Rend. Sem. Mat. Univ. Padova, Vol. 135 (2016), 39–61
DOI 10.4171/RSMUP/135-3

Projection of a nonsingular plane quintic curve and the dihedral group of order eight

T. Takahashi (*) (**)

Abstract – Let C be a nonsingular plane quintic curve over the complex number field $\mathbb{C}$, and let $\pi_P: C \to \mathbb{P}^1$ be a projection from $P \in C$. Let L_P be the Galois closure of the field extension $\mathbb{C}(C)/\mathbb{C}(\mathbb{P}^1)$ induced by π_P, where $\mathbb{C}(C)$ and $\mathbb{C}(\mathbb{P}^1)$ are the rational function fields of C and $\mathbb{P}^1$, respectively. We call the point P a D_4-point if the Galois group of $L_P/\mathbb{C}(\mathbb{P}^1)$ is isomorphic to the dihedral group D_4 of order eight. In this paper, we prove that the number of D_4-points for C equals 0, 1, 3, 5, or 15, and show that the curve with 15 D_4-points is projectively equivalent to the Fermat quintic curve.

Mathematics Subject Classification (2010). 14H50, 14H05, 14H37.

Keywords. Galois group of a projection, plane quintic curve, the dihedral group of order eight.

1. Introduction

We shall work over the complex number field $\mathbb{C}$. Let $C \subset \mathbb{P}^2$ be a nonsingular plane curve of degree $d \geq 2$, and let $\pi_P: C \to \mathbb{P}^1$ be the projection from a point $P \in \mathbb{P}^2$. The projection π_P induces an extension $\pi_P^*: \mathbb{C}(\mathbb{P}^1) \hookrightarrow \mathbb{C}(C)$, where $\mathbb{C}(C)$ and $\mathbb{C}(\mathbb{P}^1)$ are the rational function fields of C and $\mathbb{P}^1$, respectively. We denote by K and K_P the function field $\mathbb{C}(C)$ and its subfield $\pi_P^*(\mathbb{C}(\mathbb{P}^1))$, respectively. Let L_P be the Galois closure of K/K_P, and let $\mathrm{Gal}(L_P/K_P)$ be the Galois group of the field extension L_P/K_P.

The study of the projections π_P is an interesting issue of nonsingular plane curves. Indeed, a classical theorem of Noether and later results (see e.g. [5]) assure

(*) *Indirizzo dell'A.*: Takeshi Takahashi, Division of General Education, Nagaoka National College of Technology, 888 Nishikatakai, Nagaoka, Niigata 940-8532, Japan.
E-mail: takeshi@nagaoka-ct.ac.jp

(**) *Indirizzo attuale.*: Takeshi Takahashi, Information Engineering, Faculty of Engineering, Niigata University, Niigata 950-2181, Japan.
E-mail: takeshi@ie.niigata-u.ac.jp

that the minimum degree of a morphism $C \to \mathbb{P}^1$ equals $d-1$, and all the maps of degree $d-1 \geq 2$ (resp. $d \geq 5$) are projections $\pi_P : C \to \mathbb{P}^1$ from some point $P \in C$ (resp. $P \in \mathbb{P}^2 \setminus C$). Thus it is natural to investigate the Galois group $\mathrm{Gal}(L_P/K_P)$ associated to projections π_P.

Cukierman [1] has shown that if C is a general plane curve of degree d, then for every point $P \in \mathbb{P}^2 \setminus C$, the Galois group $\mathrm{Gal}(L_P/K_P)$ is isomorphic to the symmetric group on d letters. Without any assumption of generality, Miura and Yoshihara [3, 8] have shown that if $P \in C \setminus C^\#$ (resp. $P \in \mathbb{P}^2 \setminus (C \cup C^\#)$), then the Galois group $\mathrm{Gal}(L_P/K_P)$ is isomorphic to the symmetric group on $d-1$ letters (resp. on d letters), where $C^\#$ is the union of all multitangent lines to C. Moreover, Pirola and Schlesinger [7] have shown that there are only finitely many points $P \in \mathbb{P}^2 \setminus C$ for which the Galois group $\mathrm{Gal}(L_P/K_P)$ is not isomorphic to the symmetric group on d letters.

In [3, 8], the notion of "Galois point" has been introduced: a point P is said to be a *Galois point* if the extension K/K_P is Galois. Miura and Yoshihara have shown that if $P \in C$ (resp. $P \in \mathbb{P}^2 \setminus C$) is a Galois point, then the Galois group $\mathrm{Gal}(K/K_P)$ is isomorphic to the cyclic group of order $d-1$ (resp. d). Furthermore, they have determined the number and distribution of Galois points.

As an extension of these studies, we would like to treat the case in which the Galois group L_P/K_P is not isomorphic to a cyclic group or a full symmetric group. In this paper, we study points $P \in C$ lying on a nonsingular plane curve C of degree $d = 5$, such that the Galois group $\mathrm{Gal}(L_P/K_P)$ of the projections $\pi_P : C \to \mathbb{P}^1$ is isomorphic to the dihedral group D_4 of order 8.

DEFINITION 1.1. A point $P \in C$ is called a *D_4-point* if the Galois group of L_P/K_P is isomorphic to the dihedral group D_4 of order 8. We denote the set of all D_4-points as $\Delta(C, D_4)$, and $\delta(C, D_4)$ is the number of elements in this set, i.e.,

$$\Delta(C, D_4) = \{P \in C \mid P \text{ is a } D_4\text{-point}\}, \quad \delta(C, D_4) = \#\Delta(C, D_4).$$

Our main theorem is as follows.

THEOREM 1.2. *Let C be a nonsingular plane quintic curve. Then,*

$$\delta(C, D_4) \in \{0, 1, 3, 5, 15\}.$$

Moreover,

(1) *if $\delta(C, D_4) = 3$ or 5, then all D_4-points are collinear*;

(2) *equality $\delta(C, D_4) = 15$ holds if and only if C is projectively equivalent to the Fermat quintic curve*; *in this case C possesses three disjoint 5-tuples of collinear D_4-points.*

Furthermore, we present explicit examples showing that all the values of $\delta(C, D_4)$ listed above occur.

In order to prove Theorem 1.2, the main techniques rely on the description of D_4-points presented in [2]. In particular, since the projection $\pi_P : C \to \mathbb{P}^1$ from a D_4-point factors through a double covering $C \to C'$ of a genus two curve C', we deduce that each D_4-point $P \in C$ induces an involution $\iota_P \in \mathrm{Aut}(C)$. Then we achieve Theorem 1.2 by studying the action of the subgroup of $\mathrm{Aut}(C)$ generated by those involutions on the points of C.

It is worth noting that, in the light of [2, 8] and Theorem 1.2, the only case to discuss for completing the description of projections from points on plane quintic curves is when the Galois group is isomorphic to the alternating group A_4. Unfortunately, our techniques do not apply as projections from A_4-points do not factor through intermediate curves.

The plan of the paper is the following. In Section 2, we give some preliminary results on D_4-points for a nonsingular plane quintic curve. Section 3 concerns examples assuring that the numbers of D_4-points listed in Theorem 1.2 occur. In Section 4, we prove Theorem 1.2.

2. Preliminary results

In this section, we are aimed at presenting the preliminary results necessary to study D_4-points.

Hereafter, $C \subset \mathbb{P}^2$ is a nonsingular curve of degree $d = 5$, and we use the following notation.

Notation. We denote by $\mathrm{Aut}(C)$ the group of automorphisms of C, and by id_C the identity automorphism. Consider a point $P \in C$ and the Galois closure L_P associated to the projection $\pi_P : C \to \mathbb{P}^1$. Then, $\widetilde{C}_P$ is the nonsingular projective curve having L_P as rational function field, and $g(P)$ denotes its genus. We denote by $T_P C$ the tangent line to C at P. Given two plane curves A and B, let $I_P(A, B)$ denote their intersection multiplicity at P. Let $(X : Y : Z)$ be the homogeneous coordinates of the projective plane $\mathbb{P}^2$. Then we denote by $f_i(X, Y)$ any homogeneous polynomial of degree i. Given a homogeneous polynomial $f \in \mathbb{C}[X, Y, Z]$, we denote by $V(f)$ the plane curve defined by $f = 0$, and by $F(5) := V(X^5 + Y^5 + Z^5)$ the Fermat quintic curve. By $\mathrm{Aut}(\mathbb{P}^2)$ we mean the group of projective linear transformation of $\mathbb{P}^2$, and for any $T \in \mathrm{Aut}(\mathbb{P}^2)$ we define its fixed locus $\mathrm{Fix}(T) := \{P \in \mathbb{P}^2 \mid T(P) = P\}$. Finally, we denote by $\#S$ the number of elements of a finite set S.

The following theorem summarizes Miura's results on D_4-point (see [2]).

THEOREM 2.1. *Let C be a nonsingular plane quintic curve.*

(1) *A point $P \in C$ is a D_4-point if and only if P and the defining equation of C can be expressed as*

$$P = (0:0:1) \quad and \quad f_1(X,Y)Z^4 + f_3(X,Y)Z^2 + f_5(X,Y) = 0$$

by taking a suitable projective transformation, where $f_i(X,Y) \neq 0$, for $i = 1, 3, 5$.

(2) *If $P \in C$ is a D_4-point, then there exists an intermediate field K' of the extension K/K_P such that the degrees $[K : K'] = 2$ and $[K' : K_P] = 2$. Moreover, the nonsingular projective curve whose function field is isomorphic to K' has genus 2.*

(3) *If $P \in C$ is a D_4-point, then $g(P)$ equals 16 or 17. Let $\delta_{16}(C)$ (resp. $\delta_{17}(C)$) be the number of D_4-points satisfying $g(P) = 16$ (resp. $g(P) = 17$). Then, $5\delta_{16}(C) + 6\delta_{17}(C) \leq B(C)$, where $B(C)$ is the number of lines bitangent to C.*

REMARK 2.2. Every automorphism of a nonsingular plane curve C of degree $d \geq 4$ can be extended to a projective transformation of $\mathbb{P}^2$ ([6, Corollary 5.3.19]), i.e., for every $\sigma \in \mathrm{Aut}(C)$, there exists $T \in \mathrm{Aut}(\mathbb{P}^2)$ such that $T|_C = \sigma$. Note that $T \neq T' \implies T|_C \neq T'|_C$. Indeed, $T|_C = T'|_C \implies \mathrm{Fix}(T \circ T'^{-1}) \supset C$. Because $\mathrm{Fix}(T \circ T'^{-1})$ is the union of some linear spaces, we have that $\mathrm{Fix}(T \circ T'^{-1}) = \mathbb{P}^2$. Hence, $T \circ T'^{-1}$ is the identity, and so $T = T'$. In this paper, we shall often express $\sigma \in \mathrm{Aut}(C)$ as an element of $\mathrm{Aut}(\mathbb{P}^2)$, and we shall represent it by its representation matrix in $\mathrm{PGL}(3, \mathbb{C})$. We note that an element $A \in \mathrm{PGL}(3, \mathbb{C})$ induces a projective transformation $(\widetilde{X} : \widetilde{Y} : \widetilde{Z}) \mapsto (X : Y : Z)$ so that $(X, Y, Z)^{\mathrm{tr}} = A\,(\widetilde{X}, \widetilde{Y}, \widetilde{Z})^{\mathrm{tr}}$, where $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})^{\mathrm{tr}}$ and $(X, Y, Z)^{\mathrm{tr}}$ are transposed matrices of $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ and of (X, Y, Z), respectively.

REMARK 2.3. For an element $\sigma \in \mathrm{Aut}(\mathbb{P}^2)$, let A be its representation matrix. Then, a point $Q = (a : b : c)$ is a fixed point of σ, i.e., $\sigma(Q) = Q$, if and only if the vector (a, b, c) is an eigenvector of A. Hence, $\mathrm{Fix}(\sigma)$ is one of the following:

(1) $\mathrm{Fix}(\sigma)$ consists of one point, two points, or one line;

(2) $\mathrm{Fix}(\sigma)$ consists of three non-collinear points;

(3) $\mathrm{Fix}(\sigma)$ consists of one point and one line;

(4) $\mathrm{Fix}(\sigma) = \mathbb{P}^2$.

In particular, if the order of σ is finite, then $\mathrm{Fix}(\sigma)$ is of type (2), (3), or (4), because A is diagonalizable.

By Theorem 2.1, we have the following lemmas.

LEMMA 2.4. *If $P \in C$ is a D_4-point for C, then $I_P(C, T_P C) = 3$ or $I_P(C, T_P C) = 5$.*

PROOF. By Theorem 2.1 (1), we may assume that

$$P = (0 : 0 : 1)$$

and the defining equation of C is $f_1(X, Y)Z^4 + f_3(X, Y)Z^2 + f_5(X, Y) = 0$. Then, $T_P C$ is given by $f_1(X, Y) = 0$. Hence, we have the conclusion. $\square$

LEMMA 2.5. *If $P \in C$ is a D_4-point for C, then there exists a unique involution $\iota_P \in \mathrm{Aut}(C)$ such that*

(1) *$\iota_P \neq \mathrm{id}_C$ and it extends to an involution of $\mathbb{P}^2$, that is $\iota_P^2 = \mathrm{id}_{\mathbb{P}^2}$;*

(2) *$\iota_P(P) = P$ and $\iota_P(\ell) = \ell$ for every line ℓ passing through the point P;*

(3) *ι_P has fixed locus $\mathrm{Fix}(\iota_P) = \{P\} \cup \ell_P$, where ℓ_P is a line not passing through P.*

PROOF. By Theorem 2.1 (1), we may assume that $P = (0 : 0 : 1)$ and the defining equation of C is $f_1(X, Y)Z^4 + f_3(X, Y)Z^2 + f_5(X, Y) = 0$. Then, we put

$$\iota_P \colon (X : Y : Z) \longmapsto (-X : -Y : Z).$$

Hence $\iota_P \in \mathrm{Aut}(C)$ is an involution satisfying properties (1), (2), and (3). In particular, the line $\ell_P \subset \mathrm{Fix}(\iota_P)$ has equation $Z = 0$ and it does not pass through P.

To check that ι_P is unique, we assume that there exists another involution $\iota'_P \in \mathrm{Aut}(C)$ satisfying the same properties. Then, let G be the group generated by ι_P and ι'_P. Both ι_P and ι'_P correspond to automorphisms of K that fix every element of K_P. Hence, we see that the order of G satisfies $\#G \leq [K : K_P] = 4$. As K/K_P is not Galois, we have $\#G = 2$, and therefore, $\iota_P = \iota'_P$. $\square$

We call ι_P the *involution associated to the D_4-point P*, and ℓ_P the *line of fixed points of ι_P*.

Remark 2.6. It follows from Theorem 2.1 (2) that the projection $\pi_P : C \to \mathbb{P}^1$ factors through a double covering $\varphi_P : C \to C'$ and a hyperelliptic map $\psi_P : C' \to \mathbb{P}^1$, where C' is a curve of genus 2. Then it is easy to check that ι_P is the involution associated to the map φ_P, sending $Q \in C$ to the other point $Q' \in C$ such that $\varphi_P(Q') = \varphi_P(Q)$. In particular, ι_P fixes 6 points on C (P and $C \cap \ell_P$), and they coincide with the ramification points of φ_P.

Lemma 2.7. *Assume that $P := (0:0:1) \in C$ and*

$$\iota := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(C).$$

Then, the defining equation of C must be expressed as

$$f_1(X,Y)Z^4 + f_3(X,Y)Z^2 + f_5(X,Y) = 0.$$

In particular, P is a Galois point when $f_3(X,Y) = 0$, and P is a D_4-point otherwise.

Proof. Let the defining equation of C be

$$F(X,Y,Z) := \sum_{i=0}^{5} f_i(X,Y)Z^{5-i} = 0.$$

Because $P \in C$, P is not a singular point, and C is irreducible, we have that $f_0(X,Y) = 0$, $f_1(X,Y) \neq 0$, $f_5(X,Y) \neq 0$. As $\iota \in \mathrm{Aut}(C)$, we have that $\iota^* F = \lambda F$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Hence,

$$\begin{aligned} &-f_1(X,Y)Z^4 + f_2(X,Y)Z^3 - f_3(X,Y)Z^2 + f_4(X,Y)Z - f_5(X,Y) \\ &\quad = \lambda F(X,Y,Z), \end{aligned}$$

so that $\lambda = -1$ and $f_2(X,Y) = f_4(X,Y) = 0$. By [2, Proposition 3.6] (resp. Theorem 2.1), if $f_3(X,Y) = 0$ (resp. $f_3(X,Y) \neq 0$), then P is a Galois point (resp. a D_4-point). $\square$

Lemma 2.8. *Let P and Q be two D_4-points for C. Then, $\iota_P(Q) \neq P$, Q is a D_4-point, and the three points P, Q, and $\iota_P(Q)$ are collinear.*

Proof. By Theorem 2.1 and Lemma 2.5, we may assume that $P = (0:0:1)$,

$$C : f_1(X,Y)Z^4 + f_3(X,Y)Z^2 + f_5(X,Y) = 0,$$

and $\ell_P\colon Z = 0$. Because $\iota_P(P) = P$, we have that $\iota_P(Q) \neq P$. Assume that $\iota_P(Q) = Q$. Then $Q \in \ell_P$, and so $f_5(Q) = 0$. Therefore, the tangent line $T_Q C$ has equation

$$\frac{\partial f_5}{\partial X}(Q)X + \frac{\partial f_5}{\partial Y}(Q)Y = 0,$$

and hence $P \in T_Q C$. Thus $I_Q(C, T_Q C) = 2$ or 4, which contradicts Lemma 2.4. Therefore $\iota_P(Q) \neq Q$.

By Lemma 2.5, we see that the three points P, Q, and $\iota_P(Q)$ are collinear. □

Lemma 2.9. *For every line ℓ, $\#(\Delta(C, D_4) \cap \ell) \in \{0, 1, 3, 5\}$.*

Proof. Assume that there exist two D_4-points P_1 and P_2 on ℓ. Then, by Lemma 2.8, we obtain a third D_4-point $\iota_{P_1}(P_2)$ on ℓ.

Assume that there exist four D_4-points P_i $(i = 1, 2, 3, 4)$ on ℓ. By Lemma 2.8, we may assume that $\iota_{P_1}(P_2) = P_3$. Then, we see that $\iota_{P_1}(P_4) \neq P_i$ $(i = 1, 2, 3, 4)$. Hence, we have a fifth D_4-point $\iota_{P_1}(P_4)$ on ℓ. □

Lemma 2.10. *Let P_1 and P_2 be two D_4-points for C, and let ℓ be the line passing through the points P_1 and P_2. If $\#(\Delta(C, D_4) \cap \ell) = 3$ (resp. 5), then the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ equals 3 (resp. 5).*

Proof. By Lemma 2.8, we have the third D_4-point $P_3 := \iota_{P_1}(P_2)$. With a suitable projective transformation, we may assume that $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 1 : 1)$, and $\ell\colon X = 0$. By Lemmas 2.4 and 2.8, we see that ℓ is not a tangent line to C at P_i, where $i = 1, 2, 3$. Indeed, if ℓ were the tangent line to C at P_i, then ℓ would be also tangent at $\iota_{P_j}(P_i)$, where $j \neq i$. Thus the intersection number would be $I(C, \ell) > 5$, which is a contradiction. Hence, we have that $\#(C \cap \ell) = 4$ or 5. Let $P_4 := (0 : \alpha : 1) \in (C \cap \ell) \setminus \{P_1, P_2, P_3\}$ and $P_5 := \iota_{P_1}(P_4)$, where $\alpha \in \mathbb{C} \setminus \{0, 1\}$. Note that $\#(C \cap \ell) = 4 \implies P_4 = P_5$. From $\iota_{P_1}(P_1) = P_1$, $\iota_{P_1}(P_2) = P_3$, $\iota_{P_1}(P_3) = P_2$, we infer that

$$\iota_{P_1}|_\ell = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C})$$

and $P_5 = (0 : \alpha : \alpha - 1)$. Here, we have two cases, $\iota_{P_2}(P_1) = P_3$ or $\iota_{P_2}(P_1) \in \{P_4, P_5\}$.

If $\iota_{P_2}(P_1) = P_3$, we can infer from $\iota_{P_2}(P_2) = P_2$, $\iota_{P_2}(P_1) = P_3$, $\iota_{P_2}(P_3) = P_1$, $\iota_{P_2}(P_4) = P_5$ that

$$\iota_{P_2}|_\ell = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C})$$

and $\alpha^2 - \alpha + 1 = 0$. Hence, we may assume that $P_4 = (0 : -\omega : 1)$ and $P_5 = (0 : 1 : -\omega)$, where ω is a primitive cubic root of unity. We see now that the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ equals 3. If P_4 is a D_4-point, then there exists the involution ι_{P_4} such that $\iota_{P_4}(P_4) = P_4$, ι_{P_4} acts on $\{P_1, P_2, P_3, P_5\}$, and $\iota_{P_4}^2 = \mathrm{id}_C$. However, we see that there does not exist such an element $\iota_4|_\ell$ in $\mathrm{PGL}(2, \mathbb{C})$. Therefore, $\#(\Delta(C, D_4) \cap \ell) = 3$.

If $\iota_{P_2}(P_1) \in \{P_4, P_5\}$, then because $\iota_{P_1} \circ \iota_{P_2}$ acts transitively on the five points $P_1, \ldots, P_5$, we have that the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ equals 5, and both P_4 and P_5 are D_4-points. □

Remark 2.11. By the proof of Lemma 2.10, if $\#(\Delta(C, D_4) \cap \ell) \geq 2$ for a line ℓ, then $\#(C \cap \ell) = 5$.

Proposition 2.12. *Let P and Q be two D_4-points for C. Then, there exists an automorphism $\sigma \in \mathrm{Aut}(C)$ such that $\sigma(P) = Q$.*

Proof. Let $R := \iota_P(Q)$, so that P, Q and R are distinct collinear points by Lemma 2.8. If $\iota_Q(R) = P$, then $\iota_P \circ \iota_Q(P) = Q$. If $\iota_Q(R) \neq P$, then let $R' := \iota_Q(R)$ and $R'' := \iota_Q(P)$. In particular, P, Q, R, R' and R'' are collinear D_4-points, and hence $\iota_P(R') = R''$. Thus we have that $\iota_P \circ \iota_Q \circ \iota_P \circ \iota_Q(P) = Q$. □

By Proposition 2.12, if there exist two D_4-points P and Q for C, then $\widetilde{C}_P$ is isomorphic to $\widetilde{C}_Q$, and in particular $g(P) = g(Q)$.

We can improve the inequality stated in Theorem 2.1 (3) as follows.

Lemma 2.13. *Let P be a D_4-point for C. Then, we have*

$$A_2 + A_3 + B_1 = 6,$$

where A_2 is the number of lines that intersect C transversally at P and meet C with multiplicity 4 at another point, A_3 is the number of lines that meet C with multiplicity 5 at P, and B_1 is the number of lines that intersect C transversally at P and are tangent to C at two other distinct points.

Proof. Let ℓ be a line passing through P. By Lemma 2.4 we have that $I_P(C, \ell) \in \{1, 3, 5\}$. Moreover, Lemma 2.5 assures that $\ell \neq \ell_P$, and that ι_P permutes points on $(C \cap \ell) \setminus \mathrm{Fix}(\iota_P)$. In particular, for any $Q \in (C \cap \ell) \setminus \mathrm{Fix}(\iota_P)$ we have that $I_Q(C, \ell) = I_{\iota_P(Q)}(C, \ell)$. Thus only the following cases may occur:

(1) the line ℓ intersects C transversally at P and four other distinct points that are not on ℓ_P;

(2) the line ℓ intersects C transversally at P and two other distinct points, and it is tangent to C at another point lying on ℓ_P (let A_0 be the number of these lines);

(3) the line ℓ is tangent to C at P with multiplicity 3, and intersects C transversally at two other distinct points that are not on ℓ_P (let A_1 be the number of these lines);

(4) the line ℓ intersects C transversally at P, and is tangent to C with multiplicity 4 at another point lying on ℓ_P (let A_2 be the number of these lines);

(5) The line ℓ is tangent to C at P with multiplicity 5 (let A_3 be the number of these lines);

(6) the line ℓ intersects C transversally at P, and is tangent to C at two other distinct points that are not on ℓ_P (let B_1 be the number of these lines);

(7) the line ℓ is tangent to C at P with multiplicity 3, and is tangent to C at another point lying on ℓ_P (let B_2 be the number of these lines).

Consider the morphism $\pi_P\colon C \to \mathbb{P}^1$, where the genus $g(C) = 6$ and $\deg \pi_P = 4$. Then, by the Riemann–Hurwitz formula

$$2g(C) - 2 = (\deg \pi_P)(2g(\mathbb{P}^1) - 2) + \deg R,$$

where R is the ramification divisor, we have that

$$\deg R = A_0 + A_1 + 3A_2 + 3A_3 + 2B_1 + 2B_2 = 18.$$

Since the tangent line to C at P is unique, we have $A_1 + A_3 + B_2 = 1$. Since $\#(C \cap \ell_P) = 5$ (cf. Remark 2.6), we have $A_0 + A_2 + B_2 = 5$. From these three equations, we have $A_2 + A_3 + B_1 = 6$. □

Lemma 2.14. $\delta(C, D_4) \leq 19$.

Proof. Let a_1 be the number of tangent lines that have contacts of order 3 at a point and intersect C transversally at two other points, and let a_i $(i = 2, 3)$ be the number of tangent lines that have contacts of order $i + 2$. Let b_1 be the number of bitangent lines that have contacts at two points, both of order 2, and let b_2 be the number of bitangent lines that have contacts at two points of orders 2 and 3. Let C^* be the dual curve of C. Then, the degree of C^* is

$$d^* = d(d-1) = 20,$$

and its geometric genus equals $(d-1)(d-2)/2 = 6$ as the dual map $C \to C^*$ is birational. Moreover, the number of double points of C^* (resp. triple points, singular points with multiplicity 4) equals $a_1 + b_1$ (resp. $a_2 + b_2, a_3$), and every singular point of C^* can be resolved by one blowing up (cf. [6, Section 1.5]). By the genus formula, we have $a_1 + b_1 + 3(a_2 + b_2) + 6a_3 = 165$, and by the flex formula ([6, Example 1.5.11]), we have $a_1 + b_2 + 2a_2 + 3a_3 = 45$.

By Remark 2.11 and Lemma 2.13, we have $6\delta(C, D_4) \le a_2 + a_3 + b_1$. Therefore,

$$\begin{aligned} 6\delta(C, D_4) &\le a_2 + a_3 + b_1 \\ &\le a_2 + 3a_3 + b_1 + 2b_2 \\ &= (a_1 + b_1 + 3(a_2 + b_2) + 6a_3) - (a_1 + b_2 + 2a_2 + 3a_3) \\ &= 120, \end{aligned}$$

so that $\delta(C, D_4) \le 20$.

We remark that if P and Q are distinct D_4-points, then $\iota_P(P) = P$ and $\iota_P(Q) \ne Q$ by Lemma 2.8. Thus $\delta(C, D_4)$ must be odd. Hence, we have $\delta(C, D_4) \le 19$. $\square$

3. Examples

This section concerns the existence of plane quintic curves satisfying $\delta(C, D_4) \in \{0, 1, 3, 5, 15\}$. In particular, we show that for any value of $\delta(C, D_4)$ listed above, there exists a nonsingular plane quintic curve having the prescribed number of D_4-points.

We have the classical theorem: the general smooth plane curve of degree $d > 3$ has no non-trivial automorphism. Hence, by Lemma 2.5, the general plane quintic curve does not possess D_4-points, that is $\delta(C, D_4) = 0$.

On the other hand, Miura and Yoshihara proved the following result on D_4-points on the Fermat quintic curve $F(5)$: $X^5 + Y^5 + Z^5 = 0$ (see [4, Theorem 1]).

THEOREM 3.1 ([4]). *If $P \in F(5)$ is not a flex of $F(5)$, then the Galois group* $\mathrm{Gal}(L_P/K_P)$ *is isomorphic to the symmetric group of* 4 *letters and* $g(P) = 85$. *On the contrary, if $P \in F(5)$ is a flex, then P is a D_4-point and $g(P) = 16$.*

In particular, the curve $F(5)$ has 15 flexes, which lie on the three lines $X = 0$, $Y = 0$ and $Z = 0$. For example, it is easy to check that the point $P = (0 : -1 : 1)$ is a D_4-point for $F(5)$. Indeed, by taking the projective transformation

$T\colon (\widetilde{X} : \widetilde{Y} : \widetilde{Z}) \mapsto (X : Y : Z)$ given by $X = \widetilde{X}, Y = \widetilde{Y} + \widetilde{Z}, Z = \widetilde{Y} - \widetilde{Z}$, we have that $T((0:0:1)) = P$ and $T^*(F(5))\colon 10\widetilde{Y}\widetilde{Z}^4 + 20\widetilde{Y}^3\widetilde{Z}^2 + \widetilde{X}^5 + 2\widetilde{Y}^5 = 0$. Thus $F(5)$ satisfies $\delta(C, D_4) = 15$, and there are three disjoint 5-tuples of collinear D_4-points.

Finally, we present three explicit examples of nonsingular plane quintic curves satisfying $\delta(C, D_4) = 1$, 3 and 5, respectively.

Example 3.2. Let C be the plane curve defined by the equation

$$X^5 + Y^5 + Y^4 Z + Y Z^4 + Z^5 = 0$$

and $P := (0 : -1 : 1)$. Then, C is a nonsingular plane quintic curve, $\delta(C, D_4) = 1$ and $\Delta(C, D_4) = \{P\}$.

We can prove that $\delta(C, D_4) = 1$ as follows. First, we can show easily that the point P is a D_4-point. By taking the projective transformation T given by $X = \widetilde{X}, Y = \widetilde{Y} + \widetilde{Z}, \quad Z = \widetilde{Y} - \widetilde{Z}$, we have that $T((0 : 0 : 1)) = P$ and $T^*(C)\colon (4\widetilde{Y})\widetilde{Z}^4 + (24\widetilde{Y}^3)\widetilde{Z}^2 + (\widetilde{X}^5 + 4\widetilde{Y}^5) = 0$. Hence, by Theorem 2.1, P is a D_4-point. Moreover, we have the involution ι_P as follows:

$$\iota_P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From a comparison of the Hessians of the defining equations, we infer that C is not projectively equivalent to $F(5)$. By [8, Theorem 4' and Proposition 5'], we have that the point $Q := (1 : 0 : 0)$ is the unique outer Galois point for C. Thus any $\sigma \in \mathrm{Aut}(C)$ satisfies $\sigma(Q) = Q$. Then, by systematic direct computations, we can see that ι_P is the only non-trivial involution of C fixing Q. By Lemma 2.5, we conclude that $\delta(C, D_4) = 1$.

Example 3.3. Let C be the plane curve defined by the equation

$$X^5 + Y^4 Z + Y Z^4 = 0$$

and $P_i := (0 : -\omega^{i-1} : 1)$, where $i = 1, 2, 3$ and ω is a primitive cubic root of unity. Then, C is a nonsingular plane quintic curve, $\delta(C, D_4) = 3$ and $\Delta(C, D_4) = \{P_1, P_2, P_3\}$.

We can prove that $\delta(C, D_4) = 3$ as follows. First, we can show easily that 3 points P_1, P_2, P_3 are D_4-points. For example, by taking the projective transformation T given by $X = \widetilde{X}, Y = \widetilde{Y} + \widetilde{Z}, Z = \widetilde{Y} - \widetilde{Z}$, we have that $T((0 : 0 : 1)) = P_1$

and $T^*(C)\colon (-6\widetilde{Y})\widetilde{Z}^4 + (4\widetilde{Y}^3)\widetilde{Z}^2 + (\widetilde{X}^5 + 2\widetilde{Y}^5) = 0$. Hence, by Theorem 2.1, P_1 is a D_4-point. Moreover, we have that the involutions ι_{P_1} and ι_{P_2} as follows:

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}.$$

Hence, the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ equals 3, where $\ell\colon X = 0$. From a comparison of the Hessians of the defining equations, we infer that C is not projectively equivalent to $F(5)$. By Lemma 2.10 and Theorem 1.2, we see that $\delta(C, D_4) = 3$.

Example 3.4. Let C be the plane curve defined by the equation

$$X^5 + X^3YZ + Y^5 + Z^5 = 0$$

and $P_i := (0 : -\zeta^{i-1} : 1)$, where $i = 1, \dots, 5$ and ζ is a primitive fifth root of unity. Then, C is a nonsingular plane quintic curve, $\delta(C, D_4) = 5$ and $\Delta(C, D_4) = \{P_1, \dots, P_5\}$.

We can prove that $\delta(C, D_4) = 5$ as follows. First, we can show easily that 5 points $P_1, \dots, P_5$ are D_4-points. For example, by the projective transformation T given by $X = \widetilde{X}$, $Y = \widetilde{Y} + \widetilde{Z}$, $Z = \widetilde{Y} - \widetilde{Z}$, we have that $T((0 : 0 : 1)) = P_1$ and $T^*(C)\colon (10\widetilde{Y})\widetilde{Z}^4 + (-\widetilde{X}^3 + 20\widetilde{Y}^3)\widetilde{Z}^2 + (\widetilde{X}^5 + \widetilde{X}^3\widetilde{Y}^2 + 2\widetilde{Y}^5) = 0$. Hence, by Theorem 2.1, P_1 is a D_4-point. Moreover, since P_1 is an ordinary flex and $F(5)$ has no ordinary flex, C is not projectively equivalent to $F(5)$. By Theorem 1.2, we see that $\delta(C, D_4) = 5$.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. In particular, the proof follows almost straightforwardly from Lemmas 2.8 and 2.9, and from the following results.

Lemma 4.1. *Assume that there exist five collinear D_4-points for C, as well as one other D_4-point. Then, C is projectively equivalent to $F(5)$. In particular, $\delta(C, D_4) = 15$.*

Lemma 4.2. *Assume that $\#(\Delta(C, D_4) \cap \ell) \leq 3$ for every line ℓ. Then, $\delta(C, D_4) \leq 3$.*

In Subsections 4.1 and 4.2, we prove Lemmas 4.1 and 4.2, respectively. Subsection 4.3 is devoted to conclude the proof of Theorem 1.2.

4.1 – *Proof of Lemma* 4.1

Assume that there exist five collinear D_4-points P_i $(i = 1, \dots, 5)$ for C, as well as another D_4-point P_0. Taking a suitable projective transformation, we may assume that the five collinear points P_i $(i = 1, \dots, 5)$ are on the line $\ell \colon X = 0$. Let $\sigma := \iota_{P_1} \circ \iota_{P_2} \in \mathrm{Aut}(C)$. We will find other D_4-points using the automorphisms ι_{P_i} $(i = 0, \dots, 5)$ and σ.

CLAIM 4.3. *The three lines ℓ_{P_1}, ℓ_{P_2}, ℓ are not concurrent, i.e., they do not meet at one point.*

PROOF. Assume that $\ell_{P_1} \cap \ell_{P_2} \cap \ell = \{Q\}$. Taking a suitable projective transformation, we may assume that $P_1 = (0 : 0 : 1)$, $P_2 = (0 : 1 : 0)$, and $Q = (0 : 1 : 1)$. Then, as $\iota_{P_1}(P_1) = P_1$, $\iota_{P_1}(Q) = Q$, $\iota_{P_1}^2 = \mathrm{id}_\ell$, $\iota_{P_2}(P_2) = P_2$, $\iota_{P_2}(Q) = Q$, $\iota_{P_2}^2 = \mathrm{id}_\ell$, we have that

$$\iota_{P_1}|_\ell = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \iota_{P_2}|_\ell = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Hence, the order of $\iota_{P_1}|_\ell \circ \iota_{P_2}|_\ell$ is not finite, which contradicts Lemma 2.10. □

Let Q_1 be the intersection of ℓ_{P_1} and ℓ_{P_2}. Then, $Q_1 \notin \ell$ and $Q_1 \in \mathrm{Fix}(\sigma)$. Consider the morphism $\nu \colon \ell \to \ell/\langle\sigma\rangle$, where the genera of ℓ and $\ell/\langle\sigma\rangle$ are $g(\ell) = g(\ell/\langle\sigma\rangle) = 0$, and $\deg \nu = 5$ by Lemma 2.10. Then Riemann–Hurwitz formula $2g(\ell) - 2 = (\deg \nu)(2g(\ell/\langle\sigma\rangle) - 2) + \deg R$ assures that the ramification divisor R consists of at least two points $Q_2, Q_3 \in \mathrm{Fix}(\sigma) \cap \ell$. Taking a suitable projective transformation, we may assume that $Q_1 = (1 : 0 : 0)$, $Q_2 = (0 : 1 : 0)$, $Q_3 = (0 : 0 : 1)$. Then, $\sigma \in \mathrm{PGL}(3, \mathbb{C})$ is expressed as a diagonal matrix.

Let $G := \langle\sigma\rangle$, $H := \langle\sigma^5\rangle$, $G|_\ell := \langle\sigma|_\ell\rangle$. Then, by Lemma 2.10, the order of G_ℓ equals 5. Hence, we have the exact sequence $1 \to H \to G \to G|_\ell \to 1$ and $\#G|_\ell = 5$.

CLAIM 4.4. $\#G = 5$.

PROOF. We show $\sigma^5 = \mathrm{id}_C$. Because $Q_1 = (1 : 0 : 0) \in \ell_{P_1} \cap \ell_{P_2}$, and $\iota_{P_1}(\ell) = \ell$, $\iota_{P_2}(\ell) = \ell$, we have that the involutions ι_{P_1} and ι_{P_2} can be expressed as matrices:

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & \smash{\raisebox{0.5ex}{A_1}} \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & \smash{\raisebox{0.5ex}{A_2}} \end{pmatrix},$$

where $A_1, A_2 \in \mathrm{GL}(2,\mathbb{C})$. We remark that ι_{P_i} $(i = 1,2)$ has a 1-dimensional eigenspace corresponding to P_i and a 2-dimensional eigenspace corresponding to ℓ_{P_i} by Remark 2.3 and Lemma 2.5. The eigenvalues of ι_{P_i} $(i = 1,2)$ are equal to 1 and -1, since

$$\iota_{P_i}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{A_i^2} \end{pmatrix},$$

which induces $\mathrm{id}_{\mathbb{P}^2}$, must be a unit matrix. The eigenvalue of ι_{P_i} $(i = 1,2)$ belonging to $Q_1 = (1:0:0) \in \ell_{P_i}$ equals 1. Thus, we infer that the eigenvalues of A_i $(i = 1,2)$ are equal to 1 and -1. In particular, $\det A_i = -1$ $(i = 1,2)$.

Because

$$\sigma^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{(A_1A_2)^5} \end{pmatrix}, \quad \sigma^5|_\ell = \mathrm{id}_\ell, \quad \det(A_1A_2)^5 = 1,$$

we have that

$$\sigma^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Assume that $\sigma^5 \neq \mathrm{id}_C$. Then, by Lemma 2.7, the defining equation of C can be expressed as $f_1(Y,Z)X^4 + f_3(Y,Z)X^2 + f_5(Y,Z) = 0$. In particular, we have that $f_5(P_1) = 0$ as $P_1 \in C \cap \{X = 0\}$, and the line $\ell(P_1, Q_1)$ passing through Q_1 and P_1 is given by a factor of f_5. From $I_{P_1}(C, \ell(P_1, Q_1)) \geq 2$, we see that $\ell(P_1, Q_1) = T_{P_1}C$ and $I_{P_1}(C, T_{P_1}C) = 2$ or $I_{P_1}(C, T_{P_1}C) = 4$. This contradicts Lemma 2.4. $\square$

Since σ is represented by a diagonal matrix in $\mathrm{PGL}(3,\mathbb{C})$ and $\sigma \neq \mathrm{id}_{\mathbb{P}^2}$, Remark 2.3 assures that its fixed locus $\mathrm{Fix}(\sigma)$ consists either of three non-collinear points, or of one point and one line. In the latter case we have:

Claim 4.5. *If* $\mathrm{Fix}(\sigma)$ *consists of one point and one line, then* Q_2 *and* Q_3 *are Galois points.*

Proof. Because $Q_1, Q_2, Q_3 \in \mathrm{Fix}(\sigma)$, we may assume that

$$\mathrm{Fix}(\sigma) = \{Q_2\} \cup \ell(Q_1, Q_3),$$

where $\ell(Q_1, Q_3)$ is the line passing through Q_1 and Q_3. By Claim 4.4, the morphism $C \to C/\langle\sigma\rangle$ is a cyclic covering of degree 5, and its ramification divisor consists of 5 total ramification points $\ell(Q_1, Q_3) \cap C$. From the Riemann–Hurwitz formula, we see that the quotient $C/\langle\sigma\rangle$ is isomorphic to $\mathbb{P}^1$. Hence, by [5, Proposition 2.3.6], the covering $C \to C/\langle\sigma\rangle$ is obtained as a projection. Because $\sigma(\ell') = \ell'$ for every line ℓ' passing through Q_2, we see that the center of the projection $C \to C/\langle\sigma\rangle$ is the point Q_2. Therefore, Q_2 is a Galois point.

From $\sigma \circ \iota_{P_2}(Q_2) = \iota_{P_1}(Q_2) = \iota_{P_1} \circ \sigma(Q_2) = \iota_{P_2}(Q_2)$ and $\ell_{P_1} \cap \ell_{P_2} \cap \ell = \emptyset$, we infer that $\iota_{P_2}(Q_2) = Q_3$. Hence, the point Q_3 is also a Galois point. □

In particular, $Q_2, Q_3 \in \mathbb{P}^2 \setminus C$. Therefore, [8, Theorem 4′ and Proposition 5′] assure that if $\mathrm{Fix}(\sigma)$ consists of one point and one line, then C is projectively equivalent to $F(5)$. Hence we only need to prove the assertion of Lemma 4.1 when $\mathrm{Fix}(\sigma) = \{Q_1, Q_2, Q_3\}$ consists of three non-collinear points.

Claim 4.6. *For a point $P \in \mathbb{P}^2 \setminus \{Q_1, Q_2, Q_3\}$, the five points $P, \sigma(P), \sigma^2(P), \sigma^3(P), \sigma^4(P)$ are collinear if and only if $P \in V(XYZ)$. Furthermore, if $P \notin V(XYZ)$, then no three of these five points can be collinear.*

Proof. Assume that $P \in V(XYZ)$. Because $\mathrm{Fix}(\sigma) = \{Q_1, Q_2, Q_3\}$, we have $\sigma(V(X)) = V(X)$, $\sigma(V(Y)) = V(Y)$, $\sigma(V(Z)) = V(Z)$. Hence, $P, \sigma(P), \sigma^2(P), \sigma^3(P), \sigma^4(P)$ are collinear.

Assume that $P \notin V(XYZ)$. Let $P = (1 : a : b)$, where $ab \neq 0$. As $\sigma^5 = \mathrm{id}_C$ by Claim 4.4, σ can be expressed as $(X : Y : Z) \to (X : \zeta Y : \zeta^i Z)$, where $i = 2, 3, 4$. Then, the three points $P = (1 : a : b)$, $\sigma(P) = (1 : \zeta a : \zeta^i b)$, $\sigma^2(P) = (1 : \zeta^2 a : \zeta^{2i} b)$ cannot be collinear. □

Claim 4.7. *There exist 15 D_4-points $P_1, \ldots, P_{15}$ such that each five points $P_{i+1}, \ldots, P_{i+5}$ $(i = 0, 5, 10)$ are collinear.*

Proof. The five D_4-points $P_1, \ldots, P_5$ are collinear by assumption. Using the involution induced by D_4-point P_0 not collinear to them, we have that the points $\iota_{P_0}(P_1), \ldots, \iota_{P_0}(P_5)$ are also collinear D_4-points. Let $P_{5+i} := \iota_{P_0}(P_i)$ $(i = 1, \ldots, 5)$. Then, $\iota_{P_6}(P_1), \ldots, \iota_{P_6}(P_5)$ are also collinear D_4-points. Let $P_{10+i} := \iota_{P_6}(P_i)$ $(i = 1, \ldots, 5)$. Note that $P_1, \ldots, P_{15}$ are distinct by Lemma 2.8 (and the point P_{11} is actually P_0). □

Claim 4.8. *If $\{P_6, \ldots, P_{15}\} \not\subset V(YZ)$, then $\delta(C, D_4) \geq 30$.*

Proof. We may assume that $\{P_6, \dots, P_{10}\} \not\subset V(YZ)$. Let

$$S_i := \{P_i, \sigma(P_i), \dots, \sigma^4(P_i)\} \quad (i = 6, \dots, 10).$$

We show $S_i \cap S_j = \emptyset$ for $i \neq j$. Assume that $S_6 \cap S_7 \neq \emptyset$. Then, there exists a point $P \in S_6 \cap S_7$, and then $\{P, \sigma(P), \dots, \sigma^4(P)\} = S_6 = S_7$. As $\iota_{P_6}(P_6) = P_6$, we have that $\iota_{P_6}(S_6) = S_6$. Hence, $\iota_{P_6}(S_7) = S_6 = S_7 \ni P_6, P_7, \iota_{P_6}(P_7)$, where the three points $P_6, P_7, \iota_{P_6}(P_7)$ are collinear. By Claim 4.6, $P_6, P_7 \in V(YZ)$. Hence, $S_6 = C \cap V(Y)$ or $S_6 = C \cap V(Z)$. Because $P_6, P_7 \in S_6$ and $P_6, \dots, P_{10}$ are collinear, we have that $P_6, \dots, P_{10} \in V(YZ)$, which is a contradiction. By the same argument as above, we see that $S_i \cap S_j = \emptyset$ for $i \neq j$.

The number of points in $\{P_1, \dots, P_5\} \cup S_6 \cup \cdots \cup S_{10}$ equals 30. □

By Lemma 2.14, and Claims 4.7 and 4.8, we have that the D_4-points $P_1, \dots, P_{15}$ are on $V(XYZ)$.

Claim 4.9. $\delta(C, D_4) = 15$.

Proof. If $\delta(C, D_4) > 15$, then we have 15 D_4-points $P_1, \dots, P_{15}$ on $V(XYZ)$, and another D_4-point $P_{16} \notin V(XYZ)$. By Claim 4.4 and $\mathrm{Fix}(\sigma) = \{Q_1, Q_2, Q_3\}$, we have that P_i, where $i = 1, \dots, 15$, and $\sigma^j(P_{16})$, where $j = 0, \dots, 4$, are distinct D_4-points. This contradicts Lemma 2.14. □

Claim 4.10. *Taking a suitable projective transformation, we may assume that*

$$\{P_1, \dots, P_{15}\} = F(5) \cap V(XYZ).$$

Proof. We may assume that $\{P_1 \dots, P_5\} \subset V(X)$, $\{P_6 \dots, P_{10}\} \subset V(Y)$, $\{P_{11} \dots, P_{15}\} \subset V(Z)$. Moreover, we may assume that $P_1 = (0 : -1 : 1)$ and $P_6 = (-1 : 0 : 1)$. Note that $\iota_{P_i}(\Delta(C, D_4)) = \Delta(C, D_4)$, for any $i \in \{1, \dots, 15\}$ by Claim 4.9, and $Q_1 = (1 : 0 : 0)$, $Q_2 = (0 : 1 : 0)$, $Q_3 = (0 : 0 : 1)$. Thus $\iota_{P_i}(\{Q_1, Q_2, Q_3\}) = \{Q_1, Q_2, Q_3\}$ for any $i \in \{1, \dots, 15\}$. Because $\iota_{P_i}(P_j) \neq P_j$ for $i \neq j$ ($i, j \in \{1, \dots, 15\}$), and $\iota_{P_i}(\{P_1, \dots, P_5\}) = \{P_1, \dots, P_5\}$ for any $i \in \{1, \dots, 5\}$, we have that $\iota_{P_i}(Q_1) = Q_1$ and $\iota_{P_i}(Q_2) = Q_3$, for any $i \in \{1, \dots, 5\}$.

Because $\iota_{P_1}(Q_1) = Q_1$, $\iota_{P_1}(Q_2) = Q_3$, $\iota_{P_1}(P_1) = P_1$, and $\iota_{P_1}^2 = \mathrm{id}_C$, we have that

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

By Lemma 2.5, the eigenspace corresponding P_1 is 1-dimensional. Thus we have that

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, we may assume that $P_2 = (0 : a : 1)$, $P_3 = (0 : 1 : a)$, $P_4 = (0 : b : 1)$, and $P_5 = (0 : 1 : b)$, where $a, b \in \mathbb{C} \setminus \{0\}$ and $a \neq b$. Moreover, we may assume that $\iota_{P_2}(P_1) = P_4$ and $\iota_{P_2}(P_3) = P_5$. Then, as $\iota_{P_2}(Q_1) = Q_1$, $\iota_{P_2}(Q_2) = Q_3$, $\iota_{P_2}(P_2) = P_2$, $\iota_{P_2}^2 = \mathrm{id}_C$, and $\iota_{P_2}(P_1) = P_4$, and by Lemma 2.5 (i.e., the eigenspace corresponding to P_2 is 1-dimensional), we have that

$$\iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -1/a & 0 \end{pmatrix}$$

and $b = -a^2$. Because $\iota_{P_2}(P_3) = P_5$, we have that $a^5 = -1$. Hence, we have that $P_1 = (0 : -1 : 1)$, $P_2 = (0 : -\zeta : 1)$, $P_3 = (0 : -\zeta^4 : 1)$, $P_4 = (0 : -\zeta^2 : 1)$, and $P_5 = (0 : -\zeta^3 : 1)$, where ζ is a primitive fifth root of unity.

By the same argument as above, we have that $P_6 = (-1 : 0 : 1)$, $P_7 = (-\zeta : 0 : 1)$, $P_8 = (-\zeta^4 : 0 : 1)$, $P_9 = (-\zeta^2 : 0 : 1)$, and $P_{10} = (-\zeta^3 : 0 : 1)$. By $\iota_{P_1}(V(Y) \cap C) = V(Z) \cap C$, we have that $P_{11} = (-1 : 1 : 0)$, $P_{12} = (-\zeta : 1 : 0)$, $P_{13} = (-\zeta^4 : 1 : 0)$, $P_{14} = (-\zeta^2 : 1 : 0)$, and $P_{15} = (-\zeta^3 : 1 : 0)$. Note that $F(5) \cap V(XYZ)$ is the set of these 15 points. □

By a similar argument to that in the proof of Claim 4.10, we have that

$$\iota_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\zeta \\ 0 & -\zeta^4 & 0 \end{pmatrix}, \quad \iota_{P_6} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let the defining equation of C be expressed by $F(X, Y, Z) = \sum_{i=0}^{5} X^{5-i} f_i(Y, Z)$. Because $Q_1 = (1 : 0 : 0) \notin C$, we may assume that $f_0(Y, Z) = 1$. As $\iota_{P_i}(C) = C$ $(i = 1, 2)$, we have that $\iota_{P_i}^* F = \lambda_i F$, where $\lambda_i \in \mathbb{C} \setminus \{0\}$. Using the matrices above as a representations of ι_{P_i} $(i = 1, 2, 6)$, we have that $\lambda_1 = \lambda_2 = 1$. Hence, each f_i $(i = 1, \dots, 5)$ is symmetric and $\iota_{P_2}^* f_i = f_i$. Therefore, $F = X^5 + \alpha X^3 YZ + \beta XY^2Z^2 + \gamma Y^5 + \gamma Z^5$, where $\alpha, \beta, \gamma \in \mathbb{C}$. As $P_6 = (-1 : 0 : 1) \in C$, we have $\gamma = 1$, and because $\iota_{P_6}^* F = \lambda_6 F$ for some $\lambda_6 \in \mathbb{C} \setminus \{0\}$, we have that $\alpha = \beta = 0$. Namely, C is the Fermat quintic curve. This completes the proof of Lemma 4.1.

4.2 – *Proof of Lemma* 4.2

In order to prove Lemma 4.2, we assume that $\#(\Delta(C, D_4) \cap \ell) \leq 3$ for every line $\ell \subset \mathbb{P}^2$. Aiming for a contradiction, we assume further that $\delta(C, D_4) > 3$.

Claim 4.11. *There exists a finite subset $S \subset \Delta(C, D_4)$ such that $\#S = 9$ and $\#(S \cap \ell) = 0, 1,$ or 3 for every line ℓ.*

Proof. Let L_{ijk} be the line passing through the collinear points P_i, P_j, P_k.

We can take two D_4-points P_1 and P_2. By Lemma 2.8, we can find a third point $P_3 := \iota_{P_1}(P_2)$ such that P_1, P_2, P_3 are collinear. By our assumption, $\Delta(C, D_4) \cap L_{123} = \{P_1, P_2, P_3\}$. Because $\delta(C, D_4) > 3$, there exists a point $P_0 \in \Delta(C, D_4) \setminus L_{123}$. Let $P_{i+3} := \iota_{P_0}(P_i)$ $(i = 1, 2, 3)$. Then, the three points $\{P_4, P_5, P_6\}$ (resp. the points $\{P_0, P_1, P_4\}$, $\{P_0, P_2, P_5\}$, and $\{P_0, P_3, P_6\}$) are collinear D_4-points, and

$$\Delta(C, D_4) \cap L_{456} = \{P_4, P_5, P_6\}$$

(resp. $\Delta(C, D_4) \cap L_{014} = \{P_0, P_1, P_4\}$, $\Delta(C, D_4) \cap L_{025} = \{P_0, P_2, P_5\}$, and $\Delta(C, D_4) \cap L_{036} = \{P_0, P_3, P_6\}$). Note that $P_0, \dots, P_6$ are seven distinct points. Let $P_7 := \iota_{P_5}(P_1)$ and $P_8 := \iota_{P_5}(P_3)$. Then, as L_{157} (resp. L_{358}) $\neq L_{123}, L_{456}, L_{014}, L_{025}, L_{036}$, we see that P_7 (resp. P_8) $\neq P_0, \dots, P_6$. Because $P_5 \notin L_{123}$, we see that $P_7 \neq P_8$. Hence, $P_0, \dots, P_8$ are nine distinct points. Let $S := \{P_0, \dots, P_8\}$.

We show that each set of three points $\{P_0, P_7, P_8\}$, $\{P_2, P_4, P_8\}$, $\{P_2, P_6, P_7\}$, $\{P_1, P_6, P_8\}$, and $\{P_3, P_4, P_7\}$ are collinear.

Consider the three points $\{P_0, P_7, P_8\}$. Because

$$\Delta(C, D_4) \cap L_{025} = \{P_0, P_2, P_5\},$$

we have that $\iota_{P_5}(P_2) = P_0$. Hence, $P_0, P_7, P_8 \in \iota_{P_5}(L_{123})$, i.e., $\{P_0, P_7, P_8\}$ are collinear, and $\Delta(C, D_4) \cap L_{078} = \{P_0, P_7, P_8\}$.

Next, consider the three points $\{P_2, P_4, P_8\}$. As

$$\Delta(C, D_4) \cap L_{014} = \{P_0, P_1, P_4\}$$

(resp. $\Delta(C, D_4) \cap L_{157} = \{P_1, P_5, P_7\}$), we have that $\iota_{P_1}(P_4) = P_0$ (resp. $\iota_{P_1}(P_5) = P_7$). Hence, $\iota_{P_1}(P_6) \in \iota_{P_1}(L_{456}) = L_{078}$, and so $\iota_{P_1}(P_6) = P_8$. Moreover, as $\iota_{P_1}(P_3) = P_2$ and $\iota_{P_1}(P_0) = P_4$, we have that $\{P_2, P_4, P_8\} \subset \iota_{P_1}(L_{036})$. Hence, $\{P_2, P_4, P_8\}$ are collinear, and $\Delta(C, D_4) \cap L_{248} = \{P_2, P_4, P_8\}$.

By the same argument as above, we see that $\{P_2, P_6, P_7\} = \iota_{P_3}(\{P_0, P_1, P_4\}) \subset \iota_{P_3}(L_{014})$ (resp. $\{P_1, P_6, P_8\} = \iota_{P_3}(\{P_0, P_2, P_5\}) \subset \iota_{P_3}(L_{025})$ and $\{P_3, P_4, P_7\} = \iota_{P_1}(\{P_0, P_2, P_5\}) \subset \iota_{P_1}(L_{025})$), and $\{P_2, P_6, P_7\}$ (resp. $\{P_1, P_6, P_8\}, \{P_3, P_4, P_7\}$) are collinear.

Therefore, $\#(S \cap \ell) = 0, 1,$ or 3 for every line ℓ. □

Let $P_0, \ldots, P_8$ and L_{ijk} be those in the proof of Claim 4.11.

Claim 4.12. *The three lines L_{123}, L_{456}, L_{078} are not concurrent, i.e., they do not meet at one point.*

Proof. Assume that $L_{123} \cap L_{456} \cap L_{078} = \{Q\}$. Because $\iota_{P_i}(L_{123}) = L_{456}$ and $\iota_{P_i}(L_{456}) = L_{123}$ $(i = 0, 7)$, we have that $\iota_{P_0}(Q) = Q$ and $\iota_{P_7}(Q) = Q$. Let $C \cap L_{078} := \{P_0, P_7, P_8, Q_1, Q_2\}$. Then, as $\iota_{P_0}(Q_1) = Q_2$, we have that $Q \neq Q_1, Q_2$. Note that $\iota_{P_0}(L_{078}) = L_{078}$ and $\iota_{P_7}(L_{078}) = L_{078}$. Let $\sigma := (\iota_{P_7} \circ \iota_{P_0})|_{L_{078}} \in \mathrm{PGL}(2, \mathbb{C})$. Since $\iota_{P_0}(Q_1) = \iota_{P_7}(Q_1) = Q_2$, we have $Q, Q_1, Q_2 \in \mathrm{Fix}(\sigma) := \{R \in L_{078} \mid \sigma(R) = R\}$. Hence, $\sigma = \mathrm{id}_{L_{078}}$. However, we have that $\sigma(P_0) = P_8$, which is a contradiction. □

Claim 4.13. *Taking a suitable projective transformation, we may assume the following*:

$$P_0 = (-1 : 1 : 0), \quad P_7 = (-\omega^2 : 1 : 0), \quad P_8 = (-\omega : 1 : 0),$$
$$P_1 = (-1 : 0 : 1), \quad P_2 = (-\omega : 0 : 1), \quad P_3 = (-\omega^2 : 0 : 1),$$
$$P_4 = (0 : -1 : 1), \quad P_5 = (0 : -\omega : 1), \quad P_6 = (0 : -\omega^2 : 1),$$

where ω is a primitive cubic root of unity.

Proof. Let

$$\{Q_1\} := L_{123} \cap L_{456}, \quad \{Q_2\} := L_{456} \cap L_{078}, \quad \{Q_3\} := L_{078} \cap L_{123}.$$

By Claim 4.12, Q_1, Q_2, Q_3 are not collinear, distinct points. Taking a suitable projective transformation, we may assume that

$$Q_1 = (0 : 0 : 1), \quad Q_2 = (0 : 1 : 0), \quad Q_3 = (1 : 0 : 0).$$

Then,

$$L_{456} = V(X), \quad L_{123} = V(Y), \quad L_{078} = V(Z).$$

Note that $Q_i \notin \{P_0, \ldots, P_8\}$ $(i = 1, 2, 3)$. Let $P_i = (0 : a_i : 1)$ $(i = 4, 5, 6)$, where $a_i \in \mathbb{C} \setminus \{0\}$. Taking the projective transformation

$$(X : Y : Z) \longmapsto (X : -1/a_4 Y : Z),$$

we may assume that $a_4 = -1$. From $\iota_{P_4}(L_{456}) = L_{456}$, $\iota_{P_4}(L_{123}) = L_{078}$, $\iota_{P_4}^2 = \mathrm{id}_C$, $\iota_{P_4}(P_4) = P_4$, and $P_4 \notin \ell_{P_4}$, we infer that

$$\iota_{P_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, $P_6 = (0 : a_6 : 1) = \iota_{P_4}(P_5) = (0 : 1 : a_5)$, and so $a_5 a_6 = 1$.

By a similar argument to that above (note that $\iota_{P_5}(P_4) = P_6$), we infer that

$$\iota_{P_5} = \begin{pmatrix} \sqrt{-a_6} & 0 & 0 \\ 0 & 0 & -a_6 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence, $P_5 = (0 : a_5 : 1) = (0 : -a_6 : a_5)$, and so $a_5^2 = -a_6$. We have that $a_5^3 = -1$. Note that $P_5 \neq P_4$, that is, $a_5 \neq -1$. Hence, we may put $a_5 = -\omega$ and $P_4 = (0 : -1 : 1)$, $\quad P_5 = (0 : -\omega : 1)$, $P_6 = (0 : -\omega^2 : 1)$.

Again, using a similar argument, we may assume that $P_1 = (-1 : 0 : 1)$, $P_2 = (-\omega : 0 : 1)$, and $P_3 = (-\omega^2 : 0 : 1)$. As $P_0 \in L_{078} \cap L_{014}$, $P_7 \in L_{078} \cap L_{267}$, $P_8 \in L_{078} \cap L_{358}$, we have that $P_0 = (-1 : 1 : 0)$, $P_7 = (-\omega^2 : 1 : 0)$, and $P_8 = (-\omega : 1 : 0)$. $\square$

Claim 4.14. *Assume that $P_0, \ldots, P_8$ are the points in Claim 4.13. Then,*

$$\iota_{P_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \iota_{P_7} = \begin{pmatrix} 0 & \omega^2 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \iota_{P_8} = \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_{P_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \iota_{P_2} = \begin{pmatrix} 0 & 0 & \omega^2 \\ 0 & \omega & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \iota_{P_3} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & \omega^2 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\iota_{P_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \iota_{P_5} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \iota_{P_6} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. From $\iota_{P_0}(L_{078}) = L_{078}$, $\iota_{P_0}(L_{123}) = L_{456}$, $\iota_{P_0}(L_{456}) = L_{123}$, $\iota_{P_0}(P_1) = P_4$, and $\iota_{P_0}(P_4) = P_1$, we infer that

$$\iota_{P_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By a similar argument to that above, we can determine the other ι_{P_i}. □

Claim 4.15. *Assume that $P_0, \dots, P_8$ are the points in Claim 4.13. Then,* $(0:0:1)$*,* $(0:1:0)$*,* $(1:0:0) \in C$.

Proof. Note that $\#(C \cap L_{078}) = 5$ by Remark 2.11. Let

$$C \cap L_{078} = \{P_0, P_7, P_8, Q_1, Q_2\}.$$

As neither Q_1 nor Q_2 are D_4-points, we have that $\iota_{P_i}(Q_1) = Q_2$ $(i = 0, 7, 8)$. Let $\sigma := \iota_{P_7} \circ \iota_{P_0}$. Then, $Q_1, Q_2 \in \mathrm{Fix}(\sigma) \cap L_{078}$. Because

$$\sigma = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

we have that $\mathrm{Fix}(\sigma) \cap L_{078} = \{(1:0:0), (0:1:0)\}$. Hence

$$\{(1:0:0), (0:1:0)\} = \{Q_1, Q_2\} \subset C.$$

Thus $\iota_{P_4}((0:1:0)) = (0:0:1) \in C$. □

In order to complete the proof of Lemma 4.2, let us consider the the points $P_0, \dots, P_8$ listed in Claim 4.13, and let

$$\sigma := \iota_{P_7} \circ \iota_{P_0} \in \mathrm{Aut}(C)$$

with representation matrix given by (41). Then, we obtain the cyclic Galois covering $C \to C/\langle\sigma\rangle$ of degree 3, whose ramification points are the three points $(0:0:1), (0:1:0), (1:0:0) \in C$. However, by the Riemann–Hurwitz formula, this is a contradiction. Hence, we have $\delta(C, D_4) \leq 3$, which concludes Lemma 4.2.

4.3 – *Proof of Theorem 1.2*

Let us prove $\delta(C, D_4) \in \{0, 1, 3, 5, 15\}$. Clearly, $\delta(C, D_4) \neq 2$ by Lemma 2.8.

If $\delta(C, D_4)$ were equal to 4, then $\delta(C, D_4)$ would consist of 4 collinear points $P_1, \ldots, P_4$. Indeed, P_1, P_2 and

$$P_3 := \iota_{P_1}(P_2)$$

would be collinear by Lemma 2.8, and the same would hold for P_1, P_4 and $\iota_{P_1}(P_4) \in \{P_2, P_3\}$. Thus we would get a contradiction as Lemma 2.9 would imply $\delta(C, D_4) \geq 5$.

Finally, if $\delta(C, D_4) \geq 6$, then Lemmas 2.9 and 4.2 give that there exists a line ℓ passing exactly through five D_4-points. So there exists another D_4-point not lying on ℓ, and Lemma 4.1 implies that $\delta(C, D_4) = 15$ and C is projectively equivalent to the Fermat quintic curve $F(5)$. Conversely, if C is projectively equivalent to the Fermat quintic curve $F(5)$, by the analysis of Section 3 we deduce that $\delta(C, D_4) = 15$, and C possesses three disjoint 5-tuples of collinear D_4-points.

In order to conclude the proof of Theorem 1.2, we only need to show that if $\delta(C, D_4) = 3$ or 5, then $\Delta(C, D_4)$ lies on a line. The case $\delta(C, D_4) = 3$ follows straightforwardly from Lemma 2.8, whereas Lemmas 2.9 and 4.2 give the assertion when $\delta(C, D_4) = 5$.

Acknowledgements. The author expresses his sincere thanks to Professor Takeshi Harui and the referee for their valuable comments and suggestions. This work was supported by JSPS KAKENHI Grant Number 25400059.

References

[1] F. Cukierman, *Monodromy of projections,* 15th School of Algebra (Portuguese) (Canela, 1998), Mat. Contemp. **16** (1999), pp. 9–30.

[2] K. Miura, *Field theory for function fields of plane quintic curves, Algebra Colloq.* 9 (2002), no. 3, pp. 303–312.

[3] K. Miura – H. Yoshihara, *Field theory for function fields of plane quartic curves,* J. Algebra **226** (2000), no. 1, pp. 283–294.

[4] K. Miura – H. Yoshihara, *Field theory for the function field of the quintic Fermat curve,* Comm. Algebra **28** (2000), no. 4, pp. 1979–1988.

[5] M. Namba, *Families of meromorphic functions on compact Riemann surfaces,* Lecture Notes in Mathematics, 767, Springer-Verlag, Berlin, 1979.

[6] M. Namba, *Geometry of projective algebraic curves,* Monographs and Textbooks in Pure and Applied Mathematics, 88, Marcel Dekker, New York, 1984.

[7] G. P. Pirola – E. Schlesinger, *Monodromy of projective curves,* J. Algebraic Geom. **14** (2005), no. 4, pp. 623–642.

[8] H. Yoshihara, *Function field theory of plane curves by dual curves,* J. Algebra **239** (2001), no. 1, pp. 340–355.

Manoscritto pervenuto in redazione il 27 novembre 2013.