Mordell-Lang in positive characteristic

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ABSTRACT - We give a new proof of the Mordell-Lang conjecture in positive characteristic for finitely generated subgroups. We also make some progress towards the full Mordell-Lang conjecture in positive characteristic.

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1. Introduction

This article is concerned with the full Mordell-Lang conjecture in positive characteristic, which is the following statement:

Conjecture 1.1. Let L be an algebraically closed field of characteristic p>0. Let A be a semiabelian variety over L, let $X\subset A$ be an irreducible subvariety and let $\Gamma\subset A(L)$ be a subgroup of finite rank. If $X(L)\cap \Gamma$ is Zariski dense in X, then there exist a semiabelian variety B over $\bar{\mathbb{F}}_p$, a subvariety Y of B over $\bar{\mathbb{F}}_p$, a homomorphism $h: B_L \to A/\mathrm{Stab}_A(X)$ with finite kernel and an element $a\in (A/\mathrm{Stab}_A(X))(L)$ such that $X/\mathrm{Stab}_A(X)=h(Y)+a$.

Here $\operatorname{Stab}_A(X)$ denotes the translation stabilizer of X in A. We call irreducible subvarieties X satisfying the conclusion of the above conjecture *special*.

We develop a new approach to Conjecture 1.1. Using this approach we give a new algebro-geometric proof of

Theorem 1.2 (Hrushovski, [18]). Conjecture 1.1 holds in case the group Γ is finitely generated.

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This was first proved by Hrushovski in [18] using model-theoretic methods. An algebro-geometric proof of this result has previously been given by Rössler [24].

The general case of Conjecture 1.1 is still open. We show that in case A is an ordinary abelian variety it can be reduced to the following conjecture, which is essentially a special case of Conjecture 1.1 (c.f. Section 6):

Conjecture 1.3. Let L_0 be a field which is finitely generated over \mathbb{F}_p and let L_0^{per} be a perfect closure of L_0 . Let A be a semiabelian variety over L_0 and $X\subset A_{L_0^{\mathrm{per}}}$ an irreducible subvariety. If $X(L_0^{\mathrm{per}})$ is Zariski dense in $X_{L_0^{\mathrm{per}}}$, then a translate of X by an element of $A(L_0^{\mathrm{per}})$ is defined over L_0 .

We give an overview of the structure of this article. Since we will study $X \subset A$ as above through the completions $\hat{X} \subset \hat{A} \cong \operatorname{Spf}(R[[x_1,\ldots,x_n]])$ along the origin when both X and A are defined over the valuation ring R of a local field of positive characteristic, in Section 2 we provide a number of facts about closed formal subschemes of $\operatorname{Spf}(R[[x_1,\ldots,x_n]])$. In fact we study more generally closed subschemes of the formal spectrum of the mixed power series ring $R[[x_1,\ldots,x_n]]\langle y_1,\ldots,y_m\rangle$ since this seems to be the natural setting for these results.

In Section 3 we collect some facts about special subvarieties and give a criterion for subvarieties to be special (Theorem 3.10).

In Section 4 we set up our method. In Subsection 4.1 we collect some facts about so-called completely slope divisible p-divisible groups. These are p-divisible groups which possess a filtration such that on each graded piece, a power of the relative Frobenius coincides, up to an isomorphism, with a power of the multiplication-by-p morphism.

In Subsection 4.2, we construct our central tool, a certain Frobenius morphism F: Let R be the valuation ring of a local field K of positive characteristic, let k be the residue field of R, let \bar{K} be an algebraic closure of K and let $R^{\mathrm{per}} \subset \bar{K}$ be the perfection of R. Let A be a semiabelian scheme over R. Denote by \hat{A} the completion of A along the zero section of the special fiber. This is a p-divisible group over the formal spectrum $\mathrm{Spf}(R)$ of R. Assume that \hat{A} is completely slope divisible. Then the facts from Subsection 4.1 yield a canonical isomorphism $(\hat{A}_k)_{\mathrm{Spf}(R^{\mathrm{per}})} \cong \hat{A}_{\mathrm{Spf}(R^{\mathrm{per}})}$. By transfering the Frobenius endomorphism of \hat{A}_k with respect to the finite field k via this isomorphism, we obtain an endomorphism $F_{\hat{A}}$ of $\hat{A}_{\mathrm{Spf}(R^{\mathrm{per}})}$. Its significance lies in the following characterization of special subvarieties of $A_{\bar{K}}$, where for a subvariety \mathcal{X} of A containing the zero section we denote by $\hat{\mathcal{X}}$ its completion along the zero section of the special fiber:

THEOREM 1.4 (see Theorem 4.17). Let $X \subset A_K$ be an irreducible subvariety. Then the following are equivalent:

- (i) The subvariety $X_{\bar{K}}$ is special in $A_{\bar{K}}$.
- (ii) There exist $x \in X(\overline{K})$ and $n \ge 1$ such that the schematic closure \mathcal{X} of X x in \mathcal{A} satisfies $F_{\underline{A}}^n(\widehat{\mathcal{X}}) \subset \widehat{\mathcal{X}}$.

This can be considered as a formal analogue of the classification of subvarieties of semiabelian subvarieties which are invariant under an isogeny due to Pink and Rössler (c.f. Theorem 3.7). Finally, in Subsection 4.3, we show that given a finitely generated field L_0 and a semiabelian variety A over L_0 , one can always embed L_0 into a local field over which A up to isogeny can be spread out to a semiabelian scheme A as above.

In Section 5, using the methods from Section 4, we prove Theorem 1.2: First we reduce to the situation considered above which allows us to define F. Then using Theorem 1.4 we show that Theorem 1.2 follows from a variant of the following formal Mordell-Lang result:

THEOREM 1.5 (see Theorem 5.1). Let \mathscr{G} be a formal group over k which as a formal scheme is isomorphic to $\operatorname{Spf}(k[[x_1,\ldots,x_n]])$. Let $\Gamma\subset\mathscr{G}(R)$ a finitely generated subgroup. Let $\mathscr{X}\subset\mathscr{G}_{\operatorname{Spf}(R)}$ be a closed formal subscheme. If \mathscr{X} is the minimal closed formal subscheme containing $\mathscr{X}(R)\cap\Gamma$, then there exist closed formal subschemes $\mathscr{X}_1,\ldots,\mathscr{X}_m$ of $\mathscr{G}_{\operatorname{Spf}(R)}$ defined over a finite field extension of k and elements $\gamma_1,\ldots,\gamma_m\in\mathscr{G}(R)$ such that $\mathscr{X}=\cup_j\mathscr{X}_j+\gamma_j$.

Theorem 1.5 is proven by the same method which was used by Abramovich and Voloch in [2] to prove Theorem 1.2 in the case that the ambient semiabelian variety is defined over a finite field: First one reduces to the case that $\mathscr X$ is irreducible in a suitable sense. Then after a suitable translation one may assume that $\mathscr X(R) \cap p^i \Gamma$ is dense in $\mathscr X$ for all i. Denote by R^{p^i} the subring of R consisting of all p^i -th powers. Since $p^i \Gamma \subset \mathscr S(R^{p^i})$ it follows that $\mathscr X$ is defined over R^{p^i} for all $i \geqslant 0$. From this it follows that $\mathscr X$ is defined over R^{p^i} .

In Section 6 we show that in case A is an ordinary abelian variety Conjecture 1.1 can be reduced to Conjecture 1.3 by combining our method with a reduction due to Ghioca, Moosa and Scanlon. This depends crucially on the fact that in case A is ordinary, the endomorphism $F_{\hat{\mathcal{A}}}$ of $\hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ described above can already be defined over $\mathrm{Spf}(R)$. The argument proceeds similarly to the proof of Theorem 1.2 sketched above by reduction to an analogous statement (see Theorem 2.44) for formal group schemes.

Notation. We will frequently work over a local field K of characteristic p > 0. For such a K, we will always fix an algebraic closure \bar{K} which we endow with the unique extension of the valuation on K.

We will denote by R (resp. \bar{R}) the valuation ring of K (resp. \bar{K}), by m (resp. m) the maximal ideal of R (resp. \bar{R}) and by k (resp. \bar{k}) the residue field of R (resp. \bar{R}).

We will denote by \hat{K} the completion of K. By a complete overfield $K' \subset \hat{K}$ of K we mean a field which is complete with respect to the valuation induced from \hat{K} . The valuation ring of such a K' will be denoted R' and the formal scheme associated to R' equipped with the valuation topology will be denoted by $\mathrm{Spf}(R')$. We will denote by K^{per} the perfection of K inside K and by K^{per} the valuation ring of K^{per} .

For $i \ge 0$ we let K^{p^i} be the field consisting of p^i -th powers of elements of K and R^{p^i} the valuation ring of K^{p^i} .

By a subvariety of a scheme which is of finite type over some field we mean a geometrically reduced closed subscheme.

2. Formal Schemes

Let K be a local field of characteristic p > 0.

By an adic ring we mean the same as in [15, 7.1.9], that is a complete and separated topological ring whose topology is defined by an ideal J. We will also call such a ring a J-adic ring.

For R' the valuation ring of a complete overfield $K' \subset \hat{R}$ and $n, m \ge 0$ we denote by $C'_{n,m}$ the ring $R'[[x_1,\ldots,x_n]]\langle y_1,\ldots,y_m\rangle$ which consists of those power series $\sum_{I\in(\mathbb{Z}^{\geqslant 0})^n,J\in(\mathbb{Z}^{\geqslant 0})^m}a_{IJ}x^Iy^J$ with coefficients $a_{IJ}\in R'$ such that for each I the coefficients a_{IJ} converge to zero as J goes to infinity. We endow $C'_{n,m}$ with the topology defined by the ideal $J'_{n,m}$ generated by m and the variables x_1,\ldots,x_n . This makes $C'_{n,m}$ into an adic ring. For R'=R we let $C_{n,m}:=C'_{n,m}$ and $J_{n,m}:=J'_{n,m}$.

By formal schemes, we mean the same as in [15, Section 10]. In this section, we are concerned with affine formal schemes \mathcal{X} over $\operatorname{Spf}(R)$ defined by the following class of rings:

DEFINITION 2.1 (c.f. [19, Section 2.1] and [4, Section 1]). A topological R-algebra C is of formally finite type if it is adic and if for some ideal of definition J the quotients C/J^i are of finite type over R for all $i \ge 0$.

DEFINITION 2.2. We denote by AFS_R the full subcategory of the category of formal schemes over Spf(R) whose objects are the formal schemes of the form Spf(C) for C a topological R-algebra of formally finite type.

LEMMA 2.3 ([4, Lemma 1.2]). For a J-adic R-algebra C the following are equivalent:

- (i) The ring C is of formally finite type.
- (ii) The ring C/J^2 is finitely generated over R.
- (iii) The ring C is topologically isomorphic over R to a quotient of $C_{n,m}$ for some $n, m \ge 0$.

REMARK 2.4. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$. By the remark after [15, Definition 10.14.2] closed formal subschemes of \mathscr{X} correspond to ideals of C. Thus by Lemma 2.3 a formal scheme over $\operatorname{Spf}(R)$ is in AFS_R if and only if it admits a closed embedding into $\operatorname{Spf}(C_{n,m})$ for some $n,m \geqslant 0$.

DEFINITION 2.5 (c.f. [5, Def. III.2.8.1]). A continuous homomorphism $h: G \to G'$ of topological groups is strict if the quotient topology on the image of h induced from G coincides with the subspace topology induced from G'. A continuous homomorphism of topological rings or modules is strict if the underlying homomorphism of topological groups is strict.

The following summarizes properties of topological R-algebras of formally finite type:

Proposition 2.6. Let C and C' be topological R-algebras of formally finite type.

- (i) The Jacobson radical of C is an ideal of definition, in fact it is the largest ideal of definition. In particular there is a unique topology on the ring C which makes C into a topological R-algebra of formally finite type.
- (ii) Every homomorphism $C \to C'$ is continuous.
- (iii) Every homomorphism $C \rightarrow C'$ is strict.
- (iv) Each ideal of C is closed.
- (v) The ring $C \otimes_R K$ is Jacobson.
- (vi) For each maximal ideal \mathfrak{n} of $C \otimes_R K$ the quotient $(C \otimes_R K)/\mathfrak{n}$ is a finite field extension of K.

PROOF. For (i) and (ii) see [19, Lemma 2.1]. For (iii) and (iv) see [4, Lemma 1.1]. For (v) see [19, Proposition 2.16] and for (vi) see [19, Lemma 2.3]. \Box

We will also have to work with formal schemes $\mathscr{X}_{\mathrm{Spf}(R')}$ for $\mathscr{X} \in \mathrm{AFS}_R$ and R' the valuation ring of a complete overfield $K' \subset \hat{K}$ and with closed formal subschemes of such formal schemes. However, in [15] the notion of a formal subscheme is only defined for locally Noetherian formal schemes, and valuation rings R' as above are in general not Noetherian. Thus we make the following ad hoc definition:

DEFINITION 2.7. A morphism $\operatorname{Spf}(C) \to \operatorname{Spf}(C')$ of affine formal schemes is a closed embedding if the corresponding homomorphism $C' \to C$ is surjective and strict. In this case we will say that $\operatorname{Spf}(C)$ is a closed formal subscheme of $\operatorname{Spf}(C')$.

Thus closed formal subschemes of Spf(C') correspond to closed ideals of C'. In case C' is Noetherian, this definition coincides with the one from [15] by the remark after [15, Definition 10.14.2].

DEFINITION 2.8. Let R' be the valuation ring of a complete overfield $K' \subset \tilde{R}$. We denote by AFS_{R'} the full subcategory of the category of formal schemes over $\mathrm{Spf}(R')$ whose objects are those affine formal schemes which admit a closed embedding into $\mathrm{Spf}(C'_{n,m})$ for some $n,m \ge 0$.

LEMMA 2.9. Let C be a Noetherian J-adic ring and C' a J'-adic ring. Let $C \to C'$ be a ring homomorphism such that $JC' \subset J'$ and such that for each $i \geqslant 0$ the induced homomorphism $C/J^i \to C'/(J')^i$ is faithfully flat.

Let $0 \to M' \to M \to M'' \to 0$ a sequence of finitely generated C-modules. We endow these modules with the J-adic topology. Then the sequence $0 \to M' \to M \to M'' \to 0$ is exact if and only if $0 \to M' \hat{\otimes}_C C' \to M \hat{\otimes}_C C' \to M'' \hat{\otimes}_C C' \to 0$ is exact. If this holds, the homomorphism $M \hat{\otimes}_C C' \to M'' \hat{\otimes}_C C'$ is strict.

PROOF. Assume that $0 \to M' \to M \to M'' \to 0$ is exact.

For $i \geqslant 0$ let $M'_i := J^i M \cap M'$. Since C is Noetherian, by [6, Theorem III.3.2.2] the topology on M' defined by the M'_i is the J-adic topology. This together with the fact that $JC' \subset J'$ implies that $(M_i \otimes_C (J')^i C')_{i\geqslant 0}$ is a fundamental system of neighborhoods of the identity in $M' \otimes_C C'$. Thus $M' \hat{\otimes}_C C'$ can be written as

$$M' \hat{\otimes}_C C' = \varprojlim_i M' / M'_i \otimes_C C' / (J')^i.$$

For $i \ge 0$ there is an exact sequence $0 \to M'/M'_i \to M/J^iM \to M''/J^iM'' \to 0$ of C/J^i -modules. Since $C/J^i \to C'/(J')^i$ is flat, this induces an exact sequence

$$0 \to M'/M'_i \otimes_{C/J^i} C'/(J')^i \to M/J^i M \otimes_{C/J^i} C'/(J')^i \to M''/J^i M'' \otimes_{C/J^i} C'/(J')^i \to 0.$$

The transition morphisms $M'/M'_i \otimes_C C'/(J')^i \to M'/M'_{i-1} \otimes_C C'/(J')^{i-1}$ are surjective. By [3, Proposition 10.2] this surjectivity implies that the sequence $0 \to M' \hat{\otimes}_C C' \to M \hat{\otimes}_C C' \to M'' \hat{\otimes}_C C' \to 0$ obtained by taking the inverse limit of the above exact sequences is again exact.

To prove the other direction of the claim, by a direct verification it suffices to show that if M is a non-zero finitely generated C-module endowed with the J-adic topology, the ring $M \hat{\otimes}_C C'$ is non-zero. As above we can write $M \hat{\otimes}_C C'$ as $\varprojlim_i M/J^i M \otimes_{C/J^i} C'/(J')^i$ with surjective transition homomorphisms $M/J^i M \otimes_{C/J^i} C'/(J')^i \to M/J^{i-1} M \otimes_{C/J^{i-1}} C'/(J')^{i-1}$. As it is finitely generated over the complete Noetherian ring C, the module M is complete. Hence the modules $M/J^i M$ are non-zero. Thus by the faithful flatness of $C/J^i \to C'/(J')^i$ the modules $M/J^i M \otimes_{C/J^i} C'/(J')^i$ are non-zero. Hence the surjectivity of the transition morphisms implies that $M \hat{\otimes}_C C'$ is non-zero.

It remains to prove the claim about strictness. Since $JC' \subset J'$, the topology on $M \hat{\otimes}_C C'$ and $M' \hat{\otimes}_C C'$ is the J'-adic topology. Hence if the homomorphism $M \hat{\otimes}_C C' \to M' \hat{\otimes}_C C'$ is surjective, its strictness follows from the fact that $\pi((J')^i(M \hat{\otimes}_C C')) = (J')^i(M' \hat{\otimes}_C C')$.

LEMMA 2.10. Let R' be the valuation ring of a completely valued overfield $K' \subset \hat{K}$ of K. For $n, m, i \geqslant 0$ the ring homomorphism $C_{n,m}/J_{n,m}^i \to C'_{n,m}/(J'_{n,m})^i$ induced by the inclusion $C_{n,m} \hookrightarrow C'_{n,m}$ is faithfully flat.

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PROOF. The homomorphism in question is

$$(R/\mathfrak{m}^{i})[x_{1}, \dots, x_{n}, y_{1}, \dots, y_{n}]/(x_{1}, \dots, x_{n})^{i}$$

 $\to (R'/(\mathfrak{m}R')^{i})[x_{1}, \dots, x_{n}, y_{1}, \dots, y_{n}]/(x_{1}, \dots, x_{n})^{i}$
 $\cong R' \otimes_{R} ((R/\mathfrak{m}^{i})[x_{1}, \dots, x_{n}, y_{1}, \dots, y_{n}]/(x_{1}, \dots, x_{n})^{i}).$

Thus the claim follows from the faithful flatness of $R \to R'$.

LEMMA 2.11. Let $\mathscr{K} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$ and $\mathscr{K}' = \operatorname{Spf}(C')$ a closed formal subscheme of \mathscr{K} defined by an ideal I of C. Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. Then $\mathscr{K}'_{\operatorname{Spf}(R')}$ is the closed formal subscheme of $\mathscr{K}_{\operatorname{Spf}(R')}$ defined by the ideal $I(C \hat{\otimes}_R R')$ of $C \hat{\otimes}_R R'$. This ideal is equal to $I \hat{\otimes}_R R'$.

PROOF. Pick a strict continuous surjection $C_{n,m} \to C$ for some $n, m \geqslant 0$. Note that $M \hat{\otimes}_R R' = M \hat{\otimes}_{C_{n,m}} C'_{n,m}$ for any topological C-module M. By Proposition 2.6 the topology on C' is the same as the topology defined by $J_{n,m}C'$. Hence by Lemma 2.9 applied to $C_{n,m} \to C'_{n,m}$, which is possible by Lemma 2.10, there is an exact sequence $0 \to I \hat{\otimes}_R R' \to C \hat{\otimes}_R R' \to (C/I) \hat{\otimes}_R R' \to 0$ and the homomorphism $C \hat{\otimes}_R R' \to (C/I) \hat{\otimes}_R R'$ is strict. Thus \mathscr{L}' is the closed formal subscheme of $\mathscr{L}_{\mathrm{Spf}(R')}$ defined by the ideal $I \hat{\otimes}_R R'$ in $C \hat{\otimes}_R R'$. Since C is Noetherian, there is a surjective homomorphism of C-modules $C^{\oplus k} \to I$ for some $k \geqslant 0$. Again by Lemma 2.9 this induces a surjection $(C \hat{\otimes}_R R')^{\oplus k} \cong C^{\oplus k} \hat{\otimes}_R R' \to I \hat{\otimes}_R R'$ which implies $I \hat{\otimes}_R R' = I(C \hat{\otimes}_R R')$.

DEFINITION 2.12. Let $\mathscr{X} = \operatorname{Spf}(C)$ be an affine formal scheme and $\mathscr{X}_1, \ldots, \mathscr{X}_m$ be closed formal subschemes of \mathscr{X} defined by closed ideals I_1, \ldots, I_m of C. We say that \mathscr{X} is the union of the formal subschemes \mathscr{X}_i if the intersection of the ideals I_i is the zero ideal of C.

2.1 – Points over \bar{R}

DEFINITION 2.13. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$. We define $\mathscr{X}(R)$ to be the set of homomorphisms $C \to \overline{R}$ of R-algebras.

Lemma 2.14. Let C be a topological R-algebra of formally finite type and let $h: C \to \overline{R}$ be a homomorphism of R-algebras.

- (i) The homomorphism h factors through the valuation ring R' of a finite field extension $K' \subset \overline{K}$ of K.
- (ii) The homomorphism h is continous.

PROOF. (i): The homomorphism h induces a homomorphism $C \otimes_R K \to \bar{R} \otimes_R K \to \bar{K}$ with the last homomorphism given by multiplication. Its kernel is

a maximal ideal \mathfrak{n} of $C \otimes_R K$. By Proposition 2.6 the quotient $(C \otimes_R K)/\mathfrak{n}$ is a finite field extension of K. This implies (i).

(ii): By (i) it suffices to show that if R' is the valuation ring of a finite field extension $K' \subset \overline{K}$ of K any homomorphism $h: C \to R'$ of R-algebras is continuous. Since R' is of formally finite type this is a special case of Proposition 2.6 (ii).

Caution: The ring \bar{R} is not complete with respect to the valuation topology. Thus there is no formal scheme $\mathrm{Spf}(\bar{R})$ and $\mathscr{X}(\bar{R})$ cannot be considered as $\mathscr{X}(\mathrm{Spf}(\bar{R}))$. The set $\mathscr{X}(\bar{R})$ is also not the same as $\mathscr{X}(\mathrm{Spf}(\bar{R}))$.

Our interest in the set $\mathscr{X}(R)$ comes from the following situation, to which we will later apply the results of this section: Let \mathcal{A} be a semiabelian scheme of dimension g over R and $\mathcal{X} \subset \mathcal{A}$ a closed subscheme containing the zero section. Let s be the closed point of the zero section of s. Let s and s be the formal schemes associated to the completions of the local rings s0 and s0 with respect to their maximal ideals. Then s1 s2 Spf(s1 [[s1, . . . , s3]]) so that s2 s3 and s4 and s5 is naturally identified with the set points in s5 which map to 0 in the special fiber (c.f. Subsection 2.6).

Note that the formation of $\mathcal{X}(\bar{R})$ is functorial in \mathcal{X} .

For a finite field extension $K' \subset \bar{K}$ of K with valuation ring R' and $\mathscr{X} \in \operatorname{AFS}_R$ we denote by $\mathscr{X}(R')$ the set of homomorphisms $\mathscr{X} \to \operatorname{Spf}(R')$ over $\operatorname{Spf}(R)$. There is a natural inclusion $\mathscr{X}(R') \hookrightarrow \mathscr{X}(\bar{R})$ and Lemma 2.14 (i) implies:

LEMMA 2.15. Let $\mathscr{Z} \in AFS_R$. The set $\mathscr{Z}(\bar{R})$ is the union $\mathscr{Z}(\bar{R}) = \bigcup_{K'} \mathscr{Z}(R')$ where K' varies over all finite field extensions of K contained in \bar{K} .

REMARK 2.16. For $\mathscr{X} \in \operatorname{AFS}_R$, the set $\mathscr{X}(\bar{R})$ can be described more concretely as follows: For any $r_1, \ldots, r_n \in \bar{\mathfrak{n}}$ and $s_1, \ldots, s_n \in \bar{R}$ there exists a unique continous homomorphism of R-algebras $C_{n,m} \to \bar{R}$ which sends the x_i to r_i and the y_i to s_i and each homomorphism $C_{n,m} \to \bar{R}$ is of this form. Thus associating to an element $h \in \operatorname{Spf}(C_{n,m})(\bar{R})$ the images of the x_i and the y_i gives a bijection $\operatorname{Spf}(C_{n,m})(\bar{R}) \xrightarrow{\sim} \bar{\mathfrak{n}}^{\oplus n} \oplus \bar{R}^{\oplus m}$. Any closed subscheme \mathscr{X} of $\operatorname{Spf}(C_{n,m})$ is cut out by a family of formal power series $\{f_i \mid i \in I\} \subset C_{n,m}$. Each formal power series $f \in C_{n,m}$ induces a function $\bar{\mathfrak{n}}^{\oplus n} \oplus \bar{R}^{\oplus m} \to R$. The above bijection identifies $\mathscr{X}(\bar{R})$ with the set of points in $\bar{\mathfrak{n}}^{\oplus n} \oplus \bar{R}^{\oplus m}$ on which the f_i are zero.

DEFINITION 2.17. Let $\Gamma := \operatorname{Aut}_R(\bar{R}) \cong \operatorname{Aut}_K(\bar{K})$. For $\mathscr{K} \in \operatorname{AFS}_R$, we let Γ act on $\mathscr{K}(\bar{R})$ from the left by

$$\begin{split} \varGamma \times \mathscr{X}(\bar{R}) &\to \mathscr{X}(\bar{R}) \\ (\gamma, h) &\mapsto \gamma \cdot h \colon \varGamma (\mathscr{X}, \mathcal{O}_{\mathscr{X}}) \xrightarrow{h} \bar{R} \xrightarrow{\gamma} \bar{R}. \end{split}$$

Lemma 2.18. Let C be a topological R-algebra of formally finite type. Every homomorphism $h: C \to \overline{K}$ of R-algebras factors through \overline{R} .

PROOF. The homomorphism h induces a homomorphism $C \otimes_R K \to \overline{R} \otimes_R K \to \overline{K}$ with the last homomorphism given by multiplication. Its kernel is a maximal ideal \mathfrak{n} of $C \otimes_R K$. By Proposition 2.6 the quotient $(C \otimes_R K)/\mathfrak{n}$ is a finite field extension of K. Hence it suffices to consider homomorphism $h: C \to K'$ of R-algebras where K' is a finite overfield of K contained in \overline{K} . Assume that there is such an h whose image is not contained in R'. Then the image of $h \otimes_R R': C \otimes_R R' \to K' \otimes_R R' \to K'$ is also not contained in R'. Hence after replacing K by K' we may assume K' = K. Since the image of h contains R, one sees that it must be all of K. Hence the kernel of h is maximal. Let π be a uniformizer of R. Since π is topologically nilpotent in C and since by Proposition 2.6 the Jacobson radical J of C is an ideal of definition, the element π is contained in J and hence in the kernel of h. This contradicts $h(\pi) = \pi$.

For a ring C, we denote by Max(C) the set of maximal ideals of C.

PROPOSITION 2.19. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$. Let ψ be the map $\mathscr{X}(\bar{R}) \to \operatorname{Max}(C \otimes_R K)$ which associates to $h \in \mathscr{X}(\bar{R})$ the kernel of the induced homomorphism $h \otimes_R K : C \otimes_R K \to \bar{R} \otimes_R K \cong \bar{K}$.

- (i) The map ψ makes $\text{Max}(C \otimes_R K)$ into the set-theoretic quotient of $\mathscr{X}(\bar{R})$ by the action of Γ .
- (ii) Let $\mathscr{Y} = \operatorname{Spf}(C')$ be a closed formal subscheme of \mathscr{X} . Then there is a commutative diagram

$$\mathcal{Y}(\bar{R}) \xrightarrow{\psi'} \operatorname{Max}(C' \otimes_R K)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{X}(\bar{R}) \xrightarrow{\psi} \operatorname{Max}(C \otimes_R K)$$

in which ψ' is the analogue of ψ for \mathscr{Y} and $\operatorname{Max}(C' \otimes_R K) \to \operatorname{Max}(C \otimes_R K)$ is induced by the surjection $C \to C'$.

PROOF. (i): By Lemma 2.18 any homomorphism $C \otimes_R K \to \bar{K}$ of K-algebras maps C to \bar{R} . Hence the assignment $h \mapsto h \otimes_R K : C \otimes_R K \to \bar{K}$ gives a bijection between $\mathscr{X}(\bar{R})$ and the set of homomorphisms $\tilde{h}: C \otimes_R K \to \bar{K}$ of K-algebras.

The kernel of any such homomorphism \tilde{h} is a maximal ideal of $C \otimes_R K$. On the other hand, for any maximal ideal \mathfrak{n} of $C \otimes_R K$, the quotient $(C \otimes_R K)/\mathfrak{n}$ is a finite field extension K' of K by Proposition 2.6. Thus giving \tilde{h} as above amounts to giving a maximal ideal \mathfrak{n} of $C \otimes_R K$ and an embedding of $(C \otimes_R K)/\mathfrak{n}$ into \bar{K} over K. Any two such embeddings are conjugate under Γ .

By combining the above facts one gets (i).

(ii) follows by a direct verification.

DEFINITION 2.20. Let R' be the valuation ring of a completely valued overfield K' of K. Let $\mathscr{E} = \operatorname{Spf}(C)$ be an affine formal scheme over $\operatorname{Spf}(R')$.

- (i) The formal scheme \mathcal{X} is *reduced* if the ring C is reduced.
- (ii) The formal scheme \mathcal{L} is *flat over* R' if the ring C is flat over R'.
- (iii) We denote by $AFS_{R'}^{rf}$ the full subcategory of $AFS_{R'}$ whose objects are those formal schemes which are reduced and flat over R.

LEMMA 2.21. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$. Let \mathscr{X}^{rf} be the closed formal subscheme of \mathscr{X} defined by the ideal $\{c \in C \mid \exists n \geqslant 0 : (\operatorname{cm})^n = 0\}$. This formal scheme is reduced and flat over R and the natural map $\mathscr{X}^{\text{rf}}(\bar{R}) \to \mathscr{X}(\bar{R})$ is a bijection.

PROOF. Direct verification using the fact that an R-module is flat if and only if it has no int-torsion.

PROPOSITION 2.22. Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \operatorname{AFS}_R^{rf}$ be two closed formal subschemes of $\mathscr{X} \in \operatorname{AFS}_R$. If $\mathcal{Y}_1(\bar{R}) \subset \mathcal{Y}_2(\bar{R})$, then $\mathcal{Y}_1 \subset \mathcal{Y}_2$.

PROOF. Let C be the ring of global sections of $\mathscr X$ and I_1,I_2 the ideals defining $\mathscr Y_1,\mathscr Y_2$. We want to prove $I_2\subset I_1$. Let π be a uniformizer of R. The fact that the $\mathscr Y_i$ are flat over R means that π is not a zero-divisor in C/I_i . This implies that it is enough to prove $I_2\otimes_R K\subset I_1\otimes_R K$ inside $C\otimes_R K=C[1/\pi]$. Since by assumption the ideals I_i are radical, so are the ideals $I_i\otimes_R K$. Since by Proposition 2.6 the ring $C\otimes_R K$ is Jacobson it suffices to prove that each maximal ideal of $C\otimes_R K$ which contains $I_1\otimes_R K$ also contains $I_2\otimes_R K$. This follows from the fact that $\mathscr Y_1(\bar R)\subset \mathscr Y_2(\bar R)$ and Proposition 2.19.

For $i\!\geqslant\!0$, we endow the ring $\bar{R}/(\mathfrak{m}\bar{R})^i$ with the quotient topology induced from the valuation topology on \bar{R} , with respect to which it is adic. Hence there is a formal scheme $\mathrm{Spf}(\bar{R}/(\mathfrak{m}\bar{R})^i)$ and for $\mathscr{K}\in\mathrm{AFS}_R$ we denote by $\mathscr{K}(\bar{R}/(\mathfrak{m}\bar{R})^i)$ the set of morphisms $\mathrm{Spf}(\bar{R}/(\mathfrak{m}\bar{R})^i)\to\mathscr{K}$ over $\mathrm{Spf}(R)$. There is a natural map $\mathscr{K}(\bar{R})\to\mathscr{K}(\bar{R}/(\mathfrak{m}\bar{R})^i)$ for all $i\!\geqslant\!0$.

COROLLARY 2.23. Let $\mathscr{X} \in AFS_R$. The set $\mathscr{X}(\bar{R})$ is nonempty if and only if for all $i \geqslant 0$ the set $\mathscr{X}(\bar{R}/\mathfrak{m}^i\bar{R})$ is nonempty.

PROOF. The "only if" direction is clear. Conversely, assume that $\mathscr{X}(\bar{R})$ is empty. Let $\mathscr{X}^{\mathrm{rf}} \in \mathrm{AFS}^{\mathrm{rf}}_R$ be the closed formal subscheme given by Lemma 2.21. Since $\mathscr{X}^{\mathrm{rf}}(\bar{R}) = \mathscr{X}(\bar{R})$ is empty, it follows from Proposition 2.22 that $\mathscr{X}^{\mathrm{rf}}$ is the empty formal scheme. Hence, if we let $C := \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ and π is a uniformizer of R, it follows from the definition of $\mathscr{X}^{\mathrm{rf}}$ that for each $c \in C$ there exists $n \geqslant 0$ such that $(c\pi)^n = 0$. For c = 1 we get that there is an $n \geqslant 0$ such that the image of π^n in C is zero. Hence for all i > n there is no homomorphism $C \to \bar{R}/(\mathrm{nn}\bar{R})^i$ of R-algebras since for such i the image of π^n in $\bar{R}/(\mathrm{nn}\bar{R})^i$ is not zero. This proves the claim.

LEMMA 2.24. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R$ be be the union of closed subschemes $\mathscr{X}_1, \ldots, \mathscr{X}_m$. Then $\mathscr{X}(\bar{R}) = \bigcup_i \mathscr{X}_i(\bar{R})$.

PROOF. Let $I_1, \ldots, I_m \subset C$ be the ideals defining the \mathscr{X}_i . Let $h: C \to \overline{R}$. We want to show that $h(I_i) = 0$ for some i. If this is not the case we pick $0 \neq r_i \in h(I_i)$. Then the product r of the r_i is a non-zero element which lies in $h(I_i)$ for all i. For each i pick $c_i \in I_i$ such that $h(c_i) = r$. Then the product of the c_i lies in the intersection of the I_i which by assumption is zero. Thus by applying h to this product we get $r^m = 0$ and hence r = 0, which is a contradiction.

2.2 - Irreducibility

DEFINITION 2.25. Let R' be the valuation ring of a finite field extension K' of K. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}^{\operatorname{rf}}_{R'}$ be non-empty.

- The formal scheme \mathcal{X} is *irreducible* if and only if Spec(C) is irreducible.
- An irreducible component of \mathcal{S} is a maximal irreducible closed formal subscheme.
- The formal scheme \mathscr{X} is *geometrically irreducible* if and only if $\mathscr{X}_{R''}$ is irreducible for all valuation rings R'' of finite field extensions K'' of K'.

Note that the irreducible components of \mathscr{X} correspond to the irreducible components of $\operatorname{Spec}(C)$, that is to the minimal prime ideals of C. In particular there are finitely many such components. Also, since C is reduced, the intersection of all its minimal prime ideals is the zero ideal. Thus \mathscr{X} is the union of its irreducible components in the language of Definition 2.12.

PROPOSITION 2.26. Let $\mathscr{X} \in \operatorname{AFS}_R^{rf}$ be non-empty. Each irreducible component of \mathscr{X} is reduced and flat over R.

PROOF. Let $C := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. It suffices to show that each irreducible component of $\operatorname{Spec}(C)$ is reduced and flat over R. Reducedness is clear. As R is a discrete valuation ring, a scheme X over $\operatorname{Spec}(R)$ is flat if and only if its generic fiber is schematically dense in X. By a direct verification, if $\operatorname{Spec}(C)$ satisfies this condition, then so does any irreducible component of $\operatorname{Spec}(C)$.

LEMMA 2.27. Let $\mathscr{X} = \operatorname{Spf}(C) \in \operatorname{AFS}_R^{rf}$ be irreducible. If $\mathscr{X}_1, \ldots, \mathscr{X}_m$ are closed formal subschemes of \mathscr{X} such that \mathscr{X} is the union of the \mathscr{X}_i , then $\mathscr{X} = \mathscr{X}_i$ for some i.

PROOF. Let $I_1, \ldots, I_m \subset C$ be the ideals defining the \mathcal{X}_i . By assumption their intersection is zero and C is integral. If all I_i were non-zero, we could pick ele-

ments $0 \neq x_i \in I_i$ whose product would be zero. Thus one of the I_i is zero, which is what we wanted.

In [9, Section 7], de Jong gives a construction, due to Berthelot, of a "generic fiber" functor from AFS_R to the category of quasi-separated rigid analytic spaces over K. We denote this functor by $\mathscr{K} \mapsto \mathscr{K}^{\mathrm{rig}}$. It can be described on objects as follows: The formal scheme $\mathrm{Spf}(C_{n,m})$ is sent to the product of the open n-dimensional unit disc D_K^n over K and the closed m-dimensional unit disc B_K^m over K. A closed formal subscheme \mathscr{K} as above is cut out by a family of power series $\{f_i \mid i \in I\} \subset C_{n,m}$. These f_i induce global sections of $D_K^n \times B_K^m$, and $\mathscr{K}^{\mathrm{rig}}$ is the closed rigid analytic subspace of $D_K^m \times B_K^n$ cut out by these global sections.

In [7], Conrad introduces the notion of irreducibility of a quasi-separated rigid analytic space and that of an irreducible component of such a space. He also shows that this notion is well-behaved under the functor $\mathscr{X} \mapsto \mathscr{X}^{\text{rig}}$. The following is a slight reformulation of [7, Theorem 2.3.1]:

THEOREM 2.28. Let $\mathscr{Z} \in \operatorname{AFS}_R^{rf}$ be non-empty and $\mathscr{Z}_1, \ldots, \mathscr{Z}_m$ be the irreducible components of \mathscr{Z} . The closed rigid analytic subvarieties \mathscr{Z}_i^{rig} of \mathscr{Z}^{rig} are the irreducible components of \mathscr{Z}^{rig} .

PROOF. In fact, [7, Theorem 2.3.1] says the following: Let $C := \Gamma(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ and let \tilde{C} be the normalization of C, that is the integral closure of C in its total ring of fractions. Let I_1, \ldots, I_n be the preimages in C of the minimal prime ideals of \tilde{C} . Let $\mathscr{Y}_1, \ldots, \mathscr{Y}_n$ be the closed formal subschemes of \mathscr{X} defined by the ideals I_i . Then $\mathscr{Y}_1^{\mathrm{rig}}, \ldots, \mathscr{Y}_n^{\mathrm{rig}}$ are the irreducible components of $\mathscr{X}^{\mathrm{rig}}$.

Since \tilde{C} is the normalization of C, the preimages of the minimal prime ideals of \tilde{C} are exactly the minimal prime ideals of C. Thus Theorem 2.28 is a reformulation of [7, Theorem 2.3.1].

Corollary 2.29. A non-empty formal scheme $\mathscr{X} \in \operatorname{AFS}_R^{rf}$ is irreducible if and only if the rigid analytic space \mathscr{X}^{rig} is irreducible.

PROPOSITION 2.30. Let $\mathscr{X} \in \operatorname{AFS}_R^{rf}$ be non-empty. There exists a finite field extension $K' \subset \overline{K}$ of K with valuation ring R' such that the irreducible components of $\mathscr{X}_{\operatorname{Spf}(R')}$ are geometrically irreducible.

PROOF OF PROPOSITION 2.30. In [7, Section 3.4], Conrad calls a quasi-separated rigid analytic space X over K geometrically irreducible if for all completely valued overfields K' of K the rigid analytic space $X_{K'}$ is irreducible. By [7, Theorem 3.4.3], for any quasi-separated rigid analytic space X over K having finitely many irreducible components, there exists a finite field extension $K' \subset \overline{K}$ of K with valuation ring K' such that $\mathcal{X}_{K'}^{\text{rig}}$ has finitely many irreducible components which are geometrically irreducible. Using the compatibility of the functor $\mathcal{X} \mapsto \mathcal{X}^{\text{rig}}$ with base

change to finite extensions of K (c.f. [9, 7.2.6]) the claim follows from this result and Theorem 2.28.

2.2.1 - Formal Schematic Image

DEFINITION 2.31. Let $f: \mathscr{X} \to \mathscr{X}'$ be a morphism of affine formal schemes. We define the *formal schematic image* $f(\mathscr{X})$ of f to be the intersection of all closed formal subschemes of \mathscr{X}' through which f factors.

Thus $f(\mathscr{X})$ is the smallest closed formal subscheme of \mathscr{X}' through which f factors. If $\mathscr{X}=\mathrm{Spf}(C)$ and $\mathscr{X}'=\mathrm{Spf}(C')$ then the ideal corresponding to $f(\mathscr{X})$ is the kernel of the homomorphism $C'\to C$ corresponding to $\mathscr{X}\to\mathscr{X}'$.

LEMMA 2.32. Let \mathscr{G} , \mathscr{G}' be connected p-divisible groups over $\operatorname{Spf}(R)$ considered as formal schemes. Let $f:\mathscr{G}\to\mathscr{G}'$ be an isogeny and $\mathscr{Y}\subset\mathscr{G}'$ be a closed formal subscheme. Let $\mathscr{E}:=f^{-1}(\mathscr{Y}):=\mathscr{Y}\times_{\mathscr{G}'}\mathscr{G}$ be its preimage in \mathscr{G} . Then \mathscr{Y} is the formal schematic image of \mathscr{E} in \mathscr{G}' .

PROOF. Let $C := \Gamma(\mathscr{G}, \mathcal{O}_{\mathscr{G}})$ and $C' := \Gamma(\mathscr{G}', \mathcal{O}_{\mathscr{G}})$. By [14, Proposition 4.4], the rings C and C' are isomorphic to $R[[x_1, \ldots, x_n]]$ for some $n \ge 0$. First we want to show that $C' \to C$ is flat. Since $R[[x_1, \ldots, x_n]]$ is a regular local ring, by [12, Theorem 18.16] for this it suffices to show that $\dim(C') = \dim(C/\mathfrak{n}'C) + \dim(C')$ where \mathfrak{n}' is the maximal ideal of C'. This follows from the fact that $C/\mathfrak{n}'C$ is finite over k. Thus as a finite flat module over the local ring C', the ring C is finite free over C'.

Let $I' \subset C'$ be the ideal defining \mathscr{Y} . Then \mathscr{E} is the formal spectrum of $C'/I' \hat{\otimes}_{C'} C$. Since $C' \to C$ is finite free we have $C'/I' \hat{\otimes}_{C'} C = C'/I' \otimes_{C'} C \cong C/I'C$.

Now let \mathscr{Y}' be the formal schematic image of \mathscr{E} in \mathscr{G}' . It is contained in \mathscr{Y} . Thus it is defined by an ideal \tilde{I}' containing I'. Its preimage in \mathscr{E} must coincide with \mathscr{E} . Thus the induced homomorphism $C'/I'\otimes_{C'}C\to C'/\tilde{I}'\otimes_{C'}C$ is an isomorphism. Since $C'\to C$ is finite free this implies that $C'/I'\to C'/\tilde{I}'$ is an isomorphism. This means $\mathscr{Y}'=\mathscr{Y}$.

2.3 - Base Change and Formal Schematic Closure

PROPOSITION 2.33. Let $T \subset \mathfrak{M}^{\oplus n} \oplus R^{\oplus m}$ for some $n, m \geqslant 0$. Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. Let $I \subset C_{n,m}$ and $I' \subset C'_{n,m}$ be the ideals consisting of those power series which vanish at all elements of T. Then $IC'_{n,m} = I'$.

Proof. The ideal I is characterized by the left exact sequence

$$0 o I o C_{n,m} \overset{(\operatorname{ev}_t)_{t \in T}}{\longrightarrow} \prod_{t \in T} R$$

where for $t \in T$ we let $\operatorname{ev}_t: C_{n,m} \to R$ be the function given by evaluation at t.

Let $D \cong C_{n,m}/I$ be the image of $C_{n,m}$ in $\prod_{t \in T} R$ endowed with the quotient topology from $C_{n,m}$. This is a topological R-algebra of formally finite type by Proposition 2.6. Lemma 2.11 yields an exact sequence

$$(2.34) 0 \to IC'_{n,m} \to C'_{n,m} \to D \hat{\otimes}_R R' \to 0.$$

Let D_i be the kernel of $D \to \prod_{t \in T} R \to \prod_{t \in T} R/\mathfrak{m}^i$. We have $D = \varprojlim_i D/J^i_{n,m}D$ and the rings $D/J^i_{n,m}D$, being quotients of $C_{n,m}/J^i_{n,m}$, are Artinian. Furthermore $\cap_{i \geqslant 0} D_i = 0$. By [6, Prop. III.2.7.8] these facts imply that there exists a sequence $(j(i))_{i \geqslant 0}$ of positive integers going to infinity such that $D_i \subset J^{i(i)}_{n,m}D$. Since $J^i_{n,m}D \subset D_i$ this shows that the topology on D defined by the D_i is the $J_{n,m}$ -adic topology. Thus $D \hat{\otimes}_R R'$ can be written as $\lim_i D/D_i \otimes_R R'$.

The inclusion $D \hookrightarrow \prod_{t \in T} R$ induces injections $D/D_i \hookrightarrow \prod_{t \in T} R/\mathfrak{m}^i$ and hence injections $D/D_i \otimes_R R' \hookrightarrow (\prod_{t \in T} R/\mathfrak{m}^i) \otimes_R R'$. Hence we get a homomorphism

$$(2.35) \quad D \hat{\otimes}_R R' = \varprojlim_i (D/D_i \otimes_R R') \to \varprojlim_i ((\prod_{t \in T} R/\mathfrak{m}^i) \otimes_R R')$$

$$\to \varprojlim_i (\prod_{t \in T} R'/(\mathfrak{m}R')^i) \cong \prod_{t \in T} R'$$

and by a direct verification the composition of this homomorphism with the homomorphism $C'_{n,m} \to D \hat{\otimes}_R R'$ from (2.34) is the homomorphism which evaluates a power series at the elements of T.

We want to show that the homomorphism (2.35) is injective. That the first arrow in (2.35) is injective follows from the choice of the D_i , the flatness of $R \to R'$ and the left exactness of the inverse limit functor. By the left exactness of inverse limits, in order to prove injectivity of the second arrow in (2.35), it is enough to show that

$$(2.36) \qquad (\prod_{t \in T} R/\mathfrak{m}^i) \otimes_R R' \to \prod_{t \in T} R'/(\mathfrak{m}R')^i$$

is injective for all i. Since any element of $(\prod_{t\in T} R/\mathfrak{m}^i) \otimes_R R'$ is contained in the image of $(\prod_{t\in T} R/\mathfrak{m}^i) \otimes_R R''$ for the valuation ring R'' of a finite field extension $K'' \subset K'$ of K for this step we may assume that K' is finite over K. But then R' is finite free over R and hence (2.36) is even an isomorphism in this case.

Thus by combining (2.34) and (2.35) we get a left exact sequence

$$0 \to IC'_{n,m} \to C'_{n,m} \stackrel{(\text{ev}_t)_{t \in T}}{\longrightarrow} \prod_{t \in T} R'$$

which shows $IC'_{n,m} = I'$.

DEFINITION 2.37. Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. Let \mathscr{X} be an affine formal scheme over $\operatorname{Spf}(R')$ and $T \subset \mathscr{X}(R')$. We define the formal schematic closure of T in \mathscr{X} to be the intersection of all closed subschemes \mathscr{Y} of \mathscr{X} for which $T \subset \mathscr{Y}(R')$.

If \mathscr{E}' is the formal schematic closure of T, then we say that T is formal-schematically dense in \mathscr{E}' .

Thus the formal schematic closure of T is the smallest closed subscheme of $\mathscr X$ which contains T.

COROLLARY 2.38. Let $\mathscr{X} \in \operatorname{AFS}_R$ and let $T \subset \mathscr{X}(R)$. Let \mathscr{Y} be the formal schematic closure of T in \mathscr{X} . Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. Then $\mathscr{Y}_{\operatorname{Spf}(R')}$ is the formal schematic closure of $T \subset \mathscr{X}(R) \subset \mathscr{X}(R')$ inside $\mathscr{X}_{\operatorname{Spf}(R')}$.

PROOF. It suffices to prove this for $\mathscr{E} = \operatorname{Spf}(C_{n,m})$ in which case it is a reformulation of Proposition 2.33.

2.4 - Transporters

Let \mathscr{G} be an abelian group object in the category AFS_R. We will need the existence of (strict) transporters in \mathscr{G} :

Construction 2.39. Let $\mathscr{X}, \mathscr{Y} \subset \mathscr{G}$ be two closed formal subschemes. For $i \geqslant 0$ let \mathscr{G}_i be the i-th infinitesimal neighborhood of the zero section in \mathscr{G} and let $\mathscr{X}_i, \mathscr{Y}_i := \mathscr{G}_i \cap \mathscr{X}, \mathscr{G}_i \cap \mathscr{Y}$. These are finite schemes over R/\mathfrak{m}^i . The group structure on \mathscr{G} makes \mathscr{G}_i into a group scheme over R/\mathfrak{m}^i and \mathscr{X}_i and \mathscr{Y}_i are closed subschemes of \mathscr{G}_i . For $i \geqslant 0$ let $\mathrm{Trans}_{\mathscr{G}_i}(\mathscr{X}_i, \mathscr{Y}_i)$ be the strict transporter in \mathscr{G}_i , that is the closed subscheme of \mathscr{G}_i whose points are those points g of \mathscr{G}_i for which $\mathscr{X}_i + g = \mathscr{Y}_i$. It exists by [3, Exemple VI.6.4.2 e)]. Then for each i one has a decreasing sequence ($\mathrm{Trans}_{\mathscr{S}_{i+j}}(\mathscr{X}_{i+j}, \mathscr{Y}_{i+j}) \cap \mathscr{G}_i)_{j\geqslant 0}$ of subschemes of \mathscr{G}_i . By noetherianity this sequence stabilizes; let $\mathrm{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})_i$ be its eventual value. Then $\mathrm{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})_{i+1} \cap \mathscr{G}_i = \mathrm{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})_i$ for all i. Hence the inductive limit of these schemes is a closed formal subscheme $\mathrm{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})$ of \mathscr{G} .

For a formal scheme \mathcal{Z} over $\operatorname{Spf}(R)$ and an affine formal scheme $\operatorname{Spf}(C)$ over $\operatorname{Spf}(R)$ we denote by $\mathcal{Z}(C)$ the set of morphisms $\operatorname{Spf}(C) \to \mathcal{Z}$ over $\operatorname{Spf}(R)$.

PROPOSITION 2.40. Let \mathscr{X} , \mathscr{Y} be closed formal subschemes of \mathscr{G} and let $\operatorname{Spf}(C)$ be an affine formal scheme over $\operatorname{Spf}(R)$. Then

$$\operatorname{Trans}_{\mathscr{G}}(\mathscr{X},\mathscr{Y})(C) = \{g \in \mathscr{G}(C) \mid g + \mathscr{X}_{\operatorname{Spf}(C)} = \mathscr{Y}_{\operatorname{Spf}(C)}\}.$$

PROOF. Let J be an ideal of definition of C. Let $g:R[[x_1,\ldots,x_n]]\to C\in \mathscr{G}(C)$. Since g is continuous, there exists $N\geqslant 0$ such that $g(J_{n,0}^N)\subset J$. Then g induces homomorphisms $g_j:R[[x_1,\ldots,x_n]]/J_{n,0}^{jN}\to C/J^j$ for each $j\geqslant 1$. Denote as in

Construction 2.39 by $\mathscr{S}_j = \operatorname{Spec}(R[[x_1,\ldots,x_n]]/J_{n,0}^j)$ the j-th infinitesimal neighbourhood of the origin in \mathscr{S} . Then $g_j \in \mathscr{S}_{jN}(C/J^j) \subset \mathscr{S}(C/J^j)$ is the image of g in $\mathscr{S}(C/J^j)$. The construction of $\operatorname{Trans}_{\mathscr{S}}(\mathscr{S},\mathscr{Y})$ implies:

$$\begin{split} g \in \operatorname{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})(C) &\iff \forall j \geqslant 0 \colon g_j \in \operatorname{Trans}_{\mathscr{G}}(\mathscr{X}, \mathscr{Y})_{Nj}(C/J^j) \\ &\iff \forall j \geqslant 0 \; \exists N_j \geqslant Nj \; \forall i \geqslant N_j \colon g_j + (\mathscr{X}_i)_{C/J^j} = (\mathscr{Y}_i)_{C/J^j} \\ &\iff g + \mathscr{X}_{\operatorname{Spf}(C)} = \mathscr{Y}_{\operatorname{Spf}(C)}. \end{split}$$

2.5 - Descent

Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. Let $\mathscr{X} \in \operatorname{AFS}_R$ and $\mathscr{X}' \subset \mathscr{X}_{\operatorname{Spf}(R')}$ a closed formal subscheme. If there exists a closed formal subscheme \mathscr{X}'' of \mathscr{X} such that $\mathscr{X}''_{\operatorname{Spf}(R')} = \mathscr{X}'$ then it follows from Lemmas 2.9, 2.10 and 2.11 that such an \mathscr{X} is unique. In this case we will say that \mathscr{X}' is defined over R.

In the following we will always endow $k[[x_1, \ldots, x_n]]$ with the (x_1, \ldots, x_n) -adic topology. Let $\mathscr{E} = \operatorname{Spf}(C)$ be a formal scheme over $\operatorname{Spec}(k)$ which is isomorphic to $\operatorname{Spf}(k[[x_1, \ldots, x_n]])$.

LEMMA 2.41. Let \mathscr{X}' be a closed formal subscheme of \mathscr{X} defined by an ideal I of C. Then $\mathscr{X}'_{\mathrm{Spf}(R)}$ is the closed formal subscheme of $\mathscr{X}_{\mathrm{Spf}(R)}$ defined by the ideal $I(C \hat{\otimes}_k R)$ of $C \hat{\otimes}_k R$. This ideal is equal to $I \hat{\otimes}_k R$.

PROOF. Since the homomorphisms $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^i \to R[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^i$ are faithfully flat, this follows from Lemma 2.9 in the same way as Lemma 2.11.

Let $\mathscr{X}'\subset \mathscr{X}_{\mathrm{Spf}(R)}$ a closed formal subscheme. If there exists a closed formal subscheme \mathscr{X}'' of \mathscr{X} such that $\mathscr{X}''_{\mathrm{Spf}(R)}=\mathscr{X}'$ then it follows from Lemmas 2.9 and 2.41 that such an \mathscr{X} is unique. In this case we will say that \mathscr{X}' is defined over k.

Let $F: \mathcal{X} \to \mathcal{X}$ be the Frobenius endomorphism of \mathcal{X} with respect to k. If \mathcal{X}' is a closed formal subscheme of $\mathcal{X}_{\mathrm{Spf}(R)}$ defined by formal power series $\{f_i \mid i \in I\} \subset R[[x_1, \ldots, x_n]]$, then the formal schematic image $F(\mathcal{X})$ of \mathcal{X} under F is defined by the power series obtained by applying F to the coefficients of the f_i .

LEMMA 2.42. Let \mathscr{X}' be a closed formal subscheme of $\mathscr{X}_{\mathrm{Spf}(R)}$. If \mathscr{X}' is defined over R^{p^i} for all $i \geqslant 0$, then \mathscr{X}' is defined over k.

PROOF. We may assume that $\mathscr{X} = \operatorname{Spf}(k[[x_1,\ldots,x_n]])$. Let $i \geq 0$. As any element of R^{p^i} is congruent to an element of k modulo \mathfrak{m}^{p^i} , the fact that \mathscr{X}' can be defined over R^{p^i} implies that the intersection of \mathscr{X}' with $\operatorname{Spec}((R/\mathfrak{m}^{p^i})[x_1,\ldots,x_n]/R)$

 $(x_1,\ldots,x_n)^{p^i}$) $\subset \mathscr{X}$ can be defined over k. As \mathscr{X}' is the direct limit of these intersections, the claim follows by varying i.

Lemma 2.43. A closed formal subscheme \mathscr{X}' of $\mathscr{X}_{\mathrm{Spf}(R)}$ is defined over k if and only if $F(\mathscr{X}) = \mathscr{X}$.

PROOF. The "only if" direction is straightforward. For the other direction, the fact that $F^i(\mathscr{E}) = \mathscr{E}$ implies that \mathscr{E} is defined over R^{p^i} for all $i \ge 0$. Thus we can conclude using Lemma 2.42.

PROPOSITION 2.44. Let \mathscr{G} be a formal abelian group scheme over k which as a formal scheme is isomorphic to $\operatorname{Spf}(k[[x_1,\ldots,x_n]])$. Let $\mathscr{X}\subset \mathscr{G}_{\operatorname{Spf}(R)}$ be a closed formal subscheme. If for each $i\geqslant 0$ there exists a finite field extension $K'\subset \bar{K}$ of K and $g\in \mathscr{G}(R')$ such that $\mathscr{X}_{\operatorname{Spf}(R')}+g$ is defined over R^{p^i} , then there exists a finite field extension $K'\subset \bar{K}$ of K with valuation ring R' and $g\in \mathscr{G}(R')$ such that $\mathscr{X}_{\operatorname{Spf}(R')}+g$ is defined over k.

PROOF. We consider the closed formal subscheme $\operatorname{Trans}_{\mathscr{S}_{\operatorname{Spf}(R)}}(\mathscr{X},F(\mathscr{X}))\subset \mathscr{S}_{\operatorname{Spf}(R)}$ given by Construction 2.39. Let $i\geqslant 0$ and pick R' as in the claim together with $g\in \mathscr{G}(R')$ such that $\mathscr{K}_{\operatorname{Spf}(R')}+g$ is defined over R^{p^i} . Let $\pi\in R$ be a uniformizer. By identifying R with $k[[\pi]]$ and considering defining equations for $\mathscr{K}_{\operatorname{Spf}(R')}+g$ with coefficients from $k[[\pi^{p^i}]]$ one sees that $\mathscr{K}_{R'}+g\equiv F(\mathscr{K}_{R'}+g)\pmod{\pi^{p^i}}$. Thus $g-F(g)\in\operatorname{Trans}_{\mathscr{G}}(\mathscr{K},F(\mathscr{K}))(\bar{R}/(\operatorname{im}\bar{R})^i)$ by Proposition 2.40. Thus Proposition 2.23 implies that there exists $g'\in\operatorname{Trans}_{\mathscr{G}}(\mathscr{K},F(\mathscr{K}))(\bar{R})$.

The morphism $\mathscr{G} \to \mathscr{G}, g \mapsto g - F(g)$ is the identity on the tangent space at the identity of \mathscr{G} and thus is an isomorphism by [17, A.4.5]. Hence g' can be written as g'' - F(g'') for some $g'' \in \mathscr{G}(\bar{R})$. By Lemma 2.15 there exists R' as in the claim such that $g'' \in \operatorname{Trans}_{\mathscr{C}_{\operatorname{Spf}(R')}}(\mathscr{K}, F(\mathscr{K}))(R')$. The fact that $g' \in \operatorname{Trans}_{\mathscr{G}}(\mathscr{K}, F(\mathscr{K}))(R')$ translates to $\mathscr{K}_{\operatorname{Spf}(R')} + g'' = F(\mathscr{K}_{\operatorname{Spf}(R')} + g'')$. By Lemma 2.43 this shows that $\mathscr{K}_{\operatorname{Spf}(R')} + g''$ is defined over k.

2.6 – Formal schemes arising from schemes

Let \mathcal{X} be a scheme locally of finite type over R together with a k-valued point s: Spec $(k) \to \mathcal{X}$ of the special fiber of \mathcal{X} . We let $\hat{\mathcal{X}}$ be the completion of \mathcal{X} along the closed subscheme s. We denote by $\mathcal{O}_{\mathcal{X},s}$ the stalk of $\mathcal{O}_{\mathcal{X}}$ at the closed point in the image of s. Then $\hat{\mathcal{X}}$ is the formal spectrum of the completion $\hat{\mathcal{O}}_{\mathcal{X},s}$ of the local ring $\mathcal{O}_{\mathcal{X},s}$ with respect to its maximal ideal.

PROPOSITION 2.45. (i) If \mathcal{X} is smooth over R at s of relative dimension n the formal scheme $\hat{\mathcal{X}}$ is isomorphic to $\operatorname{Spf}(R[[x_1,\ldots,x_n]])$.

(ii) The formal scheme $\hat{\mathcal{X}}$ is in AFS_R.

- (iii) The set $\hat{\mathcal{X}}(\bar{R})$ can be naturally identified with the set of elements of $\mathcal{X}(\bar{R})$ which map the closed point of $\operatorname{Spec}(\bar{R})$ to the closed point in the image of s.
- PROOF. (i) The fact that \mathcal{X} is smooth at s implies (in fact is equivalent to) that there exist an affine neighbourhood U of s and an étale morphism $U \to \operatorname{Spec}(R[x_1,\ldots,x_n])$ over $\operatorname{Spec}(R)$ which maps the zero section of the special fiber of $\operatorname{Spec}(R[x_1,\ldots,x_n])$ to s (c.f. [26, Tag 054L]). Such an étale morphism induces a finite étale morphism $\hat{\mathcal{O}}_{\mathcal{X},s} \to R[[x_1,\ldots,x_n]]$ of R-algebras. Since $\hat{\mathcal{O}}_{\mathcal{X},s}$, being a complete local ring, is Henselian, the fact that both $\hat{\mathcal{O}}_{\mathcal{X},s}$ and $R[[x_1,\ldots,x_n]]$ have residue field k implies that this is an isomorphism.
- (ii) From a closed embedding of an affine neighborhood of s into $\operatorname{Spec}(R[x_1,\ldots,x_n])$ for some $n\geqslant 0$ one gets a closed embedding of $\hat{\mathcal{X}}$ into $\operatorname{Spf}(R[[x_1,\ldots,x_n]])$.
 - (iii) This follows from the fact that $\hat{\mathcal{X}} = \mathrm{Spf}(\hat{\mathcal{O}}_{\mathcal{X},s})$.

PROPOSITION 2.46. Let \mathcal{X} be a reduced scheme which is flat and of finite type over R and let $s: \operatorname{Spec}(k) \to \mathcal{X}$ a k-valued point of \mathcal{X} .

- (i) The formal scheme $\hat{\mathcal{X}}$ is reduced and flat over R.
- (ii) Assume that \mathcal{X} is integral and let \mathcal{Y} be an irreducible component of $\hat{\mathcal{X}}$. The set $\mathcal{Y}(\bar{R}) \subset \mathcal{X}(\bar{R})$ is schematically dense in $\mathcal{X}_{\bar{R}}$.
- PROOF. (i) Since \mathcal{X} is reduced, so is the local ring $\mathcal{O}_{\mathcal{X},s_0}$. The ring $\mathcal{O}_{\mathcal{X},s_0}$ is also excellent. Since the completion of any excellent reduced local ring is reduced (c.f. [16, 7.8.3]) the formal scheme $\hat{\mathcal{X}}$ is reduced. Flatness follows from the flatness of $R \to \mathcal{O}_{\mathcal{X},s_0}$ and the flatness of $\mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s}$.
- (ii) Let $\mathcal{Y} \subset \mathcal{X}$ be the schematic closure of $\mathcal{Y}(\bar{R}) \subset \mathcal{X}(\bar{R})$ and let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ be the sheaf of ideals defining \mathcal{Y} .

As $\mathcal{O}_{\mathcal{X},s}$ is a Noetherian local ring, the homomorphism $\mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ is faithfully flat and for any finitely generated $\mathcal{O}_{\mathcal{X},s}$ -module M, its completion with respect to the topology induced by the maximal ideal of $\mathcal{O}_{\mathcal{X},s}$ is isomorphic to $\hat{\mathcal{O}}_{\mathcal{X},s} \otimes_{\mathcal{O}_{\mathcal{X},s}} M$. By applying this to $M = \mathcal{O}_{\mathcal{Y},s}$ one sees that $\hat{\mathcal{Y}}$ is the formal closed subscheme of $\hat{\mathcal{X}}$ corresponding to the ideal $\mathcal{I}_s\hat{\mathcal{O}}_{\mathcal{X},s}$. Since by construction $\mathcal{Y}(\bar{R}) \subset \hat{\mathcal{Y}}(\bar{R})$ Proposition 2.22 implies $\mathcal{Y} \subset \hat{\mathcal{Y}}$. Thus $\mathcal{I}_s\hat{\mathcal{O}}_{\mathcal{X},s}$ is contained in a minimal prime ideal of $\hat{\mathcal{O}}_{\mathcal{X},s}$ and hence the rings $\hat{\mathcal{O}}_{\mathcal{X},s}$ and $\hat{\mathcal{O}}_{\mathcal{Y},s}$ have the same dimension.

Using the flatness of $\mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ and the fact that the maximal ideal of $\hat{\mathcal{O}}_{\mathcal{X},s}$ is generated by the image of the maximal ideal of $\mathcal{O}_{\mathcal{X},s}$, Theorem 10.10 of [12] implies that $\dim(\mathcal{O}_{\mathcal{X},s}) = \dim(\hat{\mathcal{O}}_{\mathcal{X},s})$. Analogously we get $\dim(\mathcal{O}_{\mathcal{Y},s}) = \dim(\hat{\mathcal{O}}_{\mathcal{Y},s})$. Thus $\dim(\mathcal{Y}) \geqslant \dim(\mathcal{O}_{\mathcal{Y},s}) = \dim(\hat{\mathcal{O}}_{\mathcal{Y},s}) = \dim(\hat{\mathcal{O}}_{\mathcal{X},s})$. Since \mathcal{X} is irreducible $\dim(\mathcal{O}_{\mathcal{X},s}) = \dim(\mathcal{X})$ and thus we get $\dim(\mathcal{Y}) = \dim(\mathcal{X})$ which using the irreducibility of \mathcal{X} implies $\mathcal{Y} = \mathcal{X}$.

Let \mathcal{X}' be a second scheme locally of finite type over R together with a k-valued point s' of its special fiber. For a morphism $h: \mathcal{X} \to \mathcal{X}'$ over R which maps s to s' we denote by \hat{h} the induced momorphism $\hat{\mathcal{X}} \to \hat{\mathcal{X}}'$ of formal schemes over $\operatorname{Spf}(R)$.

PROPOSITION 2.47. Let $\mathcal{X}, \mathcal{X}'$ be schemes locally of finite type over R. Let s, s' be k-valued points of $\mathcal{X}, \mathcal{X}'$ and let $f: \mathcal{X} \to \mathcal{X}'$ be morphism over R which maps s to s' and is flat at s. Let \mathcal{Y} be the formal schematic image of the induced morphism $\hat{f}: \hat{\mathcal{X}} \to \hat{\mathcal{X}}'$. Every irreducible component of \mathcal{Y} is an irreducible component of $\hat{\mathcal{X}}'$.

PROOF. By the assumption on f the induced homomorphism $\mathcal{O}_{\mathcal{X}',s'} \to \mathcal{O}_{\mathcal{X},s}$ is flat. First we want to prove that the induced homomorphism $\hat{\mathcal{O}}_{\mathcal{X}',s'} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ is also flat. Let \mathfrak{n}' be the maximal ideal of $\mathcal{O}_{\mathcal{X}',s'}$ and $\hat{\mathfrak{n}}'$ its completion, which is the maximal ideal of $\hat{\mathcal{O}}_{\mathcal{X}',s'}$. By the local criterion for flatness (see [12, Theorem 6.8]) is suffices to show that $\mathrm{Tor}_1^{\hat{\mathcal{O}}_{\mathcal{X}',s'}}(\hat{\mathcal{O}}_{\mathcal{X}',s'}/\hat{\mathfrak{n}}',\hat{\mathcal{O}}_{\mathcal{X},s})=0$. Note that since $\hat{\mathfrak{n}}'=\mathfrak{n}'\hat{\mathcal{O}}_{\mathcal{X}',s'}$ we have $\hat{\mathcal{O}}_{\mathcal{X}',s'}/\hat{\mathfrak{n}}'\cong \mathcal{O}_{\mathcal{X}',s'}/\hat{\mathfrak{n}}'\otimes_{\mathcal{O}_{\mathcal{X}',s'}}\hat{\mathcal{O}}_{\mathcal{X}',s'}$. Using this and the flatness of $\mathcal{O}_{\mathcal{X}',s'}\to\hat{\mathcal{O}}_{\mathcal{X}',s'}$ the Proposition 3.2.9 of [27] on flat base change for Tor says

$$\operatorname{Tor}_{1}^{\hat{\mathcal{O}}_{\mathcal{X}',s'}}(\hat{\mathcal{O}}_{\mathcal{X}',s'}/\hat{\mathfrak{n}}',\hat{\mathcal{O}}_{\mathcal{X},s}) = \operatorname{Tor}_{1}^{\mathcal{O}_{\mathcal{X}',s'}}(\mathcal{O}_{\mathcal{X}',s'}/\mathfrak{n}',\hat{\mathcal{O}}_{\mathcal{X},s}).$$

The second term of this equation is zero since the homomorphism $\mathcal{O}_{\mathcal{X}',s'} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ is flat, being the composition of the two flat homomorphisms $\mathcal{O}_{\mathcal{X}',s'} \to \mathcal{O}_{\mathcal{X},s}$ and $\mathcal{O}_{\mathcal{X},s} \to \hat{\mathcal{O}}_{\mathcal{X},s}$. Thus $\hat{\mathcal{O}}_{\mathcal{X},s'} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ is also flat.

That the homomorphism $\hat{\mathcal{O}}_{\mathcal{X}',s'} \to \hat{\mathcal{O}}_{\mathcal{X},s}$ is flat implies by [12, Lemma 10.11] that it has the going down property, that is for any prime ideal \mathfrak{p} of $\hat{\mathcal{O}}_{\mathcal{X},s}$ and any prime ideal $\mathfrak{q} \subset \hat{\mathcal{O}}_{\mathcal{X}',s'} \cap \mathfrak{p}$ there exists a prime ideal $\mathfrak{p}' \subset \mathfrak{p}$ of $\hat{\mathcal{O}}_{\mathcal{X},s}$ such that $\mathfrak{q} = \hat{\mathcal{O}}_{\mathcal{X}',s'} \cap \mathfrak{p}'$. By applying this to a minimal prime ideal \mathfrak{p} of $\hat{\mathcal{O}}_{\mathcal{X},s}$ one sees that its pullback $\hat{\mathcal{O}}_{\mathcal{X}',s'} \cap \mathfrak{p}$ is a minimal prime ideal in $\hat{\mathcal{O}}_{\mathcal{X}',s'}$.

This means that the formal schematic image of any irreducible component of $\hat{\mathcal{X}}$ is an irreducible component of $\hat{\mathcal{X}}'$. This implies the claim.

We will in particular apply the above to a smooth abelian group scheme \mathcal{A} over R with s the zero section of the special fiber. Then \mathcal{A} is a formal group scheme, which, as a formal scheme, is isomorphic to $\operatorname{Spf}(R[[x_1,\ldots,x_n]])$. Note that often one denotes by $\hat{\mathcal{A}}$ the completion of \mathcal{A} along its zero section. This is a formal scheme over $\operatorname{Spec}(R)$, and $\hat{\mathcal{A}}$ as we define it is the base change of this formal scheme to $\operatorname{Spf}(R)$ along the morphism $\operatorname{Spf}(R) \to \operatorname{Spec}(R)$ given by the identity homomorphism $R \to R$.

If $[p]: \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ is an epimorphism, then $\hat{\mathcal{A}}$ is a p-divisible group over $\mathrm{Spf}(R)$ (c.f. [20, Corollary 4.5]). More precisely, it follows from [loc. cit.] that in this case $\hat{\mathcal{A}}$ is equal to the pullback to $\mathrm{Spf}(R)$ of the connected part $\mathcal{A}[p^{\infty}]^{\circ}$ of the p-divisible group $\mathcal{A}[p^{\infty}]$ of \mathcal{A} .

For a closed subscheme \mathcal{X} of \mathcal{A} containing the zero section, we denote by $\hat{\mathcal{X}}$ its completion along the zero section of the special fiber.

3. Special Subvarieties

For any group scheme \mathcal{A} over a scheme S and any closed subscheme \mathcal{X} of \mathcal{A} we denote by $\operatorname{Stab}_{\mathcal{A}}(\mathcal{X})$ the functor which associates to any scheme S' over S the set $\{a \in \mathcal{A}(S') \mid a + \mathcal{X}_{S'} = \mathcal{X}_{S'}\}$ and acts on morphisms by pullbacks. In case S is a field or a valuation ring, which will be the only relevant cases for us, this functor is representable by a closed subscheme of \mathcal{A} , c.f. [11, Exp. VIII, Ex. 6.5(e)].

Let L be an algebraically closed field of characteristic p>0 and A a semiabelian variety over L.

DEFINITION 3.1. We call a subvariety X of A special if X is irreducible and there exist a semiabelian variety B over $\bar{\mathbb{F}}_p$, a subvariety Y of B over $\bar{\mathbb{F}}_p$, a homomorphism $h: B_L \to A/\mathrm{Stab}_A(X)$ with finite kernel and an element $a \in (A/\mathrm{Stab}_A(X))(L)$ such that $X/\mathrm{Stab}_A(X) = h(Y) + a$.

Remark 3.2. This notion of a special subvariety is equivalent to Hrushovski's [18] notion of a special subvariety, as can be shown using the following Lemma:

LEMMA 3.3. Let B be a semiabelian variety over L, let $X \subset A$ be an irreducible subvariety and let $f: A \to B$ be a homomorphism. If X is a special subvariety of A, then f(X) is a special subvariety of B. If f has finite kernel and f(X) is a special subvariety of B, then X is a special subvariety of A.

PROOF. The homomorphism f induces a homomorphism $\bar{f}:A/\mathrm{Stab}_A(X)\to B/\mathrm{Stab}_B(f(X))$.

If X is special, then there exist a semiabelian variety C over $\bar{\mathbb{F}}_p$, a subvariety Y of C over $\bar{\mathbb{F}}_p$, a homomorphism $h\colon C_L\to A/\mathrm{Stab}_A(X)$ with finite kernel and an element $a\in (A/\mathrm{Stab}_A(X))(L)$ such that $X/\mathrm{Stab}_A(X)=h(Y)+a$. Let D be the connected component of the identity of the kernel of $\bar{f}\circ h$ equipped with the reduced scheme structure. This is a semiabelian subvariety of B_L . Thus it is defined over $\bar{\mathbb{F}}_p$. The homomorphism $\bar{f}\circ h$ induces a homomorphism $h'\colon C/D\to B/\mathrm{Stab}_B(f(X))$ with finite kernel such that $f(X)/\mathrm{Stab}_B(f(X))=h'(Y/D)+\bar{f}(a)$. Thus f(X) is special in B.

Now let f have finite kernel and f(X) be special in B. After replacing B by the image of f, we may assume that f is an isogeny. Then \bar{f} is also an isogeny. Pick an isogeny $g: B/\operatorname{Stab}_B(f(X)) \to A/\operatorname{Stab}_A(X)$ such that $g \circ \bar{f} = [n]$ for some $n \in \mathbb{Z}^{\neq 0}$. There exist a semiabelian variety C over $\bar{\mathbb{F}}_p$, a subvariety Y of C over $\bar{\mathbb{F}}_p$, a homomorphism $h: C_L \to B/\operatorname{Stab}_X(f(X))$ with finite kernel and an element $b \in (B/\operatorname{Stab}_B(f(X)))(L)$ such that $f(X)/\operatorname{Stab}_B(f(X)) = h(Y) + b$. Then $[n](X/\operatorname{Stab}_A(X)) = g(h(Y)) + g(b)$. Let Y' be an irreducible component of $[n]^{-1}(Y)$. Then g(h(Y')) + g(b) is an irreducible component of $[n]^{-1}([n](X/\operatorname{Stab}_A(X)))$. Since the n-torsion points of $(A/\operatorname{Stab}_A(X))(L)$ act

transitively on the irreducible components of $[n]^{-1}([n](X/\operatorname{Stab}_A(X)))$, it follows that $X/\operatorname{Stab}_A(X)$ is a translate of g(h(Y')). This shows that X is special in A.

LEMMA 3.4. Let L' be an algebraically closed overfield of L and $X \subset A$ a subvariety. Then X is special in A if and only if $X_{L'}$ is special in $A_{L'}$.

PROOF. The "only if" direction follows directly from the definition of a special subvariety. The other direction is shown in the proof of [24, Lemma 3.4]. For the sake of completeness, we sketch the argument: Let B, Y and $h:B_{L'} \rightarrow (A/\operatorname{Stab}_A(X))_{L'} = A_{L'}/\operatorname{Stab}_{A_{L'}}(X_{L'})$ as in Definition 3.1. Then by a rigidity property of morphisms of semiabelian varieties the morphism h is defined over L and since the points a satisfying the condition of Definition 3.1 form a closed subscheme of $A/\operatorname{Stab}_A(X)$, it follows from Hilbert's Nullstellensatz that such an a already exists over L.

LEMMA 3.5. Let L_0 be an arbitrary field, let A_0 be a semiabelian variety over L_0 and let $X_0 \subset A_0$ be an irreducible subvariety. Let $\Gamma \subset A_0(L_0)$ be a finitely generated subgroup such that $X_0(L_0) \cap \Gamma$ is Zariski dense in X_0 . For any semiabelian variety A'_0 over L_0 which is isogenous to A_0 , there exist a finitely generated subgroup Γ' of $A'_0(L_0)$ and an irreducible subvariety X'_0 of A'_0 such that $X'_0(L_0) \cap \Gamma'$ is Zariski dense in X'_0 and such that X_0 is special in A_0 , if and only if X'_0 is special in A_0 .

PROOF. Let $f: A_0 \to A_0'$ be an isogeny. Take $\Gamma' := f(\Gamma)$ and $X_0' := f(X_0)$. Lemma 3.3 implies that Γ' and X_0' have the required properties.

For an extension $L\hookrightarrow L'$ of algebraically closed fields, recall the notion, due to Chow, of the L'/L-trace of an abelian variety A over L' (c.f. [8]): This is an abelian variety over L, which we will denote $\mathrm{Tr}_{L'/L}A$, together with a homomorphism $\tau\colon (\mathrm{Tr}_{L'/L}A)_{L'}\to A$ which satisfies the following universal property: For each abelian variety B over L together with a homomorphism $f\colon B_{L'}\to A$, there exists a unique homomorphism $g\colon B\to \mathrm{Tr}_{L'/L}A$ (defined over L) such that $f=\tau\circ g_{L'}$. For $L\subset L'$ algebraically closed this trace always exists (see [8, 6.2]) and the map τ has finite kernel (see [8, 6.4]). Thus roughly speaking $\mathrm{Tr}_{L'/L}A$ is the largest subobject of A which can be defined over L. It is determined up to unique isomorphism and functorial in A.

In case A is abelian, using the trace of A one can describe special subvarieties as follows:

Lemma 3.6. If A is an abelian variety over L, an irreducible subvariety $X \subset A$ is special in A if and only if there exists a closed subvariety $Y \subset \mathrm{Tr}_{L/\bar{\mathbb{F}}_p} A$ defined over $\bar{\mathbb{F}}_p$ and $a \in A(L)$ such that $X + a = \tau(Y) + \mathrm{Stab}_X(A)$.

PROOF. Because the formation of $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A$ is functorial in A, there is a commutative diagram

$$\begin{array}{cccc} \operatorname{Tr}_{L/\bar{\mathbb{F}}_p} A & \xrightarrow{\tau} & A \\ & & \downarrow & & \downarrow \\ & \operatorname{Tr}_{L/\bar{\mathbb{F}}_p} (A/\operatorname{Stab}_A(X)) \xrightarrow{\tau'} & A/\operatorname{Stab}_A(X) \end{array}$$

with both abelian varieties on the left as well as the morphism between them defined over $\bar{\mathbb{F}}_p$ and with τ and τ' having finite kernel. The "if" direction follows directly from this. For the "only if" direction, we assume that X is special. Then there exist $Y \subset B$, h and a as in Definition 3.1. Since the homomorphism h factors through τ' , we may replace Y by its image in $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}(A/\mathrm{Stab}_A(X))$ and assume $B = \mathrm{Tr}_{L/\bar{\mathbb{F}}_p}(A/\mathrm{Stab}_A(X))$. Let Y' be the inverse image of Y in $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A$. Note that Y' is defined over $\bar{\mathbb{F}}_p$. If $a' \in A(L)$ is a lift of a the identity $\tau'(Y) = X/\mathrm{Stab}_A(X) + a$ implies $\tau(Y') + \mathrm{Stab}_A(X) = X + a'$.

Specialness Criteria

THEOREM 3.7 (Pink-Rössler, see [23, Theorem 3.1]). Let $\varphi: A \to A$ be an isogeny. Let $X \subset A$ be an irreducible subvariety such that $\varphi(X) = X + a$ for some $a \in A(L)$. Then $\varphi(\operatorname{Stab}_A(X)) = \operatorname{Stab}_A(X)$ and we denote the isogeny $A/\operatorname{Stab}_A(X) \to A/\operatorname{Stab}_A(X)$ induced by φ by $\bar{\varphi}$.

There exist finitely many homomorphisms $h_{\alpha}: A_{\alpha} \to A/\operatorname{Stab}_{A}(X)$ for certain $\alpha \in \mathbb{Q}^{\geqslant 0}$, where the A_{α} are semiabelian varieties endowed with isogenies $\varphi_{\alpha}: A_{\alpha} \to A_{\alpha}$ satisfying $\bar{\varphi} \circ h_{\alpha} = h_{\alpha} \circ \varphi_{\alpha}$ and irreducible subvarieties $X_{\alpha} \subset A_{\alpha}$ satisfying $\varphi_{\alpha}(X_{\alpha}) = X_{\alpha} + a_{\alpha}$ for some $a_{\alpha} \in A_{\alpha}(L)$ such that:

- If $\alpha = 0$, then φ_{α} is an automorphisms of finite order of A_{α} .
- If $\alpha > 0$, then there exist positive integers r and s such that $\alpha = r/s$ and $\varphi_{\alpha}^{s} = \operatorname{Frob}_{p^{r}}$ for some model of A_{α} over $\mathbb{F}_{p^{r}}$.
- The morphism

$$h\!:=\sum_{lpha}h_{lpha}\!:\prod_{i}A_{lpha} o A/\mathrm{Stab}_{A}(X)$$

has finite kernel and, for some point $\bar{a} \in (A/\operatorname{Stab}_A(X))(L)$,

$$X/\mathrm{Stab}_A(X) = \bar{a} + h(\prod_i X_\alpha).$$

We will use the following consequence of Theorem 3.7:

COROLLARY 3.8. Let $\varphi: A \to A$ be an isogeny whose minimal polynomial does not have any complex roots which are roots of unity. Let $X \subset A$ be an irreducible subvariety such that $\varphi(X) = X + a$ for some $a \in A(L)$. Then X is a special subvariety of A.

PROOF. The condition on the minimal polynomial of φ ensures that φ cannot act as an automorphism of finite order on any subquotient of A. Hence the term A_0 in Theorem 3.7 does not appear, and it follows using Lemma 3.3 that X is special in A.

DEFINITION 3.9. We call a polynomial $f \in \mathbb{Z}[t]$ *good* if it is monic, if $f(0) \neq 0$ and if no complex root of f is a root of unity.

THEOREM 3.10. Let $G \subset A(L)$ be a subgroup and $\Phi: G \to G$ an endomorphism such that there exists a good polynomial $f \in \mathbb{Z}[t]$ which annihilates Φ .

Let $X \subset A$ be an irreducible subvariety. If there exists a subset T of $G \cap X(L)$ which is Zariski dense in X and which satisfies $\Phi(T) \subset T$, then X is special in A.

PROOF. Write $f(t) = t^n + \sum_{i=0}^{n-1} a_i t^i$ with $a_i \in \mathbb{Z}$. Let φ be the endomorphism of A^n defined by the matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ 0 & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 1 & -a_{n-1} \end{bmatrix},$$

which satisfies $f(\varphi)=0$ and $\varphi(a,\Phi(a),\ldots,\Phi^{n-1}(a))=(\Phi(a),\Phi^2(a),\ldots,\Phi^n(a))$ for $a\in G$. Let X' be the Zariski closure of the set $\{(x,\Phi(x),\ldots,\Phi^{n-1}(x))\mid x\in T\}$ in A^n . The fact that $\Phi(T)\subset T$ implies that $\varphi(X')\subset X'$. Since $a_0=f(0)\neq 0$, the above matrix is invertible over $\mathbb Q$ and hence φ is an isogeny. Hence for each irreducible component Z of X', its image $\varphi(Z)$ is also an irreducible component of X'. Thus there exist $m_1>m_2\geqslant 0$ such that $\varphi^{\circ m_1}(Z)=\varphi^{\circ m_2}(Z)$. Hence $\varphi^{\circ m_2}(Z)$ is special in A^n by the assumption on f and Corollary 3.8. Thus by Lemma 3.3 each irreducible component of X' is special in A^n .

Let $\pi: A^n \to A$ be the projection to the first factor. The fact that T is Zariski dense in X implies $\pi(X') = X$. Since X is irreducible, some irreducible component of X' maps onto X under π . Hence Lemma 3.3 implies that X is special in A.

4. The General Setup

4.1 - Completely slope divisible p-divisible groups

First we collect some terminology and facts from [22].

Let S be a scheme over \mathbb{F}_p . Let $\operatorname{Frob}: S \to S$ be the absolute Frobenius morphism $x \mapsto x^p$. For a scheme G over S and $s \geqslant 1$ we write $G^{(p^s)} = G \times_{S,\operatorname{Frob}^s} S$. We denote by $F^s: G \to G^{(p^s)}$ the Frobenius morphism relative to S.

Let L be an algebraically closed field of characteristic p > 0 and \mathscr{G} a p-divisible group over L. For a rational number $\lambda \geqslant 0$, one calls \mathscr{G} isoclinic of slope λ if there exist integers $r \geqslant 0$ and s > 0 such that $\lambda = r/s$ and a p-divisible group \mathscr{G}' over L which is isogenous to \mathscr{G} and for which there exists an isomorphism $\psi \colon \mathscr{G}^{(p^s)} \to \mathscr{G}$ making the following diagram commute (c.f. [22, Section 1]):



Every p-divisible group \mathscr{S} over L is isogenous to a direct sum of isoclinic p-divisible group and the slopes appearing in such a direct sum determine \mathscr{S} up to isogeny. They are called the slopes of \mathscr{S} and are assembled into the Newton polyon of \mathscr{S} (see e.g. [10, IV.5]).

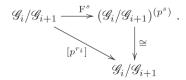
DEFINITION 4.1 (c.f. [22, Definition 1.2]). • Let $s \ge 1$ and r_1, \ldots, r_m be integers such that $s \ge r_1 > r_2 > \ldots > r_m \ge 0$. A *p*-divisible group $\mathscr G$ over a scheme S is said to be *completely slope divisible* with respect to these integers if $\mathscr G$ has a filtration by *p*-divisible subgroups

$$0 = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots \subset \mathcal{G}_m = \mathcal{G}$$

such that the following properties hold:

- (i) For $i=1,\ldots,m$ the kernel of $[p^{r_i}]:\mathscr{G}_i\to\mathscr{G}_i$ is contained in the kernel of $F^s\colon\mathscr{G}_i\to\mathscr{G}_i^{(p^s)}$.
- (ii) For $i=1,\ldots,m$ the kernel of $[p^{r_i}]: \mathscr{G}_i/\mathscr{G}_{i-1} \to \mathscr{G}_i/\mathscr{G}_{i-1}$ is equal to the kernel of $F^s: \mathscr{G}_i/\mathscr{G}_{i-1} \to (\mathscr{G}_i/\mathscr{G}_{i-1})^{(p^s)}$.
- A *p*-divisible group \mathscr{G} is completely slope divisible if there exist integers $s \ge r_1 > r_2 > \ldots > r_m \ge 0$ such that \mathscr{G} is completely slope divisible with respect to these integers.

REMARK 4.2. Let \mathscr{G} be a p-divisible group which is completely slope divisible with respect to $s \ge r_1 > r_2 > \ldots > r_m \ge 0$. Note that condition (ii) is equivalent to the existence of isomorphisms $(\mathscr{G}_i/\mathscr{G}_{i+1})^{(p^s)} \xrightarrow{\sim} \mathscr{G}_i/\mathscr{G}_{i+1}$ such that the following diagram commutes:



Thus all geometric fibers of the subquotients $\mathcal{G}_i/\mathcal{G}_{i+1}$ are isoclinic of slope r_i/s and in particular the Newton polygon of \mathcal{G} is constant on S.

REMARK 4.3. Let \mathscr{G} be a p-divisible group which is completely slope divisible with respect to $s \ge r_1 > r_2 > \ldots > r_m \ge 0$. By the remark after Definition 1.2 of [22] there exists a unique filtration $(\mathscr{G}_i)_{i=0,\ldots,m}$ satisfying the conditions above for the given integers $s \ge r_1 > r_2 > \ldots > r_m \ge 0$.

Theorem 4.4 ([22, Theorem 2.1]). Let \mathcal{G} be a p-divisible group over a integral normal Noetherian scheme S with constant Newton polygon. There exists a completely slope divisible p-divisible group over S which is isogenous to \mathcal{G} .

We call a scheme S of characteristic p > 0 perfect if for each open set U of S the endomorphism $x \mapsto x^p$ of the ring $\mathcal{O}_S(U)$ is an isomorphism.

PROPOSITION 4.5 [Oort-Zink]. Let \mathscr{G} be a p-divisible group over a perfect scheme S which is completely slope divisible with respect to integers $s \geqslant r_1 > r_2 > \ldots > r_m \geqslant 0$. Let $(\mathscr{G}_i)_i$ be a filtration as in Definition 4.1 with respect to these integers. The filtration $(\mathscr{G}_i)_i$ splits uniquely, that is there are unique sections $\mathscr{G}_i/\mathscr{G}_{i+1} \to \mathscr{G}_i$ of the quotient maps $\mathscr{G}_i \to \mathscr{G}_i/\mathscr{G}_{i+1}$.

PROOF. This is [22, Proposition 1.3]. Although the uniqueness of the splittings is not part of the statement there, it is shown in the proof given there. \Box

PROPOSITION 4.6. Let \mathcal{G} be a p-divisible group over a scheme S which is completely slope divisible. Let \mathcal{H} be a p-divisible subgroup of \mathcal{G} . Then \mathcal{H} and \mathcal{G}/\mathcal{H} are completely slope divisible.

PROOF. Let (\mathscr{G}_i) be a filtration as in Definition 4.1. Let $\mathscr{H}_i := \mathscr{H} \cap \mathscr{G}_i$. Then it follows by a direct verification that (\mathscr{H}_i) and $(\mathscr{G}_i/\mathscr{H}_i)$ have the required properties.

LEMMA 4.7 (see [22, Corollary 1.10]). Let $G \to S$ be a finite flat group scheme over a connected base scheme S. Let $\psi: G \xrightarrow{\sim} G^{(p^s)}$ be an isomorphism. Then there exists a finite étale morphism $T \to S$ and a morphism $T \to \operatorname{Spec}(\mathbb{F}_{p^s})$ such that G_T is obtained by base change from a finite group scheme H over $\operatorname{Spec}(\mathbb{F}_{p^s})$

$$H \times_{\operatorname{Spec}(\mathbb{F}_{p^s})} T \xrightarrow{\sim} G_T$$

and such that ψ_T is induced from the identity on H.

The argument in the proof of the following proposition is taken from the proof of Proposition 3.1 of [22].

PROPOSITION 4.8. Let R be a perfect strictly henselian local ring over $\bar{\mathbb{F}}_p$ and \mathscr{G} a completely slope divisible p-divisible group over R. Then there exists a p-divisible group \mathscr{G}_0 over $\bar{\mathbb{F}}_p$ such that $\mathscr{G}_{0,R}$ is isomorphic to \mathscr{G} . In case that \mathscr{G} has a single slope, it suffices that R be strictly henselian.

PROOF. In case $\mathscr G$ has multiple slopes, by Proposition 4.5 we can write $\mathscr G$ as a direct sum of completely slope divisible groups having a single slope. Thus it suffices to treat this case. Then there exist $s\geqslant r\geqslant 0$ and an isomorphism $\psi\colon \mathscr G^{(p^s)}\to\mathscr G$ such that $\psi\circ F^s=[p^r]$. For $n\geqslant 0$ denote by $\mathscr G(n)$ the kernel of $[p^n]\colon \mathscr G\to\mathscr G$. Applying Lemma 4.7 to $\mathscr G(n)$ and ψ^{-1} we obtain finite group schemes $\mathscr G_o(n)$ over $\operatorname{Spec}(\mathbb F_{p^s})$ and isomorphisms

$$\mathscr{G}(n) \cong \mathscr{G}_0(n) \times_{\operatorname{Spec}(\mathbb{F}_{n^s})} R.$$

The inductive limit of the group schemes $\mathscr{G}_0(n)$ is a p-divisible group \mathscr{G}_0 over \mathbb{F}_{p^s} which has the required property.

The following result is probably not new, but we could not find a reference.

Proposition 4.9. Let k be an algebraically closed field of positive characteristic. Let $\mathcal G$ and $\mathcal H$ be p-divisible groups over k. For any integral scheme S over k the base change map

$$\operatorname{Hom}_k(\mathcal{G}, \mathcal{H}) \to \operatorname{Hom}_S(\mathcal{G}_S, \mathcal{H}_S)$$

is a isomorphism of \mathbb{Z}_p -modules.

PROOF. Let k' be the function field of S. First we claim that for any finite flat group schemes G and H over S the natural map $\operatorname{Hom}_S(G,H) \to \operatorname{Hom}_{k'}(G_{k'},H_{k'})$ is injective: We may assume that S is affine, say $S = \operatorname{Spec}(R)$. Then G and H are the spectrum of finite flat R-algebra A_G and A_H . Since these are flat over R, the homomorphisms $A_G \to A_G \otimes_R k'$ and $A_H \to A_G \otimes_R k'$ are injective. Hence any homomorphism $A_H \to A_G$ of R-algebras is determined by its generic fiber $A_H \otimes_R k' \to A_G \otimes k'$. This shows that $\operatorname{Hom}_S(G,H) \to \operatorname{Hom}_{k'}(G_{k'},H_{k'})$ is injective. By applying this to group schemes $\mathscr{S}[p^n]$ and $\mathscr{M}[p^n]$ for $n \geqslant 0$ one gets that the homomorphism $\operatorname{Hom}_S(\mathscr{S}_S,\mathscr{M}_S) \to \operatorname{Hom}_{k'}(\mathscr{S}_{k'},\mathscr{M}_{k'})$ is injective. Thus we may assume that $S = \operatorname{Spec}(k')$. We may also assume that k' is algebraically closed.

We use the theory of Dieudonné modules. Denote by W(k) (resp. W(k')) the ring of Witt vectors of k (resp. k'), by σ the lift of Frobenius to these rings and by B(k) (resp. B(k')) their quotient field. Let $M(\mathcal{G})$ and $M(\mathcal{H})$ be the contravariant Dieudonné modules associated to \mathcal{G} and \mathcal{H} . They are free W(k)-modules endowed with a σ -linear self-map F and a σ^{-1} -linear self-map V.

A homomorphism $\mathscr{G}_{k'} \to \mathscr{H}_{k'}$ corresponds to a W(k)-linear homomorphism $M(\mathscr{H}_{k'}) = M(\mathscr{H}) \otimes_{W(k)} W(k') \to M(\mathscr{G}_{k'}) = M(\mathscr{G}) \otimes_{W(k)} W(k')$ compatible with V and F. We need to show that any such homomorphism arises from a homomorphism $M(\mathscr{H}) \to M(\mathscr{G})$. Since we are dealing with free W(k)-modules, it suffices to prove that the induced homomorphism $M(\mathscr{H}) \otimes_{W(k)} B(k') \to M(\mathscr{G}) \otimes_{W(k)} B(k')$ arises from a homomorphism $M(\mathscr{H}) \otimes_{W(k)} B(k) \to M(\mathscr{G}) \otimes_{W(k)} B(k)$. The B(k)-vector spaces $M(\mathscr{H}) \otimes_{W(k)} B(k')$ and $M(\mathscr{G}) \otimes_{W(k)} B(k')$ together with the σ -linear endomorphism induced by F are what is called an F-space in [10, Chapter IV]. By the

theorem in [10, Section IV.4], each such F-space is a direct sum of certain simple F-spaces denoted E_k^{λ} for $\lambda \in \mathbb{Q}^{\geq 0}$. Furthermore, by a proposition in [10, Section IV.3], if $\lambda \neq \lambda'$, any homomorphism $E_k^{\lambda} \to E_k^{\lambda'}$ of F-spaces is zero. Hence it suffices to prove that any endomorphism of $E_k^{\lambda} \otimes_{B(k)} B(k')$ of F-spaces arises from an endomorphism of E_k^{λ} . This follows from the description of such endomorphisms given by a proposition in [10, Section IV.3].

PROPOSITION 4.10. Let R be a discrete valuation ring of characteristic p with perfect residue field k and perfection $R \hookrightarrow R^{\mathrm{per}}$. Let $\mathscr G$ be a completely slope divisible group over R. There exists a unique isomorphism $\mathscr G_{R^{\mathrm{per}}} \cong (\mathscr G_k)_{R^{\mathrm{per}}}$ which is the identity in the fiber over k. In case $\mathscr G$ has a single slope, this isomorphism is already defined over R.

PROOF. Let $R^{\mathrm{per}} \hookrightarrow R^{\mathrm{psh}}$ be a strict henselization of R^{per} and \bar{k} the residue field of R^{psh} . Note that $k \hookrightarrow \bar{k}$ is an algebraic closure of k. Let $R^{\mathrm{per}} \hookrightarrow R'$ be a finite étale extension of R^{per} . Using the fact that the relative Frobenius morphism of R' over R^{per} is an isomorphism one sees that R' is again perfect. Hence R^{psh} is perfect. Hence by Proposition 4.8 there exists a p-divisible group \mathscr{G}' over \bar{k} such that $\mathscr{G}_{R^{\mathrm{psh}}} \cong \mathscr{G}'_{R^{\mathrm{psh}}}$. By taking the special fiber of this isomorphism, we get $\mathscr{F}_{\bar{k}} \cong \mathscr{F}'$, so that we may take $\mathscr{F}' = \mathscr{F}_{\bar{k}}$. Then Proposition 4.9 implies that there exists a unique isomorphism $\psi \colon (\mathscr{F}_{\bar{k}})_{R^{\mathrm{psh}}} \cong \mathscr{F}_{R^{\mathrm{psh}}}$ which is the identity in the special fiber. For any $\sigma \in \mathrm{Aut}(R^{\mathrm{psh}}/R^{\mathrm{per}})$, the conjugate of ψ by σ is again the identity in the special fiber and thus is equal to ψ . Thus ψ is defined over R^{per} by Galois descent.

In case \mathscr{G} has a single slope, one does not need to pass to R^{per} to split the slope filtration. Thus with the same argument as above one obtains ψ over a strict henselization of R and sees that it is defined over R by Galois descent.

Now let R be a discrete valuation ring as in Section 2 with quotient field K. The results in this subsection are formulated for p-divisible groups over $\operatorname{Spec}(R)$, however below we will work with p-divisible groups over $\operatorname{Spf}(R)$. Thus we will need the following:

PROPOSITION 4.11 ([20, Lemma 4.16]). Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. The base change functor $\mathscr{G} \mapsto \mathscr{G}_{\mathrm{Spf}(R')}$ from the category of p-divisible groups over $\mathrm{Spec}(R')$ to the category of p-divisible groups over $\mathrm{Spf}(R')$ is an equivalence.

Accordingly we define:

DEFINITION 4.12. Let R' be the valuation ring of a complete overfield $K' \subset \hat{K}$ of K. A p-divisible group over $\operatorname{Spf}(R')$ is completely slope divisible if and only if the corresponding p-divisible group over $\operatorname{Spec}(R')$ is completely slope divisible.

4.2 - Nice Semiabelian Schemes

Let *K* be a local field of characteristic p > 0.

DEFINITION 4.13. We call a semiabelian scheme \mathcal{A} over $\operatorname{Spec}(R)$ nice if \mathcal{A} is an extension of an abelian scheme by a torus over $\operatorname{Spec}(R)$ and the p-divisible group $\hat{\mathcal{A}}$ over $\operatorname{Spf}(R)$ is completely slope divisible.

Let \mathcal{A} be a nice semiabelian scheme over R.

LEMMA 4.14. Let $\mathcal{B} \subset \mathcal{A}$ be a semiabelian subgroup scheme. Then \mathcal{B} and \mathcal{A}/\mathcal{B} are nice.

PROOF. The ranks of the toric parts of \mathcal{B} and \mathcal{A}/\mathcal{B} are constant since they have constant sum and can only go up upon specialization. Thus by [13, Corollary 2.11] both \mathcal{B} and \mathcal{A}/\mathcal{B} are extensions of an abelian scheme by a torus over \mathcal{S} . Proposition 4.6 shows that the formal completions of both group schemes are again completely slope divisible.

Construction 4.15. We construct an isogeny $F_{\hat{\mathcal{A}}}:\hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}\to\hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ as follows: Propositions 4.10 and 4.11 yield a unique isomorphism $(\hat{\mathcal{A}}_k)_{\mathrm{Spf}(R^{\mathrm{per}})}\cong\hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ which is the identity on the special fiber. The p-divisible group $\hat{\mathcal{A}}_k$, being defined over the finite field k, has a Frobenius endomorphism with respect to k. Transfering the base change of this Frobenius endomorphism to $\mathrm{Spf}(R^{\mathrm{per}})$ to an endomorphism of $\hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ via the above isomorphism yields $F_{\hat{\mathcal{A}}}$.

The following summarizes the relevant properties of $F_{\hat{A}}$:

Proposition 4.16. (i) There exists a good polynomial which annihilates $F_{\hat{\mathcal{A}}}$.

- (ii) The endomorphism $F_{\hat{A}}$ is the Frobenius endomorphism with respect to a suitable model of $\hat{A}_{Spf(R^{per})}$ over the finite field k.
- (iii) Let \mathcal{B} be another nice semiabelian scheme over R. For any homomorphism $f: \mathcal{A} \to \mathcal{B}$, the induced homomorphism $\hat{f}_{\bar{R}}: \hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})} \to \hat{\mathcal{B}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ satisfies $F_{\hat{B}} \circ \hat{f}_{\mathrm{Spf}(R^{\mathrm{per}})} = \hat{f}_{\mathrm{Spf}(R^{\mathrm{per}})} \circ F_{\hat{A}}$.
- (iv) In case A has a model over k, the endomorphism $F_{\hat{A}}$ is the one induced by the Frobenius endomorphism of such a model.
- (v) If one replaces K by a finite overfield K' contained in \bar{K} , then $F_{\hat{A}}$ is replaced by $F_{\hat{A}}^N$, where N is the degree of the extension of the residue fields of R and R'

PROOF. (ii), (iv) and (v) follow directly from the construction.

- (i) By the construction of $F_{\hat{A}}$ it suffices to show that there exists a good polynomial which annihilates the Frobenius endomorphism of A_k . This follows from the Riemann hypothesis for abelian varieties, see for example [14, Fact 3.1].
- (iii) Pick isomorphisms $\hat{\mathcal{A}}_{R^{\mathrm{per}}} \cong (\hat{\mathcal{A}}_k)_{R^{\mathrm{per}}}$ and $\hat{\mathcal{B}}_{R^{\mathrm{per}}} \cong (\hat{\mathcal{B}}_k)_{R^{\mathrm{per}}}$ as in Construction 4.15. Under these identifications by Proposition 4.9 the homomorphism $\hat{f}_{R^{\mathrm{per}}}: \hat{\mathcal{A}}_{R^{\mathrm{per}}} \to \hat{\mathcal{B}}_{R^{\mathrm{per}}}$ arises by base change from its special fiber $\hat{f}_k: \hat{\mathcal{A}}_k \to \hat{\mathcal{B}}_k$. Since the latter is defined over k, it is compatible with the Frobenius endomorphisms of $\hat{\mathcal{B}}_k$ and $\hat{\mathcal{A}}_k$. This implies $F_{\hat{\mathcal{B}}} \circ \hat{f}_{R^{\mathrm{per}}} = \hat{f}_{R^{\mathrm{per}}} \circ F_{\hat{\mathcal{A}}}$.

THEOREM 4.17. Let $X \subset A_K$ be an irreducible subvariety. Then the following are equivalent:

- (i) The subvariety $X_{\bar{K}}$ is special in $A_{\bar{K}}$.
- (ii) There exist a finite field extension $K' \subset \overline{K}$ of K with valuation ring R', $x \in X(K')$ and $n \geqslant 1$ such that the schematic closure \mathcal{X} of $X_{K'} x$ in $\mathcal{A}_{R'}$ satisfies $F^n_{\hat{A}}(\widehat{\mathcal{X}}_{\mathrm{Spf}((R')^{\mathrm{per}})}) \subset \widehat{\mathcal{X}}_{\mathrm{Spf}((R')^{\mathrm{per}})}$.
- (iii) There exist $x \in X(\bar{K})$, $n \geqslant 1$ and a subset $T \subset \hat{\mathcal{A}}(\bar{R}) \cap (X x)(\bar{K})$ which is Zariski dense in $X_{\bar{K}} x$ and which satisfies $F_{\hat{\mathcal{A}}}^n(T) \subset T$.

PROOF. $(i) \Rightarrow (ii)$: Using Proposition 4.16 (iii) we see that if (ii) holds for $X/\operatorname{Stab}_{\mathcal{A}_K}(X) \subset \mathcal{A}_K/\operatorname{Stab}_{\mathcal{A}_K}(X)$, then it holds for $X \subset \mathcal{A}_{\bar{K}}$. Hence we may assume that $\operatorname{Stab}(X) = 0$. Then there exist a semiabelian variety B defined over a finite field k' containing k, a subvariety Y of B, a homomorphism $h: B_{\bar{K}} \to \mathcal{A}_{\bar{K}}$ with finite kernel and $a \in \mathcal{A}(\bar{K})$ such that $X_{\bar{K}} = h(Y_{\bar{K}}) + a$. Note that it suffices to prove (ii) after replacing K by a finite field extension contained in \bar{K} . After doing so we may assume $a \in \mathcal{A}(K)$ and k = k'. After suitably translating Y we may assume that $0 \in Y(k')$. Then $a \in X(K)$, so that after translating X by -a we may assume a = 0. Let \mathcal{X} be the schematic closure of X in \mathcal{A} .

As Y is defined over k', the set Y(k) is Zariski dense in Y, and hence the set $h(Y(\bar{k})) \subset X(\bar{K})$ is Zariski dense in $X_{\bar{K}}$. Thus there exists $y \in Y(\bar{k})$ such that \mathcal{X} is smooth over R at h(x) and such that h is flat at y. After possibly replacing k' and K by finite field extensions we may assume that $y \in Y(k')$ and hence $h(y) \in \mathcal{X}(R)$. Then after replacing Y by Y-y and \mathcal{X} by $\mathcal{X}-h(y)$ we may assume that \mathcal{X} is smooth over R at 0. By [13, Proposition 2.7] the homomorphism h extends to a homomorphism $B_{\bar{R}} \to \mathcal{A}_{\bar{R}}$. Hence after replacing K by a finite field extension contained in \bar{K} we may assume that h extends to a homomorphism $h: B_R \to \mathcal{A}$. By Proposition 2.45 the formal scheme $\hat{\mathcal{X}}$ is isomorphic to $\mathrm{Spf}(R[[x_1,\ldots,x_n]])$ for some n and hence is irreducible. Thus it follows from Proposition 2.47 that $\hat{\mathcal{X}}$ is the formal schematic image of $\hat{Y}_{\mathrm{Spf}(R)}$ under \hat{h} .

We have $F_{\hat{A}} \circ \hat{h}_{\mathrm{Spf}(R^{\mathrm{per}})} = \hat{h}_{\mathrm{Spf}(R^{\mathrm{per}})} \circ F_{\hat{B}_R}$ by Proposition 4.16. This together with the fact that $\hat{Y}_{\mathrm{Spf}(R^{\mathrm{per}})}$, being defined over k' = k, is invariant under $F_{\hat{B}_R}$ implies that $\hat{\mathcal{X}}_{\mathrm{Spf}(R^{\mathrm{per}})} = \hat{h}(\hat{Y}_{\mathrm{Spf}(R^{\mathrm{per}})})$ is invariant under $F_{\hat{A}}$. This shows (ii).

 $(ii)\Rightarrow (iii)$: By Proposition 2.46 set $T:=\hat{\mathcal{A}}(\bar{R})\cap (X_{\bar{K}}-x)(\bar{K})$ is Zariski dense in $X_{\bar{K}}-x$.

 $(iii) \Rightarrow (i)$: Using Proposition 4.16 (i), Theorem 3.10 applied to $G = \hat{\mathcal{A}}(\bar{R})$ and $\Phi = F_{\hat{A}}^n$ shows that $X_{\bar{K}} - x$ is special in $\mathcal{A}_{\bar{K}}$.

4.3 - Choice of a nice valuation

LEMMA 4.18. Let L be an algebraically closed field equipped with a non-archimedean valuation v with valuation ring Q. Let A be a semiabelian scheme over Q which is an extension of an abelian scheme over Q by a torus over Q. For $a \in A(L)$, if $na \in A(Q)$ for some $n \in \mathbb{Z}^{\neq 0}$, then $a \in A(Q)$.

PROOF. Let

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{A} \stackrel{\pi}{\longrightarrow} \mathcal{B} \longrightarrow 0$$

be exact with \mathcal{T} a torus over Q and \mathcal{B} an abelian scheme over Q. Then $\pi(a) \in \mathcal{B}(L) = \mathcal{B}(Q)$.

Since Q is strictly Henselian the flat cohomology group $\mathrm{H}^1(Q,\mathcal{T})$ is zero. Thus the point $\pi(a) \in \mathcal{B}(Q)$ lifts to a point $a' \in \mathcal{A}(Q)$. Since Q is strictly Henselian, the torus \mathcal{T} is split. Thus for $t = a - a' \in \mathcal{T}(L) \cong (L^*)^r$ we have $t^n \in (Q^*)^r$ which implies that $t \in (Q^*)^r$. Hence a lies in $\mathcal{A}(Q)$.

PROPOSITION 4.19. Let L_0 be a field which is finitely generated over \mathbb{F}_p and let A be a semiabelian variety over L_0 . There exists an embedding of L_0 into a local field K such that the semiabelian variety A_K extends to a semiabelian scheme A over the valuation ring R of K which is isogenous to a nice semiabelian scheme over R.

If we are given a finitely generated subgroup $\Gamma \subset A(L_0)$ (resp. a finite rank subgroup $\Gamma' \subset A(L_0^{\text{per}})$) we can pick v such that $\Gamma \subset A(R)$ (resp. such that $\Gamma' \subset A(R^{\text{per}})$).

PROOF. There exists a ring $R_0 \subset L_0$ which is finitely generated over \mathbb{F}_p and has quotient field L_0 such that A extends to a semiabelian scheme \mathcal{A} over R_0 . Since the Newton polygon of \mathcal{A} is generically constant, after localizing R_0 we may assume that \mathcal{A} has constant Newton polygon. Since the rank of the toric part of \mathcal{A} is generically constant, we can also assume that this rank is constant. Then \mathcal{A} is globally an extension of an abelian scheme by a torus by [13, Corollary 2.11].

If we are given a finitely generated subgroup Γ as above, after further localization we may assume that a finite generating set of Γ , and thus all of Γ , is contained in $\mathcal{A}(R_0)$.

If we are given a finite rank subgroup Γ' as above, pick a finitely generated subgroup Γ of Γ' such that Γ' consists of divison points of Γ . Then as before after further localization we may assume that $\Gamma \subset \mathcal{A}(R_0)$.

By [25, Lemma 3.1] there exists an embedding $R_0 \hookrightarrow R := \mathbb{F}_q[[t]]$ for a suitable power q of p. We pick such an embedding and let $K := \mathbb{F}_q((t))$. By Theorem 4.4, the p-divisible group $\widehat{\mathcal{A}}_{\operatorname{Spf}(R)}$ is isogenous to a completely slope divisible group $\mathscr S$ over R. Since an isogeny $\widehat{\mathcal{A}}_{\operatorname{Spf}(R)} \to \mathscr S$ is the quotient by a finite group scheme, this shows that we can find a nice semiabelian scheme $\mathscr A'$ over R which is isogenous to $\mathscr A_R$. In case we are given Γ' as above, Lemma 4.18 ensures $\Gamma' \subset \mathscr A(R^{\operatorname{per}})$. Thus the embedding has the required properties.

5. Proof of Mordell-Lang for finitely generated groups

5.1 - A formal Mordell-Lang theorem

Let K be a local field of characteristic p>0. Let $\mathscr G$ be a commutative formal group scheme over $\operatorname{Spf}(R)$ which as a formal scheme is isomorphic to $\operatorname{Spf}(R[[x_1,\ldots,x_n]])$. From an isomorphism $\mathscr G\cong\operatorname{Spf}(R[[x_1,\ldots,x_n]])$ one gets a bijection $\mathscr G(\bar R)\cong\bar{\mathfrak m}^{\oplus n}$ as in Remark 2.16. This endows $\mathscr G(\bar R)$ with a valuation topology which is independent of the chosen isomorphism. The fact that $[p]\colon\mathscr G\to\mathscr G$ acts by zero on the tangent space at the identity of $\mathscr G$ implies that for all $g\in\mathscr G(\bar R)$ the sequence $(p^ng)_{n\geqslant 0}$ converges to zero with respect to the valuation topology. This implies that the $\mathbb Z$ -module structure on $\mathscr G(\bar R)$ can be uniquely extended to a $\mathbb Z_p$ -module structure which is continuous with respect to the valuation topology.

First we prove the following Mordell-Lang statement for formal schemes in positive characteristic:

THEOREM 5.1. Let K be a local field of characteristic p. Let \mathscr{G} be a formal group over k which as a formal scheme is isomorphic to $\operatorname{Spf}(k[[x_1,\ldots,x_n]])$. Let $\bar{\Gamma} \subset \mathscr{G}(R^{\operatorname{per}})$ be a finitely generated \mathbb{Z}_p -module. Let $\mathscr{X} \subset \mathscr{S}_{\operatorname{Spf}(R^{\operatorname{per}})}$ be a closed formal subscheme.

If $\mathscr{X}(R^{\mathrm{per}}) \cap \bar{\Gamma}$ is formal-schematically dense in \mathscr{X} , then there exist a finite field extension K' of K with valuation ring R', closed formal subschemes $\mathscr{X}_1, \ldots, \mathscr{X}_m$ of $\mathscr{X}_{\mathrm{Spf}(R')}$ and elements $\gamma_1, \ldots, \gamma_m \in \bar{\Gamma}$ such that $\mathscr{X}_j + \gamma_j$ is defined over the residue field of R' and such that $\mathscr{X}_{\mathrm{Spf}((R')^{\mathrm{per}})} = \bigcup_j (\mathscr{X}_j)_{\mathrm{Spf}((R')^{\mathrm{per}})}$.

For the proof of Theorem 5.1 we need the following lemma:

LEMMA 5.2. Let K be a local field of characteristic p and q the cardinality of the residue field k of K. For $i \ge 0$ let K^{q^i} be the field consisting of q^i -th powers of elements of K and R^{q^i} the valuation ring of K^{q^i} . Let $\mathscr G$ formal group scheme over k. Then $q^i\mathscr G(R) \subset \mathscr G(R^{q^i})$.

PROOF. Let $F: \mathscr{G} \to \mathscr{G}$ be the Frobenius endomorphism of \mathscr{G} with respect to k. By [1, Section VII A.4] there exists a "Verschiebung" endormorphism $V: \mathscr{G} \to \mathscr{G}$ such that $[q] = F \circ V = V \circ F$. Using the fact that $F^i(G(R)) \subset G(R^{q^i})$ this implies the claim.

PROOF OF THEOREM 5.1. Since $\bar{\Gamma}$ is a finitely generated \mathbb{Z}_p -module, after possibly replacing K by a finite extension we may assume that $\bar{\Gamma} \subset \mathcal{G}(R)$. Since then $\mathcal{E}(R^{\mathrm{per}}) \cap \bar{\Gamma} \subset \mathcal{G}(R)$, Proposition 2.33 implies that \mathcal{E} is defined over R. Let $\mathcal{E}_1, \ldots, \mathcal{E}_m$ be the irreducible components of \mathcal{E} . Then $\mathcal{E}_i(\bar{R}) \cap \bar{\Gamma}$ is formal-schematically dense in \mathcal{E}_i for each i. It suffices to prove the claim for each \mathcal{E}_i , that is we may assume that \mathcal{E} is irreducible.

Since the group $\bar{\Gamma}/p^i\bar{\Gamma}$ is finite for all $i\!\geqslant\!0$ and since $\mathscr X$ is irreducible, it follows using Lemma 2.27 that we can choose $(\gamma_i)_{i\geqslant 0}\in \bar{\Gamma}^{\mathbb Z^{\geqslant 0}}$ such that $p^i\bar{\Gamma}\cap (\mathscr X+\gamma_i)(R)$ is formal-schematically dense in $\mathscr X+\gamma_i$ and such that $\gamma_{i+1}\equiv \gamma_i (\bmod p^i\bar{\Gamma})$ for all $i\!\geqslant\!0$. Since the finitely generated $\mathbb Z_p$ -module $\bar{\Gamma}$ is complete for the p-adic topology, there exists $\gamma\in\bar{\Gamma}$ such that $\gamma\equiv\gamma_i (\bmod p^i\bar{\Gamma})$ for $i\!\geqslant\!0$. Then $(\mathscr X+\gamma)(R)\cap p^i\bar{\Gamma}$ is formal-schematically dense in $\mathscr X+\gamma$ for all $i\!\geqslant\!0$. Let q be the number of elements of k. For $i\!\geqslant\!0$ let K^{q^i} be the field consisting of q^i -th powers of elements of K and R^{q^i} the valuation ring of K^{q^i} . Since $q^i\bar{\Gamma}\subset\mathscr S(R^{q^i})$ by Lemma 5.2, Proposition 2.33 implies that $\mathscr X+\gamma$ is defined over k.

5.2 - Proof of Mordell-Lang for finitely generated groups

Using Theorem 5.1 we can now prove Theorem 1.2.

Theorem 5.3 (Hrushovski). Let L be an algebraically closed field of positive characteristic. Let A be a semiabelian variety over L, let $X \subset A$ be an irreducible subvariety and $\Gamma \subset A(L)$ a finitely generated subgroup. If $X(L) \cap \Gamma$ is Zariski dense in X, then X is a special subvariety of A.

PROOF. Let $L_0 \subset L$ be a field which is finitely generated over \mathbb{F}_p such that A arises by base change from an abelian variety A_0 over L_0 , such that X arises by base change from a subvariety X_0 defined over L_0 and such that $\Gamma \subset A_0(L_0)$. By Proposition 4.19 there exists an embedding $L_0 \hookrightarrow K$ into a local field K and a semi-abelian scheme A over the valuation ring R of K which has generic fiber A_K , is isogenous to a nice semiabelian scheme over R and satisfies $\Gamma \subset A(R)$. Let $\mathcal X$ be the schematic closure of $X_{0,K}$ inside A.

By Lemma 3.4 it suffices to prove that $\mathcal{X}_{\bar{K}}$ is special in $\mathcal{A}_{\bar{K}}$. Using Lemma 3.5 we can replace \mathcal{A} by an isogenous semiabelian variety, so that we may assume that \mathcal{A} is nice.

Since we have an exact sequence

$$0 \to \hat{\mathcal{A}}(R) \to \mathcal{A}(R) \to \mathcal{A}(k) \to 0$$

with $\mathcal{A}(k)$ finite, after replacing \mathcal{X} by a suitable translate we may assume that $(\Gamma \cap \hat{\mathcal{A}}(R)) \cap \mathcal{X}(R)$ is schematically dense in \mathcal{X} . After replacing Γ by $\Gamma \cap \hat{\mathcal{A}}(R)$ we may thus assume $\Gamma \subset \hat{\mathcal{A}}(R)$.

Let $F_{\hat{\mathcal{A}}} \colon \hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})} o \hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ be the endomorphism given by Construction 4.15. By Proposition 4.16 (ii) there exist a p-divisible group \mathscr{G} over k and an isomorphism $\Psi \colon \mathscr{G}_{\mathrm{Spf}(R^{\mathrm{per}})} \cong \hat{\mathcal{A}}_{\mathrm{Spf}(R^{\mathrm{per}})}$ under which $F_{\hat{\mathcal{A}}}$ corresponds to the Frobenius endomorphism of \mathscr{G} with respect to k.

Let $\bar{\Gamma}$ be the closure of Γ with respect to the valuation topology on $\hat{\mathcal{A}}(\bar{R})$. Since Γ is finitely generated this is a finitely generated \mathbb{Z}_p -module. Let $\mathscr{Y} \subset \hat{\mathcal{A}}$ be the formal schematic closure of $\Gamma \cap \mathcal{X}(R)$ inside $\hat{\mathcal{A}}$. Then $\mathscr{Y}(R) \cap \bar{\Gamma}$ is formal-schematically dense in \mathscr{Y} . Thus $\mathscr{Y}(R) \cap \bar{\Gamma}$ is formal-schematically dense in $\mathscr{Y}_{\mathrm{Spf}(R^{\mathrm{per}})}$ by Corollary 2.38. Thus by Theorem 5.1 applied to $\mathscr{Y}(\mathscr{Y}_{\mathrm{Spf}(R^{\mathrm{per}})})$ and $\mathscr{Y}(\bar{\Gamma})$, there exist closed formal subschemes $\mathscr{Y}_1, \ldots, \mathscr{Y}_m$ of $\mathscr{Y}_{R^{\mathrm{per}}}$, elements $\gamma_1, \ldots, \gamma_m \in \bar{\Gamma}$ and $n \geqslant 0$ such that $F_{\hat{\mathcal{A}}}^n(\mathscr{Y}_j + \gamma_j) \subset \mathscr{Y}_j + \gamma_j$ and $\mathscr{Y} = \cup_j \mathscr{Y}_j$. Since $\Gamma \cap \mathcal{X}(R) \subset \mathscr{Y}(\bar{R})$, the set $\mathscr{Y}(\bar{R})$ is schematically dense in $\mathcal{X}_{\bar{R}}$. Lemma 2.24 implies $\mathscr{Y}(\bar{R}) = \cup_i \mathscr{Y}_i(\bar{R})$. Since $\mathcal{X}_{\bar{R}}$ is irreducible there exists i such that $\mathscr{Y}_i(\bar{R})$ is schematically dense in $\mathcal{X}_{\bar{R}}$. Thus Proposition 4.17 (iii) holds for $T := \mathscr{Y}_i(\bar{R}) + \gamma_i$. By Proposition 4.17 this implies that $\mathcal{X}_{\bar{K}}$ is special in $\mathcal{A}_{\bar{K}}$.

6. Towards full Mordell-Lang

The full Mordell-Lang conjecture in positive characteristic is the following conjecture:

Conjecture 6.1. Let L be an algebraically closed field of positive characteristic. Let A be a semiabelian variety over L, let $X \subset A$ be an irreducible subvariety and let $\Gamma \subset A(L)$ be a subgroup of finite rank. If $X(L) \cap \Gamma$ is Zariski dense in X, then X is a special subvariety of A.

In this section we show that in case A is an ordinary or supersingular abelian variety Conjecture 6.1 can be reduced to the following special case by combining our method with a reduction due to Ghioca, Moosa and Scanlon.

Conjecture 6.2. Let L_0 be a field which is finitely generated over \mathbb{F}_p , let L be an algebraic closure of L_0 and let L_0^{per} be the perfect closure of L_0 in L. Let A be an abelian variety over L_0 and $X \subset A_{L_0^{\mathrm{per}}}$ an irreducible subvariety. Assume that the canonical morphism $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A \to A$ is defined over L_0 , that there exists a finite subfield \mathbb{F}_q of L_0 over which $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A$ can be defined and that $\mathrm{Stab}_{A_{L_0^{\mathrm{per}}}}(X)$ is finite. If $X(L_0^{\mathrm{per}})$ is Zariski dense in X_0 , then a translate of X by an element of $A(L_0^{\mathrm{per}})$ is defined over L_0 .

Remark 6.3. We expect that Conjecture 6.2 holds without the condition on the field of definition of the morphism $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A \to A$, the existence of \mathbb{F}_q as above and the condition on $\mathrm{Stab}_{A_{L_0^{\mathrm{per}}}}(X)$. However we were unable to deduce Conjecture 6.2 from Conjecture 6.1 without these conditions. Note that they can always be achieved by dividing by $\mathrm{Stab}_{A_{L_0^{\mathrm{per}}}}(X)$ and by replacing L_0 by a suitable finite extension.

Remark 6.4. Consider the case of a supersingular abelian variety A. Then A is isogenous to a power of a supersingular elliptic curve (c.f. [21, Theorem 4.2]), and thus in order to prove Conjecture 6.1 for A, it suffices to prove it for powers of supersingular elliptic curves. But these can be defined over a finite field and for such abelian varieties, Conjecture 6.1 is proven in [14]. Thus the results in this section are only interesting in case A is ordinary.

LEMMA 6.5. Let L_0 be a finitely generated field of characteristic p > 0 with perfection $L_0 \hookrightarrow L_0^{\text{per}}$ and let A be a semiabelian variety over L_0 .

- (i) The group $A(L_0^{\text{per}})$ has finite rank.
- (ii) Let $\Gamma \subset A(L_0^{\mathrm{per}})$ be a subgroup and $n \geqslant 0$. The group $\Gamma/p^n\Gamma$ is finite.

PROOF. (i) For $n \ge 0$ the group $p^n A(L_0^{p^{-n}})$ is contained in $A(L_0)$. This together with the fact that $A(L_0)$ is finitely generated implies that $A(L_0^{\text{per}})$ has finite rank.

(ii) By Claim 1 in the proof of Theorem 2.2 of [14] the torsion subgroup of $A(L_0^{\rm per})$ is finite. Hence both the rank and the size of the torsion subgroup of the finitely generated groups $\Gamma_i := \Gamma \cap A(L_0^{p^{-i}})$ are bounded as $i \geqslant 0$ varies. Thus the size of the groups $\Gamma_i/p^n\Gamma_i$ is bounded as i varies. Let $\bar{\Gamma}_i$ be the image of $\Gamma_i/p^n\Gamma_i$ in $\Gamma/p^n\Gamma$. The groups $\bar{\Gamma}_i$ form an ascending sequence of finite groups of bounded size, thus for all $i \gg 0$ the $\bar{\Gamma}_i$ coincide. Since $\Gamma/p^n\Gamma$ is the union of the $\bar{\Gamma}_i$ this shows (ii).

Proposition 6.6. Conjecture 6.1 implies Conjecture 6.2 for abelian varieties A.

PROOF. After translating X by an element of $X(L_0^{\mathrm{per}})$ we may assume that $0 \in X$. By Lemma 6.5 the group $A(L_0^{\mathrm{per}})$ has finite rank. Hence by Conjecture 6.1 the subvariety X_L is special in A_L . Thus by Lemma 3.6 there exist a subvariety $Y \subset \mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A$ over $\bar{\mathbb{F}}_p$ and $a \in A(L)$ such that $X_L = \tau(Y_L) + \mathrm{Stab}_A(X)_L + a$, where τ is the natural homomorphism $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A \to A$ which by assumption is defined over L_0 . Since by assumption $\mathrm{Stab}_A(X)$ is finite and X is irreducible, there exists $a' \in A(L)$ such that $\tau(Y_L) = X_L + a'$.

The fact that $0 \in X$ implies $a' \in \operatorname{Im}(\tau)(L)$. Thus $X \subset \operatorname{Im}(\tau)$. After replacing A by $\operatorname{Im}(\tau)$ we may assume that τ is an isogeny. We fix a model of $\operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A$ over a finite field \mathbb{F}_q contained in L_0 and let $F \colon \operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A \to \operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A$ be the Frobenius endomorphism of $\operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A$ with respect to \mathbb{F}_q . Since by [8, Theorem 6.12] the homomorphism τ

is purely inseparable, the induced map $\tau\colon \mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A(L_0^{\mathrm{per}})\to A(L_0^{\mathrm{per}})$ is surjective. Let Y' be an irreducible component of the Zariski closure of $\tau^{-1}(X(L_0^{\mathrm{per}}))\subset \mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A(L_0^{\mathrm{per}})$. Let $\Gamma:=\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A(L_0)$. This is a finitely generated F-invariant subgroup of $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A(L)$ and $Y'(L)\cap (\cup_{n\geqslant 0}F^{-n}\Gamma)=Y'(L)\cap \mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A(L_0^{\mathrm{per}})$ is Zariski dense in Y'. Thus by [14, Proposition 3.9] for some $n\geqslant 0$ the set $Y'(L)\cap F^{-n}\Gamma\subset Y'(L_0^{q^{-n}})$ is Zariski dense in Y'. Thus $X(L_0^{q^{-n}})$ is Zariski dense in X. Since the group $A(L_0^{q^{-n}})/q^nA(L_0^{q^{-n}})$ is finite, there thus exists $a''\in A(L_0^{q^{-n}})$ such that $(a''+q^nA(L_0^{q^{-n}}))\cap X(L_0^{q^{-n}})$ is Zariski dense in X. Since $q^nA(L_0^{q^{-n}})\subset A(L_0)$ and since Zariski closure commutes with base change this implies that X-a'' is defined over L_0 .

6.1 - A specialness criterion

Let K be a local field of characteristic p > 0. Let A be an abelian scheme over R such that the Newton polygon of \hat{A} is constant. Denote A_K by A.

The following definition is somewhat ad hoc and adapted to our present needs:

DEFINITION 6.7. Let $i \geqslant 0$ and X a subvariety of A. Denote the schematic closure of X in A by \mathcal{X} . We say that X is K/K^{p^i} -special in A if X is irreducible and there exists an abelian variety B over K^{p^i} , a subvariety Y of B over K^{p^i} , a homomorphism $h: B_K \to A/\operatorname{Stab}_A(X)$ with finite kernel and an element $a \in (\mathcal{A}/\operatorname{Stab}_A(\mathcal{X})(\bar{R}))$ such that $(X/\operatorname{Stab}_A(X))_{\bar{K}} = h(Y_{\bar{K}}) + a$.

THEOREM 6.8. Assume that A is ordinary or supersingular. Let X be an irreducible subvariety of A containing 0. If X is K/K^{p^i} -special in A for all $i \ge 0$, then $X_{\bar{K}}$ is special in $A_{\bar{K}}$.

PROOF. Since the Newton polygon of $\hat{\mathcal{A}}$ is constant, by Proposition 4.4 the p-divisible group $\hat{\mathcal{A}}$ is isogenous to a completely slope divisible p-divisible group. Since isogenies of p-divisible groups are quotients by finite subgroup schemes, this implies that there exists a nice abelian scheme \mathcal{A}' over R together with an isogeny $f\colon \mathcal{A}\to\mathcal{A}'$. It follows directly from Definition 6.7 that f(X) is again K/K^{p^i} -special in \mathcal{A}'_K for all $i\geqslant 0$. By Lemma 3.3, the subvariety $X_{\bar{K}}$ of $A_{\bar{K}}$ is special if and only if $f(X_{\bar{K}})$ is special in $\mathcal{A}'_{\bar{K}}$. Thus we may replace \mathcal{A} by \mathcal{A}' and can assume that \mathcal{A} is nice.

Let \mathcal{X} be the schematic closure of X in \mathcal{A} . After dividing by $\operatorname{Stab}_{\mathcal{A}}(\mathcal{X})$ we may assume that $\operatorname{Stab}_{\mathcal{A}}(X) = 0$. By Proposition 2.30, after replacing K by a finite field extension we may assume that each irreducible component of $\hat{\mathcal{X}}$ is geometrically irreducible.

Let $i \geqslant 0$. By assumption there exists a abelian variety B over K^{p^i} , a subvariety $Y \subset B$, a homomorphism $h: B_K \to A$ and $a \in \hat{\mathcal{A}}(\bar{R})$ such that $X_{\bar{K}} = h(Y_{\bar{K}}) + a$. Let \mathcal{B} be the Néron model of B over R^{p^i} . Since \mathcal{A} is the Néron model of A_K , the homomorphism h extends to a homomorphism $\tilde{h}: \mathcal{B}_R \to \mathcal{A}$. Since the generic fiber

of the completion $\hat{h}:\hat{\mathcal{B}}_R\to\hat{\mathcal{A}}$ is an isogeny, Tate's conjecture implies that \hat{h} is an isogeny. Since the Newton polygon of $\hat{\mathcal{A}}$ is constant, the existence of this isogeny implies that the Newton polygon of $\hat{\mathcal{B}}$ is constant. Thus, by Theorem 4.4 and Proposition 4.11, there exists a completely slope divisible p-divisible group \mathscr{G} over $\mathrm{Spf}(R^{p^i})$ together with an isogeny $h':\mathscr{G}\to\hat{\mathcal{B}}$.

Let $\mathcal Y$ be the schematic closure of Y in $\mathcal B$. Pick a finite field extension $K'\subset \bar K$ of K with valuation ring R' such that $a\in \hat{\mathcal A}(R')$. Then $\tilde h(\mathcal Y_{R'})+a=\mathcal X_{R'}$. Let $\mathscr Y:=(h')^{-1}(\hat{\mathcal Y})\subset \mathscr F$. Then $\mathscr E_i:=\hat h(h'(\mathscr Y_{\operatorname{Spf}(R')}))+a\subset \hat{\mathcal X}_{\operatorname{Spf}(R')}$, where $\hat h(h'(\mathscr Y_{\operatorname{Spf}(R')}))$ is the formal schematic image of $\mathscr Y_{\operatorname{Spf}(R')}$. By Lemma 2.32 we have $h'(\mathscr Y_{\operatorname{Spf}(R')})=\hat{\mathcal Y}_{\operatorname{Spf}(R')}$. Thus by Proposition 2.47 each irreducible component of $\mathscr X_i$ is an irreducible component of $\hat{\mathcal X}_i$ is geometrically irreducible, it follows that $\mathscr X_i$ is the union of some of the irreducible components of $\hat{\mathcal X}$.

Using the fact that $\hat{\mathcal{A}}$, and hence \mathscr{G} , has a single slope, Propositions 4.10 and 4.11 yield unique isomorphisms $\psi \colon \mathscr{G} \xrightarrow{\sim} (\mathscr{G}_k)_{\mathrm{Spf}(R^{p^i})}$ and $\psi' \colon \hat{\mathcal{A}} \xrightarrow{\sim} (\hat{\mathcal{A}}_k)_{\mathrm{Spf}(R)}$ which are the identity in the special fiber. Under these identifications, by Proposition 4.9 the homomorphism $\hat{h} \circ h' \colon \mathscr{G}_R \to \hat{\mathcal{A}}$ arises by base change from its special fiber. Thus the identity $\mathscr{X}_i = \hat{h}(h'(\mathscr{Y})) + a$ shows that a translate of $\psi'(\mathscr{X}_i)$ by an element of $\hat{\mathcal{A}}(\bar{R})$ is defined over R^{p^i} .

As we saw above, each \mathscr{X}_i is the union of some of the irreducible components of \mathscr{X} . Since there are only finitely many such components it follows that there exists $\mathscr{X} \subset \hat{\mathscr{X}}$ such that $\mathscr{X} = \mathscr{X}_i$ for infinitely many i. Theorem 2.44 implies that there exists a finite field extension $K' \subset \bar{K}$ with valuation ring R' and $x \in \mathscr{X}(R')$ such that $\psi'(\mathscr{X}_{\mathrm{Spf}(R')} - x)$ is defined over k. By construction, the endomorphism $F_{\hat{A}}$ corresponds to the Frobenius endomorphism of \hat{A}_k under ψ' . Thus $T := \mathscr{X}(\bar{R}) - x$ satisfies $F_{\hat{A}}(T) \subset T$. By Proposition 2.46 (ii) the set T is Zariski dense in $\mathscr{X}_{\bar{K}} - x$. Thus condition (iii) of Theorem 4.17 is satisfied and Theorem 4.17 implies that $X_{\bar{K}}$ is special in $A_{\bar{K}}$.

6.2 - Proof of the reduction

Theorem 6.9. Conjecture 6.1 and Conjecture 6.2 are equivalent for abelian varieties which are ordinary or supersingular.

PROOF. One direction was already proved in Proposition 6.6 above. Now we prove that Conjecture 6.2 implies Conjecture 6.1.

By [14, Theorem 2.2], Conjecture 6.1 is implied by:

Conjecture 6.10. Let L_0 be a field which is finitely generated over \mathbb{F}_p , L an algebraic closure of L_0 and L_0^{per} the perfect closure of L_0 in L. Let A be a semi-abelian variety over L_0 , let $X \subset A$ be an irreducible subvariety and let $\Gamma \subset A(L_0^{\mathrm{per}})$ be a subgroup of finite rank. If $X(L_0^{\mathrm{per}}) \cap \Gamma$ is Zariski dense in X_L , then X_L is a special subvariety of A_L .

Furthermore, from the proof of [14, Theorem 2.2], one sees that if one only wants to prove Conjecture 6.1 for abelian varieties which are ordinary or supersingular, then it suffices to prove Conjecture 6.10 for such abelian varieties. This is what we do now.

After replacing L_0 by a finite extension we may assume that the canonical morphism τ : $\operatorname{Tr}_{L/\mathbb{F}_p}A \to A$ is defined over L_0 and that there exists a finite subfield \mathbb{F}_q of L_0 over which $\operatorname{Tr}_{L/\mathbb{F}_p}A$ can be defined. After dividing by $\operatorname{Stab}_A(X)$ we may assume that $\operatorname{Stab}_A(X) = 0$.

By Proposition 4.19 there exists an embedding of L_0 into a local field K such that the abelian variety A_K extends to an abelian scheme \mathcal{A} over the valuation ring R of K which is isogenous to a nice abelian scheme over R and such that $\Gamma \subset A(L_0^{\mathrm{per}}) \subset \mathcal{A}(R^{\mathrm{per}})$.

Let \mathcal{X} be the schematic closure of X_K inside \mathcal{A} . By Lemma 3.4 it suffices to prove that $X_{\bar{K}}$ is special in $\mathcal{A}_{\bar{K}}$.

Since there is an exact sequence

$$0 \to \hat{\mathcal{A}}(R^{\mathrm{per}}) \to \mathcal{A}(R^{\mathrm{per}}) \to \mathcal{A}(k) \to 0$$

with $\mathcal{A}(k)$ finite there exists $\gamma \in \Gamma \cap X(L_0^{\mathrm{per}})$ such that $X(\bar{K}) \cap (\gamma + (\Gamma \cap \hat{\mathcal{A}}(R^{\mathrm{per}}))$ is Zariski dense in $X_{\bar{K}}$. After replacing L_0 by a finite extension we may assume that $\gamma \in A(L_0)$. Then after translating X by $X - \gamma$ and replacing Γ by $\Gamma \cap \hat{\mathcal{A}}(R^{\mathrm{per}})$ we may assume that $\Gamma \subset \hat{\mathcal{A}}(R^{\mathrm{per}})$ and $0 \in X$.

Fix a finite subfield \mathbb{F}_q of L_0 over which $\mathrm{Tr}_{L/\bar{\mathbb{F}}_p}A_L$ can be defined. By Theorem 6.8 it now suffices to show that \mathcal{X}_K is K/K^{q^i} -special in A for all $i \geqslant 0$. Thus we fix such an i. We will work with the abelian variety $A^{(q^i)}$, which is naturally defined over $L_0^{q^i}$, together with the Verschiebung homomorphism $V:A_{L_0}^{(q^i)} \to A$. By Lemma 6.5 the group $\Gamma/q^i\Gamma$ is finite. Thus there exists $\gamma \in \Gamma$ such that $X(L_0^{\mathrm{per}}) \cap (\gamma + q^i\Gamma)$ is Zariski dense in $X_{\bar{K}}$. Let $T \subset \Gamma$ be such that $\gamma + q^iT = X(L_0^{\mathrm{per}}) \cap (\gamma + q^i\Gamma)$. Let $F:A \to A^{(q^i)}$ be the relative q^i -Frobenius and $Y \subset A_{L_0^{\mathrm{per}}}^{(q^i)}$ an irreducible component of the Zariski closure of $F(T) \subset A^{(q^i)}(L_0^{\mathrm{per}})$. The fact that $V \circ F = [q^i]$ implies $V(Y) + \gamma = X$.

By [8, Theorem 6.4] the formation of $\operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A_L$ commutes with purely inseparable base change. Thus $\tau^{(q^i)}\colon \operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A_L\cong (\operatorname{Tr}_{L/\bar{\mathbb{F}}_p}A_L)^{(q^i)}\to A_L^{(q^i)}$ is the $L/\bar{\mathbb{F}}_q$ -trace of $A_L^{(q^i)}$. This morphism is defined over $L_0^{q^i}$. Since $\operatorname{Stab}_A(X)=0$ the stabilizer $\operatorname{Stab}_{L_p^{\mathrm{per}}}(Y)$ is contained in the kernel of V and thus is finite. Thus

Conjecture 6.2 applied to $Y \subset A_{L_0^{\mathrm{per}}}^{(q^i)}$ gives an element $a \in A^{(q^i)}(L_0^{\mathrm{per}})$ such that Y+a is defined over $L_0^{q^i}$.

We consider a as an element of $A^{(q^i)}(K^{\mathrm{per}}) = \mathcal{A}^{(q^i)}(R^{\mathrm{per}})$. Since the natural map $\mathcal{A}^{(q^i)}(R^{q^i}) \to \mathcal{A}^{(q^i)}(k)$ is surjective, there exists $a' \in \mathcal{A}^{(q^i)}(R^{q^i})$ such that $a-a' \in \hat{\mathcal{A}}^{(q^i)}(R^{\mathrm{per}})$. Then Y+a-a' is defined over K^{q^i} and $V(Y+a-a')+V(a'-a)+\gamma=X$ with $V(a'-a)+\gamma\in\hat{\mathcal{A}}(R^{\mathrm{per}})$. Thus $\mathcal{X}_{\bar{K}}$ is K/K^{q^i} -special and we are done.

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