). In § 3 Theorem 1.2 is proved through the local in time existence and *a priori* estimates in the same way as in [24]. In § 4 we give a proof of Theorem 1.3 by virtue of *a priori* estimates, Theorems 1.1 and 1.2.

2. Proof of Theorem 1.1

For the initial boundary value problem (1.1) for $x \in \Omega$, $t > 0$ and (1.4), we have the following theorem on the local in time existence in [24]:

THEOREM 2.1. *Let* $n^*(|x'|) \in W_2^2(\omega)$ *satisfy* $n^*(|x'|) \geq n_*$ *with a positive constant* n_*. *Assume that* $(\phi_0, n_0) \in \widetilde{S}^4_{ap}(\omega) \times \widetilde{S}^2_{ap}(\omega)$ *satisfies the compatibility conditions* (1.6). *Then there exists a unique solution* (ϕ, n) *to problem* (1.1) *for* $x \in \Omega$, $t > 0$ *and* (1.4) *on some interval* $[0, T^*]$ *such that* $(\phi, n) \in L^2(0, T^*; \widetilde{S}^4_{ap}(\omega)) \times \widetilde{S}^{2,1}_{ap}(\omega_{T^*})$, $\partial\phi/\partial t \in L^2(0, T^*; \widetilde{S}^2_{ap}(\omega))$.

Since (1.3) are linear equations, we can easily prove the following theorem on the global in time existence:

PROPOSITION 2.1. *Let* $n^*(|x'|) \in W_2^2(\omega)$ *satisfy* $n^*(|x'|) \geq n_*$ *with a positive constant* n_*. *Assume that* $(\phi_0, n_0) \in \widetilde{S}_{ap}^4(\omega) \times \widetilde{S}_{ap}^2(\omega)$ *satisfies the compatibility conditions* (1.6). *Then for any positive number* T^{**} *there exists a unique solution* (ϕ, n) *to problem* (1.3), (1.4) *on* $[0, T^{**}]$ *such that* $(\phi, n) \in L^2(0, T^{**}; \widetilde{S}_{ap}^4(\omega)) \times \widetilde{S}_{ap}^{2,1}(\omega_{T^{**}})$, $\partial\phi/\partial t \in L^2(0, T^{**}; \widetilde{S}_{ap}^2(\omega))$.

We denote the solution (ϕ, n) established in Proposition 2.1. in the case of $c_1 = 1/\varepsilon$ by $(\phi^\varepsilon, n^\varepsilon)$. Since T^{**} in Proposition 2.1. may depend on ε, to complete the proof of Theorem 1.1, it is sufficient to show that T^{**} can be taken independently of ε.

We proceed to get *a priori* estimates of the solution $(\phi^\varepsilon, n^\varepsilon)$. Let it belong to $\big(L^2(0, T; \widetilde{S}_{ap}^4(\omega)) \cap W_2^1(0, T; \widetilde{S}_{ap}^2(\omega))\big) \times \widetilde{S}_{ap}^{2,1}(\omega_T)$ for $T > 0$. In order to get *a priori* estimates, in this section we use the following cut-off function η_s:

- Let s, $\delta \in \mathbf{R}$, $\delta > 1$ and $\eta_s(x_3) \in C^1(\mathbf{R})$ be a cut-off function such that $\eta_s \equiv 1$ on $[s, s+\delta]$, $\eta_s \equiv 0$ on $(-\infty, s-\delta] \cup [s+2\delta, +\infty)$, $0 \leq \eta_s(x_3) \leq 1$ and $|\eta_s'(x_3)| \leq c/\delta$, $|\eta_s''(x_3)| \leq c/\delta^2$ with a constant c independent of δ and $\eta_s'(x_3 + 2\delta) = -\eta_s'(x_3)$ for $x_3 \in [s-\delta, s]$.

First we prove two lemmas.

LEMMA 2.1. *Let* $f \in S_{ap}^1(\omega \times (0, t))$, $\delta > 1$. *Then for any* $\eta > 0$ *the set*

$$\left\{ \delta \in \mathbf{R} \left| \left| \int_{s-\delta}^{s+2\delta} \int_0^t \int_\omega f(x,\tau)\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3 \right| \leq \eta \right. \right\}$$

is relatively dense in $\mathbf{R}$.

PROOF. It is obvious from the inequality

$$\begin{aligned}
&\int_{s-\delta}^{s+2\delta} \int_0^t \int_\omega f(x,\tau)\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3 \\
&\quad = \int_{s-\delta}^{s} \int_0^t \int_\omega \{f(x,\tau) - f(x', x_3 + 2\delta, \tau)\}\eta_s'(x_3)\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3 \\
&\qquad \leq \frac{c(\delta+1)}{\delta} \sup_{s \in \mathbf{R}} \int_{s-1}^{s} \int_0^t \int_\omega |f(x,\tau) - f(x', x_3 + 2\delta, \tau)|\,\mathrm{d}x'\,\mathrm{d}\tau\,\mathrm{d}x_3.
\end{aligned}$$

□

LEMMA 2.2. *Let* $\psi \in S^2_{ap}(\omega)$*,* $\partial_3\psi \in S^2_{ap}(\omega)$*,* $\mathcal{M}\{\psi(x)\} = 0$ *in* $L^2(\omega)$*, and* $\delta > 1$*. Then*

$$\int_{s-\delta}^{s+2\delta} \|\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3 \le c_\dagger^2 \int_{s-\delta}^{s+2\delta} \|\partial_3\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3.$$

Here $c_\dagger$ *is a constant independent of* δ*.*

PROOF. Let X be a Hilbert space. It is well known that if $f \in S^p_{ap}(X)$ $(1 \le p < \infty)$ and $f \in C(X)$, then $f \in AP(X)$ (see, p.76 of [1]). From this it is easy to see that $\psi \in AP\big(L^2(\omega)\big)$ since the following inequality holds

$$\sup_{x_3\in\mathbf{R}}\big|\|\psi(x_3)\|_{L^2(\omega)}\big| \le 2 \sup_{x_3\in\mathbf{R}}\left\{\int_{x_3}^{x_3+1}\left(\|\psi(s)\|^2_{L^2(\omega)}+\|\partial_3\psi(s)\|^2_{L^2(\omega)}\right)\mathrm{d}s\right\}^{\frac{1}{2}}$$

which is proved by contradiction (see, p. 89 of [3]).

Now we shall prove that the set $E^* \equiv \{x_3 \in \mathbf{R}| \|\psi(x_3)\|_{L^2(\omega)} = 0\}$ is relatively dense in $\mathbf{R}$. When $\|\psi(x_3)\|_{L^2(\omega)} = 0$ for any $x_3 \in \mathbf{R}$, it is trivial. In other cases, since $\psi \in AP\big(L^2(\omega)\big)$ and $\mathcal{M}\{\psi(x)\} = 0$ in $L^2(\omega)$, we can easily prove by contradiction that there exists $y_1 \in \mathbf{R}$ such that $\|\psi(y_1)\|_{L^2(\omega)} > 0$, and there exists $y_2 \in \mathbf{R}$ such that $\|\psi(y_2)\|_{L^2(\omega)} < 0$. Hence from the intermediate value theorem, we have that there exists $y_3 \in [\min\{y_1, y_2\}, \max\{y_1, y_2\}]$ such that $\|\psi(y_3)\|_{L^2(\omega)} = 0$. From these and $\psi \in AP\big(L^2(\omega)\big)$, we can obtain that the set E^* is relatively dense in $\mathbf{R}$.

Since the set E^* is relatively dense in $\mathbf{R}$, we can introduce covering $\{K_i\}_{i=1}^n$ of $[s-\delta, s+2\delta]$ satisfying $[s-\delta, s+2\delta] = \cup_{i=1}^n K_i$, $K_i \cup K_j = \emptyset$ $(i \ne j)$ and for any i $(i = 1, 2, 3, ..., n)$ there exists $y_i \in K_i$ such that $\|\psi(y_i)\|_{L^2(\omega)} = 0$. Let $c_\dagger \equiv \max|K_i|$. Then we have for $x_3 \in K_i$,

$$|\psi(x)| = \left|\int_{y_i}^{x_3} \partial_3\psi(x)\, \mathrm{d}x_3\right| \le \sqrt{c_\dagger}\sqrt{\int_{K_i} |\partial_3\psi(x)|^2\, \mathrm{d}x_3}.$$

From this we have

$$\int_{s-\delta}^{s+2\delta} \|\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3 = \sum_{i=1}^n \int_{K_i} \|\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3$$

$$\le \sum_{i=1}^n c_\dagger^2 \int_{K_i} \|\partial_3\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3 = c_\dagger^2 \int_{s-\delta}^{s+2\delta} \|\partial_3\psi(x_3)\|^2_{L^2(\omega)}\, \mathrm{d}x_3.$$

□

It is easily seen that (1.3) is equivalent to

$$
(2.1)\qquad \begin{cases}
\dfrac{\partial}{\partial t}\left(\varDelta\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)+\left(\nabla\widetilde{\phi^\varepsilon}\times\vec{e}\right)\cdot\nabla\log n^*=c_2\varDelta^2\widetilde{\phi^\varepsilon},\\[2mm]
\varepsilon\left(\dfrac{\partial\widetilde{n^\varepsilon}}{\partial t}-(\nabla\widetilde{\phi^\varepsilon}\times\vec{e})\cdot\nabla\log n^*\right)=-\dfrac{1}{n^*}\dfrac{\partial^2}{\partial x_3^2}\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right),\\[2mm]
\dfrac{\partial\varDelta\overline{\phi^\varepsilon}}{\partial t}=c_2\varDelta^2\overline{\phi^\varepsilon},\\[2mm]
\dfrac{\partial\overline{n^\varepsilon}}{\partial t}-(\nabla\overline{\phi^\varepsilon}\times\vec{e})\cdot\nabla\log n^*=0.
\end{cases}
$$

Next we prove

LEMMA 2.3. *Let* $c_\star\equiv\sup_{x'\in\omega}|\nabla\log n^*(|x'|)|$, *and* $\delta>3c_\dagger^2$ *(*$c_\dagger$ *is a constant of Lemma* 2.2*). For any* $t\in[0,T]$, $\eta>0$ *the set*

$$
(2.2)\qquad \left\{\delta\in\mathbf{R}\,\middle|\,\varepsilon\left(\left\|\nabla\widetilde{\phi^\varepsilon}(t)\right\|_\delta^2+\left\|\widetilde{n^\varepsilon}(t)\right\|_\delta^2+\left\|\left\|\varDelta\widetilde{\phi^\varepsilon}\right\|\right\|_{t,\delta}^2\right)+\left\|\left\|\partial_3\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)\right\|\right\|_{t,\delta}^2\right.
$$
$$
\left.\le\varepsilon c\left(\left\|\nabla\widetilde{\phi_0^\varepsilon}\right\|_\delta^2+\left\|\widetilde{n_0^\varepsilon}\right\|_\delta^2+\frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\right\}
$$

is relatively dense in $\mathbf{R}$. *Here* c *is a constant independent of* ε *and* δ, *but may depends on* $c_\star$.

PROOF. Multiplying $(2.1)_1$ by $\phi^\varepsilon\eta_s$ and integrating over $\varOmega^s\equiv\omega\times(s-\delta,s+2\delta)$, we have, by virtue of the integration by parts,

$$
(2.3)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla\widetilde{\phi^\varepsilon}(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+c_2\left\|\varDelta\widetilde{\phi^\varepsilon}(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2
$$
$$
+\int_{\Omega^s}\partial_t\widetilde{n^\varepsilon}(x,t)\widetilde{\phi^\varepsilon}(x,t)\eta_s(x)\,\mathrm{d}x=\int_{\Omega^s}\{\cdots\}\eta_s'(x_3)\,\mathrm{d}x.
$$

Multiplying $(2.1)_2$ by $\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)\eta_s$ and integrating over Ω^s, we have

$$
(2.4)\qquad \varepsilon\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\widetilde{n^\varepsilon}(t)\sqrt{\eta_s}\right\|_{L^2(\Omega^s)}^2+\left\|\partial_3\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(t)\sqrt{\frac{\eta_s}{n^*}}\right\|_{L^2(\Omega^s)}^2
$$
$$
-\varepsilon\int_{\Omega^s}\partial_t\widetilde{n^\varepsilon}(x,t)\widetilde{\phi^\varepsilon}(x,t)\eta_s(x_3)\,\mathrm{d}x
$$

$$
\begin{aligned}
&= -\varepsilon \int_{\Omega^s} \left(\nabla\widetilde{\phi^\varepsilon}(x,t)\times\vec{e}\right)\cdot\nabla\log n^*(x)\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(x,t)\eta_s(x_3)\,\mathrm{d}x \\
&\qquad - \int_{\Omega^s} \frac{1}{n^*}\partial_3\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(x,t)\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(x,t)\eta_s'(x_3)\,\mathrm{d}x \\
&\le \frac{2}{3}\varepsilon c_\star\delta\left(\frac{1}{\varepsilon_1}\left\|\nabla\widetilde{\phi^\varepsilon}\right\|_\delta^2 + \varepsilon_1 c_\dagger\left\|\partial_3\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(t)\sqrt{\frac{1}{n^*}}\right\|_\delta^2\right) \\
&\qquad + 3c_\dagger^2\left\|\partial_3\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)(t)\sqrt{\frac{1}{n^*}}\right\|_\delta^2.
\end{aligned}
$$

Here we taking $\varepsilon_1 > 0$ sufficiently small. In the right most inequality, we used Lemma 2.2 and the inequality

$$
\sup_{s\in\mathbf{R}}\|f\|^2_{L^2(\Omega^s)} \le 3\delta\|f\|_\delta^2 \qquad \text{for } f\in S^2(L^2(\omega))\,. \tag{2.5}
$$

Adding (2.4) multiplied by $1/\delta$ and (2.3) multiplied by ε/δ and integrating this over $[0,T]$ and taking the supremum over $s\in\mathbf{R}$, we have (2.2) with the help of Gronwall's lemma, Lemma 2.1 and the inequalities $\|\nabla\widetilde{\phi^\varepsilon}\|_\delta \le c\|\Delta\phi^\varepsilon\|_\delta$ and

$$
\|f\|_\delta^2 \le \frac{1}{\delta}\sup_{s\in\mathbf{R}}\|f\sqrt{\eta_s}\|^2_{L^2(\Omega^s)} \qquad \text{for } f\in S^2(L^2(\omega))\,. \tag{2.6}
$$

□

Next we prove

LEMMA 2.4. *Let* $c_\star \equiv \sup_{x'\in\omega}|\nabla\log n^*(|x'|)|$, *and* $\delta > 3c_\dagger^2$ ($c_\dagger$ *is a constant of Lemma* 2.2). *For any* $t\in[0,T]$, $\eta>0$ *the set*

$$
\begin{aligned}
\Bigg\{\delta\in\mathbf{R}\,\Bigg|\,&\varepsilon\left(\left\|\mathrm{D}_x^\alpha\Delta\widetilde{\phi^\varepsilon}(t)\right\|_\delta^2 + \left\|\mathrm{D}_x^\alpha\nabla\widetilde{n^\varepsilon}(t)\right\|_\delta^2 + \left|\left|\left|\mathrm{D}_x^\alpha\nabla\Delta\widetilde{\phi^\varepsilon}\right|\right|\right|_{t,\delta}^2\right) \\
&+ \left|\left|\left|\mathrm{D}_x^\alpha\partial_3\nabla\left(\widetilde{\phi^\varepsilon}-\widetilde{n^\varepsilon}\right)\right|\right|\right|_{t,\delta}^2 \le \varepsilon c\left(\left\|\Delta\widetilde{\phi_0^\varepsilon}\right\|^2_{\tilde{S}_\delta^{|\alpha|}} + \left\|\nabla\widetilde{n_0^\varepsilon}\right\|^2_{\tilde{S}_\delta^{|\alpha|}} + \frac{\eta}{\delta}\right) \\
&+ \varepsilon c\left(\left\|\nabla\widetilde{\phi_0^\varepsilon}\right\|_\delta^2 + \left\|\widetilde{n_0^\varepsilon}\right\|_\delta^2 + \frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\Bigg\},
\end{aligned} \tag{2.7}
$$

is relatively dense in $\mathbf{R}$ ($|\alpha| = 0, 1$). Here c is a constant independent of ε and δ, but may depends on $c_\star$.

PROOF. In the similar way to Lemma 2.3, multiplying $(2.1)_1$ and $(2.1)_2$ by $\varDelta\widetilde{\phi}^\varepsilon\eta_s$ and $\varDelta\left(\widetilde{\phi}^\varepsilon - \widetilde{n}^\varepsilon\right)\eta_s$, respectively, and integrating over Ω^s, we have

$$\begin{aligned}
&\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla\widetilde{n}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\varDelta\widetilde{\phi}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\right.\\
&\qquad\left.+\,c_2\left\|\nabla\varDelta\widetilde{\phi}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)+\left\|\partial_3\nabla\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\sqrt{\frac{\eta_s}{n^*}}\right\|^2_{L^2(\Omega^s)}\\
&\quad\le \varepsilon\delta c_\star c\left\|\varDelta\widetilde{\phi}^\varepsilon\right\|^2_\delta+C_1(\delta,c_\star,c_\dagger,\varepsilon_1)\left\|\partial_3\nabla\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\sqrt{\frac{1}{n^*}}\right\|^2_\delta\\
&\qquad+C_2\left(\delta,c_\star,c_\dagger,\frac{1}{\varepsilon_1}\right)\left\|\partial_3\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\right\|^2_\delta+\varepsilon\int_{\Omega^s}\{\cdots\}\eta_s'(x_3)\,\mathrm{d}x,
\end{aligned}$$

where $\varepsilon_1>0$, C_1, C_2 are positive constants depending on δ, $c_\star$, $c_\dagger$, ε_1. If we take δ sufficiently large and ε_1 sufficiently small, then $C_1<\delta$. In the right most inequality, we used Lemma 2.2 and (2.5) and the inequality $\|\nabla\widetilde{\phi}^\varepsilon\|_\delta\le c\|\varDelta\widetilde{\phi}^\varepsilon\|_\delta$. Integrating above inequality over $[0,T]$ and taking the supremum over $s\in\mathbf{R}$, we have (2.7) ($|\alpha|=0$) with the help of Lemma 2.1 and (2.2).

In the similar way to Lemma 2.3, multiply $(2.1)_1$ by $\varDelta^2\widetilde{\phi}^\varepsilon\eta_s$ and integrate it over Ω^s, operate Laplacian to $(2.1)_2$, multiply it by $\varDelta\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)\eta_s$ and integrate it over Ω^s. From these we have

$$\begin{aligned}
&\varepsilon\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\varDelta\widetilde{n}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}+\left\|\nabla\varDelta\widetilde{\phi}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)\right.\\
&\qquad\left.+\,c_2\left\|\varDelta^2\widetilde{\phi}^\varepsilon(t)\sqrt{\eta_s}\right\|^2_{L^2(\Omega^s)}\right)+\left\|\partial_3\varDelta\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\sqrt{\frac{\eta_s}{n^*}}\right\|^2_{L^2(\Omega^s)}\\
&\quad\le\varepsilon C_1\left(\delta,c_{\star\star},\frac{1}{\varepsilon_1}\right)\left(\left\|\varDelta\widetilde{\phi}^\varepsilon\right\|^2_\delta+\left\|\nabla\varDelta\widetilde{\phi}^\varepsilon\right\|^2_\delta\right)+\varepsilon\int_{\Omega^s}\{\cdots\}\eta_s'(x_3)\,\mathrm{d}x\\
&\qquad+C_2(\delta,c_\star,c_\dagger,\varepsilon_1)\left\|\partial_3\varDelta\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\sqrt{\frac{1}{n^*}}\right\|^2_\delta\\
&\qquad+C_3\left(\delta,c_\star,c_\dagger,\frac{1}{\varepsilon_1}\right)\left(\left\|\partial_3\nabla\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\right\|^2_\delta+\left\|\partial_3\left(\widetilde{\phi}^\varepsilon-\widetilde{n}^\varepsilon\right)(t)\right\|^2_\delta\right),
\end{aligned}$$

□

where $\varepsilon_1 > 0$, $c_{\star\star} \equiv \sum_{|\alpha|\le 2} \sup_{x'\in\omega} |\mathrm{D}_{x'}^{\alpha} \nabla \log n^*(|x'|)|$, C_1 is a positive constant depending on δ, $c_{\star\star}$, ε_1, and C_2, C_3 are positive constants depending on δ, $c_\star$, $c_\dagger$, ε_1. If we take δ sufficiently large and ε_1 sufficiently small, then $C_2 < \delta$. In the right most inequality, we used Lemma 2.2, (2.5) and the inequality $\|\nabla\widetilde{\phi^{\varepsilon}}\|_{\delta} \le c\|\Delta\widetilde{\phi^{\varepsilon}}\|_{\delta}$. Integrating above inequality over $[0, T]$ and taking the supremum over $s \in \mathbf{R}$, we have (2.7) ($|\alpha| = 1$) with the help of Lemma 2.1 and (2.2), (2.6), (2.7) ($|\alpha| = 0$). □

The proof of the following lemma is easy, hence we omit it.

Lemma 2.5. *Let* $c_\star \equiv \sup_{x'\in\omega} |\nabla \log n^*(|x'|)|$. *For any* $t \in [0, T]$

$$\left\|\mathrm{D}_{x'}^{\alpha}\nabla\overline{\phi^{\varepsilon}}(t)\right\|_{L^2(\omega)}^2 \le \left\|\mathrm{D}_{x'}^{\alpha}\nabla\overline{\phi_0^{\varepsilon}}\right\|_{L^2(\omega)}^2 \mathrm{e}^{-ct},$$

$$\int_0^t \left\|\mathrm{D}_{x'}^{\alpha}\Delta\overline{\phi^{\varepsilon}}(\tau)\right\|_{L^2(\omega)} \mathrm{d}\tau \le \left\|\mathrm{D}_{x'}^{\alpha}\nabla\overline{\phi_0^{\varepsilon}}\right\|_{L^2(\omega)}^2, \quad |\alpha| = 0, 1, 2,$$

$$\|\overline{n^{\varepsilon}}(t)\|_{L^2(\omega)}^2 \le \|\overline{n_0^{\varepsilon}}\|_{L^2(\omega)}^2 + c_\star ct\left\|\nabla\overline{\phi_0^{\varepsilon}}\right\|_{L^2(\omega)}^2,$$

$$\|\mathrm{D}_{x'}^{\alpha}\overline{n^{\varepsilon}}(t)\|_{L^2(\omega)}^2 \le \|\mathrm{D}_{x'}^{\alpha}\overline{n_0^{\varepsilon}}\|_{L^2(\omega)}^2 + c_\star ct\left\|\mathrm{D}_{x'}^{\alpha}\overline{\phi_0^{\varepsilon}}\right\|_{L^2(\omega)}^2, \quad |\alpha| = 1, 2, 3,$$

$$\left\|\frac{\partial \mathrm{D}_{x'}^{\alpha}\overline{n^{\varepsilon}}}{\partial t}(t)\right\|_{L^2(\omega)}^2 \le c_\star\left\|\mathrm{D}_{x'}^{\alpha}\overline{\phi_0^{\varepsilon}}\right\|_{L^2(\omega)}^2, \quad |\alpha| = 0, 1, 2.$$

Here c is a constant independent of ε, but may depends on $c_\star$.

By the standard arguments based upon the *a priori* estimates in Lemmas 2.3-2.5 the solution can be extended up to any T. Since (1.3) are linear equations, we can easily prove Stepanov-almost-periodicity of the solution. Thus the proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into two parts. First we prove the local in time existence in the same way as in [24] in § 3.1. Second we give a proof of Theorem 1.2 with the help of *a priori* estimates in § 3.2.

3.1 – *Local in time existence and uniqueness*

3.1.1 – Auxiliary lemmas

Let X be a Hilbert space and $\psi \in S^2_{ap}(X)$. Note that for any $\xi \in \mathbf{R}$ the mean value

$$\psi_\xi = \mathcal{M}\{\psi(x)\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\} \equiv \lim_{A\to\infty} \frac{1}{2A}\int_{-A}^{A} \psi(x)\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\,\mathrm{d}x_3$$

exists in X ([10], [38]), where $\mathrm{i} = \sqrt{-1}$.

Let $\{\xi_k\}_{k\in\mathbf{N}}$ be a sequence in $\mathbf{R}$ such that $\xi_k \neq \xi_{k'}$ for $k \neq k'$. For each $m \in \mathbf{N}$, it is easy to obtain

$$\mathcal{M}\left\{\left\|\psi(x_3) - \sum_{k=1}^{m} \psi_{\xi_k}\,\mathrm{e}^{-\mathrm{i}\,\xi_k x_3}\right\|_X^2\right\} = \mathcal{M}\left\{\|\psi(x_3)\|_X^2\right\} - \sum_{k=1}^{m}\|\psi_{\xi_k}\|_X^2,$$

and hence

$$\sum_{k=1}^{m}\|\psi_{\xi_k}\|_X^2 \leq \mathcal{M}\left\{\|\psi(x_3)\|_X^2\right\}.$$

This inequality implies that for any $\varepsilon > 0$ there corresponds at most a finite number of ξ_k for which $\|\psi_{\xi_k}\|_X > \varepsilon$. From this fact it follows that every $\|\psi_{\xi_k}\|_X$ $(\neq 0)$ belongs to one of the enumerable set of inequalities

$$\|\psi_{\xi_k}\|_X > 1, \quad \frac{1}{m} \geq \|\psi_{\xi_k}\|_X > \frac{1}{m+1} \quad (m = 1, 2, 3, \dots),$$

and each of these inequalities is satisfied by at most a finite number of ξ_k. Therefore, only for at most countable $\xi \in \mathbf{R}$ the quantity ψ_ξ is a non-zero element of X. We call $\sigma(\psi) = \left\{\xi \in \mathbf{R} \,\middle|\, \|\psi_\xi\|_X \neq 0\right\}$ the spectrum of ψ, and the formal series $\sum_{\xi\in\sigma(\psi)} \psi_\xi\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$ the Bohr–Fourier series of ψ, which is written as

$$\psi \sim \sum_{\xi\in\sigma(\psi)} \psi_\xi\,\mathrm{e}^{\mathrm{i}\,\xi x_3}.$$

Then the following lemmas hold (see [1], [5], [12], [13]).

LEMMA 3.1. *If ψ, $\psi' \in S^2_{ap}(X)$ have the same Bohr–Fourier series, then*

$$\|\psi - \psi'\|_{S^2(X)} = 0.$$

Lemma 3.2. *For any $\psi \in S^2_{ap}(X)$ Parseval's identity*

$$\mathcal{M}\Big\{\|\psi(x_3)\|_X^2\Big\} = \sum_{\xi\in\sigma(\psi)} \big\|\psi_\xi\big\|_X^2$$

holds.

Let us consider a generalized trigonometric series

$$\sum_{\xi\in\Lambda} a_\xi\, \mathrm{e}^{\mathrm{i}\,\xi x}, \tag{3.1}$$

where Λ is a countable subset of $\mathbf{R}$ and $\{a_\xi\}_{\xi\in\Lambda}\subset \mathbf{C}$. Let $\{\gamma_j\}_{j\in\mathbf{N}}$ be a basis of Λ ([10]). Bochner–Fejér sum $\mathcal{S}^m(x)$ associated with (3.1) is given by

$$\mathcal{S}^m(x) = \sum_{\nu_1=-(m!)^2}^{(m!)^2} \cdots \sum_{\nu_m=-(m!)^2}^{(m!)^2} \left(1-\frac{|\nu_1|}{(m!)^2}\right)\cdots\left(1-\frac{|\nu_m|}{(m!)^2}\right)$$
$$\times\, a_\xi^* \exp\left(\mathrm{i}\sum_{j=1}^m \nu_j \frac{\gamma_j}{m!}\, x\right),$$

where for $\xi\in\Lambda$

$$a_\xi^* = \begin{cases} a_\xi & \text{if } \displaystyle\sum_{j=1}^m \nu_j \frac{\gamma_j}{m!} = \xi, \\ 0 & \text{if } \displaystyle\sum_{j=1}^m \nu_j \frac{\gamma_j}{m!} \neq \xi. \end{cases}$$

By introducing an increasing symmetric sequence $\{\Lambda_m\}_{m\in\mathbf{N}}$ of Λ converging to Λ, that is, $-\Lambda_m = \Lambda_m,\ \Lambda_m\subset\Lambda_{m+1}$ and $\Lambda = \cup_m\Lambda_m, \mathcal{S}^m(x)$ can be written as

$$\mathcal{S}^m(x) = \sum_{\xi\in\Lambda_m} d_\xi^{(m)} a_\xi\, \mathrm{e}^{\mathrm{i}\,\xi x}$$

with constants $d_\xi^{(m)}$ satisfying $0\le d_\xi^{(m)}\le 1$ and $\lim_{m\to\infty} d_\xi^{(m)} = 1$. Note that $d_\xi^{(m)}$ depend on ξ and m, but not on a_ξ ([13]).

We say that $\mathcal{F}\subset S^p_{ap}(X)$ is S^p-equi-almost-periodic if for any $\varepsilon>0$ there exists a relatively dense subset E_ε of $\mathbf{R}$ such that

$$\sup_{s\in\mathbf{R}} \int_s^{s+1} \|f(x+\sigma)-f(x)\|_X^p\, \mathrm{d}x < \varepsilon \quad \text{for } f\in\mathcal{F},\ \sigma\in E_\varepsilon.$$

It is well-known that Riesz–Fischer theorem does not hold for $S^p_{ap}(X)$ $(1 \leq p < \infty)$ ([4], [27]), while the following lemma holds true (see [11], [14]).

LEMMA 3.3. *A necessary and sufficient condition for a generalized trigonometric series* (3.1) *to be a Bohr–Fourier series of a function* $f \in S^p_{ap}(X)$ $(1 < p < \infty)$ *is that a sequence of the Bochner–Fejér sums* $\{\mathcal{S}^m(x)\}_{m\in\mathbf{N}}$ *associated with the series* (3.1) *is bounded in* $S^p(X)$ *and* S^p*-equi-almost-periodic.*

3.1.2 – Local in time existence and uniqueness

It is easily seen that $(1.7)_2$ has a unique solution $\overline{n^0}$ when $\overline{\phi^0}$ is given. Hence we consider only the problem for $(1.7)_1$. The following lemmas are well-known (see, for example, [26], [30], [33]).

LEMMA 3.4. *Let* $l \in \mathbf{R}$, $l \geq 0$, $c_2 > 0$ *and* $\xi \in \mathbf{R}$. *Assume that* $\psi_0 \in W_2^{1+l}(\omega)$ *satisfies the compatibility conditions up to order* $\max\{[l-3/2], 0\}$ *and* $f \in W_2^{l,l/2}(\omega_T)$ *satisfies* $f(x',t) = 0$ *for* $x' \in \partial\omega,\ t > 0$. *Then there exists a unique solution* $\psi \in W_2^{2+l,1+l/2}(\omega_T)$ *to problem*

$$\begin{cases} \dfrac{\partial \psi}{\partial t} - c_2(\Delta' - \xi^2)\psi = f \quad \text{for } x' \in \omega,\ t > 0, \\ \psi(x',0) = \psi_0(x') \quad \text{for } x' \in \omega, \\ \psi(x',t) = 0 \quad \text{for } x' \in \partial\omega,\ t > 0. \end{cases}$$

Moreover, this solution satisfies

$$\|\psi\|_{W_2^{2+l,1+l/2}(\omega_T)} \leq c_\xi \Big(\|\psi_0\|_{W_2^{1+l}(\omega)} + \|f\|_{W_2^{l,l/2}(\omega_T)} \Big)$$

with c_ξ being a positive constant depending on ξ.

LEMMA 3.5. *Let* $\psi \in W_2^{2+l,1+l/2}(\omega_T)$, $l \geq 0$ *and* $\xi \in \mathbf{R}$. *Then the problem*

$$\begin{cases} \Big(\Delta' - \xi^2\Big)\phi = \psi \quad \text{for } x' \in \omega,\ t > 0, \\ \phi(x',t) = 0 \quad \text{for } x' \in \partial\omega,\ t > 0 \end{cases}$$

has a unique solution $\phi \in L^2(0,T; W_2^{4+l}(\omega)) \cap W_2^{1+l/2}(0,T; W_2^2(\omega))$, *which satisfies*

$$\|\phi\|_{L^2(0,T;W_2^{4+l}(\omega))} + \|\phi\|_{W_2^{1+l/2}(0,T;W_2^2(\omega))} \leq c_\xi \|\psi\|_{W_2^{2+l,1+l/2}(\omega_T)}$$

with c_ξ *being a positive constant depending on* ξ.

Let us fix symmetric increasing sequence $\{\Lambda_m\}_{m\in\mathbf{N}}$ of $\Lambda\equiv\sigma(\phi_0^0)$ converging to Λ. For $\xi\in\Lambda$ we consider problem

$$
\begin{cases}
\dfrac{\partial}{\partial t}\Big((\Delta'-\xi^2)\phi_\xi^0-\phi_\xi^0+\overline{\phi_\xi^0}\Big)+\Big(\nabla\widetilde{\phi_\xi^0}\times\vec{e}\Big)\cdot\nabla\log n^* \\
\qquad = c_2(\Delta'-\xi^2)^2\phi_\xi^0 \quad \text{for } x'\in\omega,\ t>0, \\
\phi_\xi^0\Big|_{t=0}=\phi_{0\xi}^0 \quad \text{for } x'\in\omega, \\
\phi_\xi^0=(\Delta'-\xi^2)\phi_\xi^0=0 \quad \text{for } x'\in\partial\omega,\ t>0,
\end{cases}
\tag{3.2}
$$

where $\phi_{0\xi}^0=\mathcal{M}\{\phi_0^0\,\mathrm{e}^{-\mathrm{i}\,\xi x_3}\}$. Lemma 3.4 implies that (3.2) has a unique solution ϕ_ξ^0. Then it is obvious that $\mathcal{S}_{\phi^0}^m=\sum\limits_{\xi\in\Lambda_m}d_\xi^{(m)}\phi_\xi^0\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$ is a solution of problem

$$
\begin{cases}
\dfrac{\partial}{\partial t}\Big(\Delta\mathcal{S}_{\phi^0}^m-\mathcal{S}_{\phi^0}^m+\mathcal{S}_{\overline{\phi^0}}^m\Big)+\Big(\nabla\mathcal{S}_{\widetilde{\phi^0}}^m\times\vec{e}\Big)\cdot\nabla\log n^* \\
\qquad = c_2\Delta^2\mathcal{S}_{\phi^0}^m \quad \text{for } x\in\Omega,\ t>0, \\
\mathcal{S}_{\phi^0}^m\Big|_{t=0}=\mathcal{S}_{\phi_0^0}^m \quad \text{for } x\in\Omega, \\
\mathcal{S}_{\phi^0}^m=\Delta\mathcal{S}_{\phi^0}^m=0 \quad \text{for } x\in\Gamma,\ t>0
\end{cases}
,
\tag{3.3}
$$

where $\mathcal{S}_{\phi_0^0}^m=\sum\limits_{\xi\in\Lambda_m}d_\xi^{(m)}\phi_{0\xi}^0\,\mathrm{e}^{\mathrm{i}\,\xi x_3}$.

The following lemma holds in the same way as in Lemmas 2.3-2.4.

LEMMA 3.6. *Let* $c_\star\equiv\sup\limits_{x'\in\omega}|\nabla\log n^*(|x'|)|$ *and* $\widetilde{\phi^0}=\phi^0-\overline{\phi^0}$. *For any* $t\in[0,T]$, $\eta>0$ *the sets*

$$
\begin{aligned}
&\Big\{\delta\in\mathbf{R}\,\Big|\ \Big\|\nabla\mathcal{S}_{\phi^0}^m(t)\Big\|_\delta^2+\Big\|\mathcal{S}_{\widetilde{\phi^0}}^m(t)\Big\|_\delta^2+\Big|\Big|\Big|\Delta\mathcal{S}_{\phi^0}^m\Big|\Big|\Big|_{t,\delta}^2 \\
&\qquad\le\varepsilon c\Big(\Big\|\nabla\mathcal{S}_{\phi_0^0}^m\Big\|_\delta^2+\Big\|\mathcal{S}_{\widetilde{\phi_0^0}}^m\Big\|_\delta^2+\frac{\eta}{\delta}\Big)\mathrm{e}^{cc_\star t}\Big\}, \\
&\Big\{\delta\in\mathbf{R}\,\Big|\ \Big\|\mathrm{D}_x^\alpha\Delta\mathcal{S}_{\phi^0}^m(t)\Big\|_\delta^2+\Big\|\mathrm{D}_x^\alpha\nabla\mathcal{S}_{\widetilde{\phi^0}}^m(t)\Big\|_\delta^2+\Big|\Big|\Big|\mathrm{D}_x^\alpha\nabla\Delta\mathcal{S}_{\phi^0}^m\Big|\Big|\Big|_{t,\delta}^2 \\
&\qquad\le\varepsilon c\Big(\Big\|\Delta\mathcal{S}_{\phi_0^0}^m\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2+\Big\|\nabla\mathcal{S}_{\widetilde{\phi_0^0}}^m\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2+\frac{\eta}{\delta}\Big) \\
&\qquad\quad+\varepsilon c\Big(\Big\|\nabla\mathcal{S}_{\phi_0^0}^m\Big\|_\delta^2+\Big\|\mathcal{S}_{\widetilde{\phi_0^0}}^m\Big\|_\delta^2+\frac{\eta}{\delta}\Big)\mathrm{e}^{cc_\star t}\Big\},
\end{aligned}
$$

are relatively dense in $\mathbf{R}$ ($|\alpha| = 0, 1$)*. Here c is a constant independent of* δ*, but may depends on* $c_\star$*.*

Let $\mathcal{V}^m_{\cdot\sigma}(x,t) = \mathcal{S}^m_{\cdot}(x', x_3+\sigma, t) - \mathcal{S}^m_{\cdot}(x', x_3, t)$ for any $\sigma \neq 0$. Then $\mathcal{V}^m_{\phi^0\sigma}$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\Delta\mathcal{V}^m_{\phi^0\sigma} - \mathcal{V}^m_{\phi^0\sigma} + \mathcal{V}^m_{\overline{\phi^0}\sigma}\right) + \left(\nabla\mathcal{V}^m_{\widetilde{\phi^0}\sigma} \times \vec{e}\right)\cdot\nabla\log n^* \\ \qquad = c_2\Delta^2\mathcal{V}^m_{\phi^0\sigma} \quad \text{for } x\in\Omega,\ t>0, \\ \mathcal{V}^m_{\phi^0\sigma}\Big|_{t=0} = \mathcal{V}^m_{\phi^0_0\sigma} \quad \text{for } x\in\Omega, \\ \mathcal{V}^m_{\phi^0\sigma} = \Delta\mathcal{V}^m_{\phi^0\sigma} = 0 \quad \text{for } x\in\Gamma,\ t>0. \end{cases}$$

The following lemma holds in the same way as in Lemma 3.6.

Lemma 3.7. *Let* $c_\star \equiv \sup_{x'\in\omega}|\nabla\log n^*(|x'|)|$ *and* $\widetilde{\phi^0} = \phi^0 - \overline{\phi^0}$*. For any* $t\in[0,T]$, $\eta>0$ *the sets*

$$\left\{\delta\in\mathbf{R}\,\middle|\, \left\|\nabla\mathcal{V}^m_{\phi^0\sigma}(t)\right\|^2_\delta + \left\|\mathcal{V}^m_{\widetilde{\phi^0}\sigma}(t)\right\|^2_\delta + \left|\left|\left|\Delta\mathcal{V}^m_{\phi^0\sigma}\right|\right|\right|^2_{t,\delta} \le \varepsilon c\left(\left\|\nabla\mathcal{V}^m_{\phi^0_0\sigma}\right\|^2_\delta + \left\|\mathcal{V}^m_{\widetilde{\phi^0_0}\sigma}\right\|^2_\delta + \frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\right\},$$

$$\left\{\delta\in\mathbf{R}\,\middle|\, \left\|\mathrm{D}^\alpha_x\Delta\mathcal{V}^m_{\phi^0\sigma}(t)\right\|^2_\delta + \left\|\mathrm{D}^\alpha_x\nabla\mathcal{V}^m_{\widetilde{\phi^0}\sigma}(t)\right\|^2_\delta + \left|\left|\left|\mathrm{D}^\alpha_x\nabla\Delta\mathcal{V}^m_{\phi^0\sigma}\right|\right|\right|^2_{t,\delta} \le \varepsilon c\left(\left\|\Delta\mathcal{V}^m_{\phi^0_0\sigma}\right\|^2_{\widetilde{S}^{|\alpha|}_\delta} + \left\|\nabla\mathcal{V}^m_{\widetilde{\phi^0_0}\sigma}\right\|^2_{\widetilde{S}^{|\alpha|}_\delta} + \frac{\eta}{\delta}\right) + \varepsilon c\left(\left\|\nabla\mathcal{V}^m_{\phi^0_0\sigma}\right\|^2_\delta + \left\|\mathcal{V}^m_{\widetilde{\phi^0_0}\sigma}\right\|^2_\delta + \frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\right\},$$

are relatively dense in $\mathbf{R}$ ($|\alpha| = 0, 1$)*. Here c is a constant independent of* δ*, but may depends on* $c_\star$*.*

Now we prove that $\{\mathcal{S}^m_{\phi^0}\}_{m=1}^\infty$ forms a sequence bounded in $\widetilde{S}^{3,3/2}(\omega_T)$ and $\widetilde{S}^{3,3/2}(\omega_T)$-equi-almost-periodic with the help of Lemmas 3.6-3.7 and the well-known fact (see [4], [6], [11], [13])

$$\left\|\mathcal{S}^m_\psi\right\|_{S^p(X)} \le \|\psi\|_{S^p(X)}, \tag{3.4}$$

$$\left\|\mathcal{S}^m_\psi - \psi\right\|_{S^p(X)} \to 0 \quad \text{as } m\to\infty \tag{3.5}$$

for any $\psi\in S^p_{ap}(X)$ defined on a Banach space X ($1\le p\le\infty$).

Indeed, the boundedness of $\{\mathcal{S}^m_{\phi^0}\}_{m=1}^\infty$ in $\widetilde{S}^{3,3/2}(\omega_T)$ followed from

$$\left\|\mathcal{S}^m_{\phi^0_0}\right\|^2_{\tilde{S}^2} \leq \left\|\phi^0_0\right\|^2_{\tilde{S}^2}$$

directly derived from (3.4) and Lemma 3.6.

Let

$$\Phi^0_{0\sigma}(x) = \phi^0_0(x', x_3 + \sigma) - \phi^0_0(x', x_3)$$

for any $\sigma \neq 0$. It is easy to see that $\Phi^0_{0\sigma\xi} = \left(\mathrm{e}^{\mathrm{i}\xi\sigma} - 1\right)\phi^0_{0\xi}$ and

$$\mathcal{V}^m_{\phi^0_0\sigma}(x) = \mathcal{S}^m_{\Phi^0_{0\sigma}}(x).$$

Then (3.4) yields

$$\left\|\mathcal{V}^m_{\phi^0_0\sigma}\right\|^2_{\tilde{S}^2} \leq \left\|\Phi^0_{0\sigma}\right\|^2_{\tilde{S}^2}.$$

From this we find that $\mathcal{S}^m_{\phi^0}$ is $\widetilde{S}^{3,3/2}(\omega_T)$-equi-almost-periodic by virtue of Lemmas 3.7.

Lemmas 3.3 implies that ϕ^0 belongs to $\widetilde{S}^{3,3/2}_{ap}(\omega_T)$. Moreover ϕ^0 is unique in the same class according to Lemma 3.1. Therefore from (3.5), we have

$$\mathcal{S}^m_{\phi^0} \to \phi^0 \quad \text{in } \widetilde{S}^{3,3/2}(\omega_T) \quad \text{as } m \to \infty. \tag{3.6}$$

Since

$$\mathcal{S}^m_{\phi^0_0} \to \phi^0_0 \quad \text{in } \widetilde{S}^2(\omega) \quad \text{as } m \to \infty$$

follows from (3.5), we conclude from (3.3) and (3.6), that this ϕ^0 is a solution of the problem for $(1.7)_1$. Here the strong convergence of Galerkin approximations are obtained by using property of linear equations (p. 79 of [25]). Thus the proof of local in time existence and uniqueness is complete.

3.2 – *A priori estimates*

In the similar way as in Lemmas 2.3-2.5 we can show the following *a priori* estimates of the solution $\left(\phi^0, \overline{n^0}\right)$ established in § 3.1. Let T be an arbitrary positive number and $\left(\phi^0, \overline{n^0}\right)$ be a solution of problem (1.7) belonging to $(L^2(0,T;\widetilde{S}^4(\omega)) \cap W^1_2(0,T;\widetilde{S}^2(\omega))) \times (L^\infty(0,T;W^3_2(\omega)) \cap W^1_2(0,T;W^2_2(\omega)))$.

LEMMA 3.8. *Let* $c_\star \equiv \sup_{x'\in\omega} |\nabla \log n^*(|x'|)|$ *and* $\widetilde{\phi^0} = \phi^0 - \overline{\phi^0}$. *For any* $t \in [0,T]$, $\eta > 0$ *the sets*

$$\Big\{\delta\in\mathbf{R}\Big|\ \Big\|\nabla\widetilde{\phi^0}(t)\Big\|_\delta^2+\Big\|\widetilde{\phi^0}(t)\Big\|_\delta^2+\Big|\Big\|\varDelta\widetilde{\phi^0}\Big\|\Big|_{t,\delta}^2$$
$$\leq\varepsilon c\left(\Big\|\nabla\widetilde{\phi_0^0}\Big\|_\delta^2+\Big\|\widetilde{\phi_0^0}\Big\|_\delta^2+\frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\Big\},$$
$$\Big\{\delta\in\mathbf{R}\Big|\ \Big\|\mathrm{D}_x^\alpha\varDelta\widetilde{\phi^0}(t)\Big\|_\delta^2+\Big\|\mathrm{D}_x^\alpha\nabla\widetilde{\phi^0}(t)\Big\|_\delta^2+\Big|\Big\|\mathrm{D}_x^\alpha\nabla\varDelta\widetilde{\phi^0}\Big\|\Big|_{t,\delta}^2$$
$$\leq\varepsilon c\left(\Big\|\varDelta\widetilde{\phi_0^0}\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2+\Big\|\nabla\widetilde{\phi_0^0}\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2+\frac{\eta}{\delta}\right)$$
$$+\varepsilon c\left(\Big\|\nabla\widetilde{\phi_0^0}\Big\|_\delta^2+\Big\|\widetilde{\phi_0^0}\Big\|_\delta^2+\frac{\eta}{\delta}\right)\mathrm{e}^{cc_\star t}\Big\},$$

are relatively dense in $\mathbf{R}$ $(|\alpha|=0,1)$. *For any* $t\in[0,T]$

$$\Big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\phi^0}(t)\Big\|_{L^2(\omega)}^2\leq\Big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\phi_0^0}\Big\|_{L^2(\omega)}^2\mathrm{e}^{-ct},$$
$$\int_0^t\Big\|\mathrm{D}_{x'}^\alpha\varDelta\overline{\phi^0}(\tau)\Big\|_{L^2(\omega)}\mathrm{d}\tau\leq\Big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\phi_0^0}\Big\|_{L^2(\omega)}^2,\ \ |\alpha|=0,1,2,$$
$$\Big\|\overline{n^0}(t)\Big\|_{L^2(\omega)}^2\leq\Big\|\overline{n_0^0}\Big\|_{L^2(\omega)}^2+c_\star ct\Big\|\nabla\overline{\phi_0^0}\Big\|_{L^2(\omega)}^2,$$
$$\Big\|\mathrm{D}_{x'}^\alpha\overline{n^0}(t)\Big\|_{L^2(\omega)}^2\leq\Big\|\mathrm{D}_{x'}^\alpha\overline{n_0^0}\Big\|_{L^2(\omega)}^2+c_\star ct\Big\|\mathrm{D}_{x'}^\alpha\overline{\phi_0^0}\Big\|_{L^2(\omega)}^2,\quad |\alpha|=1,2,3,$$
$$\left\|\frac{\partial\mathrm{D}_{x'}^\alpha\overline{n^0}}{\partial t}(t)\right\|_{L^2(\omega)}^2\leq c_\star\Big\|\mathrm{D}_{x'}^\alpha\overline{\phi_0^0}\Big\|_{L^2(\omega)}^2\quad |\alpha|=0,1,2.$$

Here c *is a constant independent of* δ, *but may depends on* $c_\star$.

By the standard arguments in help of the *a priori* estimates in Lemma 3.8 the solution ϕ^0 established in § 3.1 can be extended to any time interval $[0,T]$. Since $(1.7)_1$-$(1.7)_2$ are linear equations, we can easily prove Stepanov-almost-periodicity of the solution. Thus the proof of Theorem 1.2 is complete.

4. Proof of Theorem 1.3

Subtracting (1.5) from (1.3) and denoting by $\varPhi\equiv\phi^\varepsilon-\phi^0$, $N\equiv n^\varepsilon-n^0$, we have

$$
(4.1)\quad \begin{cases}
\dfrac{\partial}{\partial t}(\varDelta\varPhi - N) + (\nabla\varPhi\times\vec{e})\cdot\nabla\log\, n^* = c_2\varDelta^2\varPhi, \\[2mm]
\varepsilon\dfrac{\partial N}{\partial t} = -\varepsilon\left(\dfrac{\partial\widetilde{n^0}}{\partial t} - \left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\log\, n^*\right) \\[2mm]
\qquad -\dfrac{1}{\bar{n}}\dfrac{\partial^2(\widetilde{\varPhi}-\widetilde{N})}{\partial x_3^2} + \varepsilon(\nabla\varPhi\times\vec{e})\cdot\nabla\log\, n^* \quad x\in\varOmega,\ t>0, \\[2mm]
\varPhi(x,0) = \phi_0^\varepsilon - \phi_0^0, \quad N(x,0) = n_0^\varepsilon - n_0^0 \quad \text{for } x\in\varOmega, \\[2mm]
\varPhi(x,t) = \varDelta\varPhi(x,t) = N(x,t) = 0 \quad \text{for } x\in\varGamma,\ T>t>0.
\end{cases}
$$

It is easily seen that $(4.1)_1$-$(4.1)_2$ is equivalent to

$$
(4.2)\quad \begin{cases}
\dfrac{\partial}{\partial t}\left(\varDelta\widetilde{\varPhi} - \widetilde{N}\right) + \left(\nabla\widetilde{\varPhi}\times\vec{e}\right)\cdot\nabla\log\, n^* = c_2\varDelta^2\widetilde{\varPhi}, \\[2mm]
\varepsilon\dfrac{\partial\widetilde{N}}{\partial t} = -\varepsilon\left(\dfrac{\partial\widetilde{n^0}}{\partial t} - \left(\nabla\widetilde{\phi^0}\times\vec{e}\right)\cdot\nabla\log\, n^*\right) \\[2mm]
\qquad -\dfrac{1}{\bar{n}}\dfrac{\partial^2\left(\widetilde{\varPhi}-\widetilde{N}\right)}{\partial x_3^2} + \varepsilon\left(\nabla\widetilde{\varPhi}\times\vec{e}\right)\cdot\nabla\log\, n^*, \\[2mm]
\dfrac{\partial\varDelta\overline{\varPhi}}{\partial t} = c_2\varDelta^2\overline{\varPhi}, \\[2mm]
\dfrac{\partial\overline{N}}{\partial t} - (\nabla\overline{\varPhi}\times\vec{e})\cdot\nabla\log\, n^* = 0 \quad \text{for } x\in\varOmega,\ t>0.
\end{cases}
$$

By virtue of Lemmas 2.3-2.5 and 3.8, the following lemma is derived from the problem for (4.2). Here we denote by c a constant independent of t and by $C(t)$ a constant dependent on both t and the bounds of ϕ^ε, n^ε, ϕ^0, n^0, which may differ at each occurrence.

LEMMA 4.1. *Let* $c_\star \equiv \sup_{x'\in\omega}|\nabla\log\, n^*(|x'|)|$, *and* $\delta > 3c_\dagger^2$ *(*$c_\dagger$ *is a constant of Lemma* 2.2*). For any* $t\in[0,T]$, $\eta>0$ *the sets*

$$
\Big\{\delta\in\mathbf{R}\,\Big|\ \varepsilon\left(\left\|\nabla\widetilde{\varPhi}(t)\right\|_\delta^2 + \left\|\widetilde{N}(t)\right\|_\delta^2 + \left|\left|\left|\varDelta\widetilde{\varPhi}\right|\right|\right|_{t,\delta}^2\right) + \left|\left|\left|\partial_3\left(\widetilde{\varPhi}-\widetilde{N}\right)\right|\right|\right|_{t,\delta}^2
\le \varepsilon C(t)\left(\left\|\nabla\widetilde{\varPhi}(0)\right\|_\delta^2 + \left\|\widetilde{N}(0)\right\|_\delta^2 + \frac{\eta}{\delta} + \varepsilon\right)\mathrm{e}^{cc_\star t}\Big\},
$$

$$\Big\{\delta \in \mathbf{R}\Big|\ \varepsilon\Big(\Big\|\mathrm{D}_x^\alpha \varDelta\widetilde{\varPhi}(t)\Big\|_\delta^2 + \Big\|\mathrm{D}_x^\alpha \nabla\widetilde{N}(t)\Big\|_\delta^2 + \Big|\Big|\Big|\mathrm{D}_x^\alpha\nabla\varDelta\widetilde{\varPhi}\Big|\Big|\Big|_{t,\delta}^2\Big)$$
$$+\Big|\Big|\Big|\mathrm{D}_x^\alpha\nabla\partial_3\Big(\widetilde{\varPhi}-\widetilde{N}\Big)\Big|\Big|\Big|_{t,\delta}^2 \le \varepsilon C(t)\Big(\Big\|\varDelta\widetilde{\varPhi}(0)\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2 + \Big\|\nabla\widetilde{N}(0)\Big\|_{\tilde{S}_\delta^{|\alpha|}}^2 + \frac{\eta}{\delta} + \varepsilon\Big)$$
$$+\varepsilon C(t)\Big(\Big\|\nabla\widetilde{\varPhi}(0)\Big\|_\delta^2 + \Big\|\widetilde{N}(0)\Big\|_\delta^2 + \frac{\eta}{\delta} + \varepsilon\Big)\mathrm{e}^{cc_\star t}\Big\},$$

$$\Big\{\delta \in \mathbf{R}\Big|\ \Big|\Big|\Big|\partial_\tau\Big(\varDelta\widetilde{\varPhi}-\widetilde{N}\Big)\Big|\Big|\Big|_{t,\delta}^2 \le \varepsilon C(t)\Big(\Big\|\varDelta\widetilde{\varPhi}(0)\Big\|_{\tilde{S}_\delta^1}^2 + \Big\|\nabla\widetilde{N}(0)\Big\|_{\tilde{S}_\delta^1}^2 + \frac{\eta}{\delta} + \varepsilon\Big)$$
$$+\varepsilon C(t)\Big(\Big\|\nabla\widetilde{\varPhi}(0)\Big\|_\delta^2 + \Big\|\widetilde{N}(0)\Big\|_\delta^2 + \frac{\eta}{\delta} + \varepsilon\Big)\mathrm{e}^{cc_\star t}\Big\},$$

$$\Big\{\delta \in \mathbf{R}\Big|\ \Big|\Big|\Big|\partial_\tau\widetilde{N}\Big|\Big|\Big|_{t,\delta}^2 \le \varepsilon C(t)\Big(\Big\|\varDelta\widetilde{\varPhi}(0)\Big\|_\delta^2 + \Big\|\nabla\widetilde{N}(0)\Big\|_\delta^2 + \frac{\eta}{\delta} + \varepsilon\Big)$$
$$+\varepsilon C(t)\Big(\Big\|\nabla\widetilde{\varPhi}(0)\Big\|_\delta^2 + \Big\|\widetilde{N}(0)\Big\|_\delta^2 + \frac{\eta}{\delta} + \varepsilon\Big)\mathrm{e}^{cc_\star t}\Big\}$$

are relatively dense in $\mathbf{R}$ $(|\alpha| = 0, 1)$. *For any* $t \in [0, T]$

$$\big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\varPhi}(t)\big\|_{L^2(\omega)}^2 \le \big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\varPhi}(0)\big\|_{L^2(\omega)}^2\mathrm{e}^{-ct},$$

$$\int_0^t \big\|\mathrm{D}_{x'}^\alpha\varDelta\overline{\varPhi}(\tau)\big\|_{L^2(\omega)}\,\mathrm{d}\tau \le \big\|\mathrm{D}_{x'}^\alpha\nabla\overline{\varPhi}(0)\big\|_{L^2(\omega)}^2, \quad |\alpha| = 0, 1, 2,$$

$$\big\|\overline{N}(t)\big\|_{L^2(\omega)}^2 \le \big\|\overline{N}(0)\big\|_{L^2(\omega)}^2 + c_\star ct\big\|\nabla\overline{\varPhi}(0)\big\|_{L^2(\omega)}^2,$$

$$\big\|\mathrm{D}_{x'}^\alpha\overline{N}(t)\big\|_{L^2(\omega)}^2 \le \big\|\mathrm{D}_{x'}^\alpha\overline{N}(0)\big\|_{L^2(\omega)}^2 + c_\star ct\big\|\mathrm{D}_{x'}^\alpha\overline{\varPhi}(0)\big\|_{L^2(\omega)}^2, \quad |\alpha| = 1, 2, 3,$$

$$\Big\|\frac{\partial\mathrm{D}_{x'}^\alpha\overline{N}}{\partial t}(t)\Big\|_{L^2(\omega)}^2 \le c_\star\big\|\mathrm{D}_{x'}^\alpha\overline{\varPhi}(0)\big\|_{L^2(\omega)}^2 \quad |\alpha| = 0, 1, 2.$$

Here c *is a constant independent of* ε *and* δ, *but may depends on* $c_\star$.

From Lemma 4.1, it is easy to see that if the initial data $(\phi_0^\varepsilon, n_0^\varepsilon) \to (\phi_0^0, n_0^0)$ as $\varepsilon \to 0$ in $\widetilde{S}^3(\omega) \times \widetilde{S}^2(\omega)$, then $(\phi^\varepsilon, n^\varepsilon) \to (\phi^0, n^0)$ as $\varepsilon \to 0$ in $L^2(0, T; \widetilde{S}^4(\omega)) \times \widetilde{S}^{2,1}(\omega_T)$ and $\varDelta\phi^\varepsilon - n^\varepsilon \to \varDelta\phi^0 - n^0$ as $\varepsilon \to 0$ in $\widetilde{S}^{2,1}(\omega_T)$. Thus the proof of Theorem 1.3 is complete.

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