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Strongly FP-injective and strongly flat functors

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Abstract – A functor $F \in (\text{mod-}R, \text{Ab})$ is called *strongly FP-injective* if F is isomorphic to some functor $- \otimes M$ in $(\text{mod-}R, \text{Ab})$ with M an FP-injective left R-module. A functor $G \in ((\text{mod-}R)^{\text{op}}, \text{Ab})$ is said to be *strongly flat* if G is isomorphic to some functor $(-, N)$ in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with N a flat right R-module. We study the properties of strongly FP-injective functors and explore the relationship between strongly FP-injective functors and strongly flat functors. Precovers and preenvelopes by strongly FP-injective and strongly flat functors are also investigated.

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1. Introduction

Given a ring R, denote by mod-R the category of finitely presented right R-modules, the two functor categories $(\text{mod-}R, \text{Ab})$ and $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with values in the category Ab of abelian groups have received extensive attention since the 1960s. Particularly they play important roles in the investigation of the model theory of modules and the representation theory of Artinian algebras (see e.g. [1, 2, 3, 4, 6, 7, 14, 17, 20, 22, 23]). Strikingly, Herzog has developed a general theory for the category $(\text{mod-}R, \text{Ab})$ and $((\text{mod-}R)^{\text{op}}, \text{Ab})$ for any ring R (see [15, 16]).

It is well known that both $(\text{mod-}R, \text{Ab})$ and $((\text{mod-}R)^{\text{op}}, \text{Ab})$ are locally finitely presented Grothendieck additive categories. There is a full and faithful right exact functor from the category R-Mod of left R-modules to $(\text{mod-}R, \text{Ab})$, which is given by the rule $M \to - \otimes M$. Similarly, the functor given by $M \to (-, M)$ from

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the category Mod-R of right R-modules to $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is a full and faithful left exact functor. In this paper, we are interested in investigating the objects of the functor category (mod-R, Ab) which correspond to the FP-injective left R-modules via the above right exact functor and the objects of the functor category $((\text{mod-}R)^{\text{op}}, \text{Ab})$ which correspond to the flat right R-modules via the above left exact functor. This paper is organized as follows.

In Section 2, we first introduce the concept of strongly FP-injective functors in (mod-R, Ab). Some basic properties of strongly FP-injective functors are obtained. For example, we prove that $-\otimes M$ is a strongly FP-injective functor in (mod-R, Ab) if and only if for every exact sequence $0 \to M \to N \to L \to 0$ of left R-modules and every finitely presented functor $G \in$ (mod-R, Ab), the induced morphism $[G, -\otimes N] \to [G, -\otimes L]$ is an epimorphism. We also prove that the class of strongly FP-injective functors in (mod-R, Ab) is closed under extensions, direct sums, direct products and pure subfunctors. Then we explore the relationship between strongly FP-injective functors in (mod-R, Ab) and strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ in the sense of [7]. It is shown that a functor F in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is strongly flat if and only if F^+ is strongly FP-injective in (mod-R, Ab). Finally we give some characterizations of left coherent rings and von Neumann regular rings in terms of strongly FP-injective and strongly flat functors.

Section 3 is devoted to precovers and preenvelopes by strongly FP-injective and strongly flat functors. We first obtain that

(1) the class of strongly FP-injective functors in (mod-R, Ab) is a preenveloping class;

(2) if R is a left coherent ring, then the class of strongly FP-injective functors in (mod-R, Ab) is a covering class;

(3) R is a left coherent ring if and only if the class of strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is a preenveloping class.

Then we obtain that

(1) a right R-module homomorphism $A \xrightarrow{f} B$ is a flat precover (resp. cover) of B in Mod-R if and only if $(-, A) \xrightarrow{(-,f)} (-, B)$ is a strongly flat precover (resp. cover) of $(-, B)$ in $((\text{mod-}R)^{\text{op}}, \text{Ab})$;

(2) a left R-module homomorphism $A \xrightarrow{f} B$ is an FP-injective preenvelope (resp. envelope) of A in R-Mod if and only if $-\otimes A \xrightarrow{-\otimes f} -\otimes B$ is a strongly FP-injective preenvelope (resp. envelope) of $-\otimes A$ in (mod-R, Ab).

Finally, for a left coherent ring R, it is shown that

(1) if $f: A \to B$ is a strongly FP-injective preenvelope of a functor A in (mod-R, Ab), then $f^+: B^+ \to A^+$ is a strongly flat precover of A^+ in ((mod-R)$^{\text{op}}$, Ab); the converse holds if R is a left Noetherian ring;

(2) if $g: C \to D$ is a strongly flat preenvelope of a functor C in ((mod-R)$^{\text{op}}$, Ab), then $g^+: D^+ \to C^+$ is a strongly FP-injective precover of C^+ in (mod-R, Ab); the converse holds if R is a right perfect ring.

Throughout this paper, R is an associative ring with identity and all modules are unitary. The character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of an R-module M is denoted by M^+. Given R-modules M and N, we use the abbreviation $(M, N) := \text{Hom}_R(M, N)$. We denote Mod-$R$ (resp. R-Mod) for the category of right (resp. left) R-modules. If F and G are objects of (mod-R, Ab) or ((mod-R)$^{\text{op}}$, Ab), then the set of morphisms $[F, G]$ consists of the natural transformations from F to G.

We next recall some known notions needed in the sequel.

A sequence of functors $F \overset{\mu}{\to} G \overset{\nu}{\to} H$ in ($\mathcal{C}$, Ab) with $\mathcal{C}$ a skeletally small additive category is called *exact* if for every $A \in \mathcal{C}$, the corresponding sequence of abelian groups $F(A) \overset{\mu_A}{\to} G(A) \overset{\nu_A}{\to} H(A)$ is exact.

A functor X of ($\mathcal{C}$, Ab) with $\mathcal{C}$ a skeletally small additive category is called *finitely presented* if $[X, -]$ commutes with direct limits, equivalently, there is an exact sequence $(A, -) \to (B, -) \to X \to 0$ with A and B in $\mathcal{C}$.

An exact sequence $0 \to X \to Y \to Z \to 0$ in ($\mathcal{C}$, Ab) with $\mathcal{C}$ a skeletally small additive category is called *pure* [6] if it induces an exact sequence of abelian groups $0 \to [P, X] \to [P, Y] \to [P, Z] \to 0$ for every finitely presented functor P of ($\mathcal{C}$, Ab). In this case, $X \to Y$ is called a *pure monomorphism* and $Y \to Z$ a *pure epimorphism*.

A functor F of ($\mathcal{C}$, Ab) with $\mathcal{C}$ a skeletally small additive category is said to be *FP-injective* if $\text{Ext}^1[X, F] = 0$ for every finitely presented functor $X \in$ ($\mathcal{C}$, Ab). By [17, Lemma 1.3], A functor F of (mod-R, Ab) is FP-injective if and only if F is isomorphic to $- \otimes M$ of (mod-R, Ab) for some left R-module M.

For unexplained concepts, we refer the reader to [10, 11, 13, 17, 21, 25, 26].

2. Definition and general results

Recall that a left R-module M is *FP-injective* [24] or *absolutely pure* [19] if $\text{Ext}^1(N, M) = 0$ for every finitely presented left R-module N, equivalently, every exact sequence $0 \to M \to N \to L \to 0$ of left R-modules is pure.

Definition 2.1. Given a ring R, a functor $F \in (\text{mod-}R, \text{Ab})$ is called *strongly FP-injective* if F is isomorphic to some functor $- \otimes M$ in (mod-R, Ab) with M an FP-injective left R-module.

Remark 2.2. Any strongly FP-injective functor in (mod-R, Ab) is clearly FP-injective, but the converse is not true. For example, the functor $- \otimes \mathbb{Z}$ in (mod-$\mathbb{Z}$, Ab) is FP-injective but is not strongly FP-injective since $\mathbb{Z}$ is not an FP-injective $\mathbb{Z}$-module.

Proposition 2.3. *Let* $- \otimes M \in (\text{mod-}R, \text{Ab})$. *Then the following conditions are equivalent*:

(1) $- \otimes M$ *is a strongly FP-injective functor in* (mod-R, Ab);

(2) *for every exact sequence* $0 \to M \to N \to L \to 0$ *of left* R*-modules and every finitely presented functor* $G \in (\text{mod-}R, \text{Ab})$, *the induced morphism* $[G, - \otimes N] \to [G, - \otimes L]$ *is an epimorphism.*

Proof. (1) $\Longrightarrow$ (2). Since M is an FP-injective left R-module, the exact sequence $0 \to M \to N \to L \to 0$ of left R-modules is pure and so we get the exact sequence $0 \to - \otimes M \to - \otimes N \to - \otimes L \to 0$ in (mod-R, Ab). Thus for every finitely presented functor $G \in (\text{mod-}R, \text{Ab})$, we have the induced exact sequence

$$[G, - \otimes N] \longrightarrow [G, - \otimes L] \longrightarrow \text{Ext}^1[G, - \otimes M] = 0.$$

So the morphism $[G, - \otimes N] \to [G, - \otimes L]$ is an epimorphism.

(2) $\Longrightarrow$ (1). It is enough to show that any exact sequence

$$(*) \qquad 0 \longrightarrow M \longrightarrow N \stackrel{g}{\longrightarrow} L \longrightarrow 0$$

of left R-modules is pure. If A is any finitely presented left R-module and $A \stackrel{f}{\to} L$ is any left R-module homomorphism, then there exists the exact sequence

$$R^m \longrightarrow R^n \longrightarrow A \longrightarrow 0$$

with $m, n \in \mathbb{N}$, which induces the exact sequence in (mod-R, Ab):

$$- \otimes R^m \longrightarrow - \otimes R^n \longrightarrow - \otimes A \longrightarrow 0.$$

Since $- \otimes R^m \cong (R^m, -)$ and $- \otimes R^n \cong (R^n, -)$ in (mod-R, Ab), we have $- \otimes A$ is a finitely presented functor in (mod-R, Ab). So there exists $\beta \colon - \otimes A \to - \otimes N$

such that the following diagram is commutative:

$$\begin{array}{ccc} & & -\otimes A \\ & \overset{\beta}{\swarrow} & \Big\downarrow {\scriptstyle -\otimes f} \\ -\otimes N & \xrightarrow[-\otimes g]{} & -\otimes L. \end{array}$$

Now we have $\beta = -\otimes h$ for some left R-module morphism $h\colon A \to N$. Thus

$$-\otimes f = (-\otimes g)\beta = (-\otimes g)(-\otimes h) = -\otimes (gh).$$

Hence $f = gh$. So ($*$) is a pure exact sequence. Thus M is FP-injective and so $-\otimes M$ is strongly FP-injective in (mod-R, Ab). □

PROPOSITION 2.4. *The class of strongly FP-injective functors in* (mod-R, Ab) *is closed under extensions, direct sums, direct products and pure subfunctors.*

PROOF. Let $0 \to -\otimes M \to G \to -\otimes L \to 0$ be an exact sequence in (mod-R, Ab) with $-\otimes M$ and $-\otimes L$ strongly FP-injective. Then G is FP-injective by [18, Proposition 2.3(1)]. Let $G = -\otimes N$. Then we get the exact sequence $0 \to M \to N \to L \to 0$ of left R-modules. So N is FP-injective since M and L are FP-injective. Hence G is a strongly FP-injective functor.

Let $\{-\otimes M_i\}$ be a family of strongly FP-injective functors in (mod-R, Ab). Then $\coprod(-\otimes M_i) \cong -\otimes \coprod M_i$ and $\prod(-\otimes M_i) \cong -\otimes \prod M_i$ by [25, Lemma 13.2]. Note that $\coprod M_i$ and $\prod M_i$ are FP-injective. So $\coprod(-\otimes M_i)$ and $\prod(-\otimes M_i)$ are strongly FP-injective functors.

Let F be a pure subfunctor of a strongly FP-injective functor $-\otimes M$ in (mod-R, Ab). Then for any finitely presented functor G in (mod-R, Ab), the pure exact sequence $0 \to F \to -\otimes M \to H \to 0$ induces the exact sequence

$$[G, -\otimes M] \longrightarrow [G, H] \longrightarrow \mathrm{Ext}^1[G, F] \longrightarrow \mathrm{Ext}^1[G, -\otimes M] = 0.$$

Since $[G, -\otimes M] \to [G, H]$ is epic, $\mathrm{Ext}^1[G, F] = 0$. Thus $F = -\otimes N$ for some left R-module N. Hence N may be viewed as a pure submodule of M and so is FP-injective. Thus F is a strongly FP-injective functor. □

Recall that a functor of $((\text{mod-}R)^{\mathrm{op}}, \mathrm{Ab})$ is *flat* if it is isomorphic to a direct limit of finitely generated projective functors, or equivalently, it is isomorphic to $(-, M)$ for some right R-module M by [6, Theorem 1.4]. In [7], Crivei called a flat functor $(-, Z)$ of $((\text{mod-}R)^{\mathrm{op}}, \mathrm{Ab})$ *strongly flat* if for every morphism $g_*\colon (-, Y) \to (-, Z)$ in $((\text{mod-}R)^{\mathrm{op}}, \mathrm{Ab})$ such that $g\colon Y \to Z$ is an epimorphism in

Mod-R, and for every finitely presented object P of $((\text{mod-}R)^{\text{op}}, \text{Ab})$, the induced abelian group homomorphism $[P, (-, Y)] \to [P, (-, Z)]$ is an epimorphism. By [7, Theorem 4.1], a functor F of $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is strongly flat if and only if F is isomorphic to some functor $(-, M)$ in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with M a flat right R-module.

We now discuss the relationship between strongly FP-injective functors in $(\text{mod-}R, \text{Ab})$ and strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$.

Let F be an additive functor from an additive category $\mathcal{C}$ to Ab. We denote by F^+ the character functor from $\mathcal{C}^{\text{op}}$ to Ab, defined by $F^+(M) = (F(M))^+$ for any $M \in \mathcal{C}$. There is a standard morphism $\delta_F: F \to F^{++}$ defined by $(\delta_F)_M(x)(\tau) = \tau(x)$ for any $M \in \mathcal{C}$, $x \in F(M)$, $\tau \in F^+(M)$.

Proposition 2.5. *A functor F in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is strongly flat if and only if F^+ is strongly FP-injective in* $(\text{mod-}R, \text{Ab})$.

Proof. $\Longrightarrow$. Let $F = (-, M) \in ((\text{mod-}R)^{\text{op}}, \text{Ab})$ with M a flat right R-module. Then M^+ is an FP-injective left R-module and so $- \otimes M^+$ is a strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$. By [5, Lemma 2], we have $(-, M)^+ \cong - \otimes M^+ \in (\text{mod-}R, \text{Ab})$. Thus $F^+ = (-, M)^+$ is a strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$.

$\Longleftarrow$. Since F^+ is an FP-injective functor in $(\text{mod-}R, \text{Ab})$, F is a flat functor in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ by [18, Proposition 2.9(2)]. Let $F = (-, N) \in ((\text{mod-}R)^{\text{op}}, \text{Ab})$. Then $F^+ = (-, N)^+ \cong - \otimes N^+ \in (\text{mod-}R, \text{Ab})$ by [5, Lemma 2]. So N^+ is an FP-injective left R-module by hypothesis. Thus N is a flat right R-module by [12, Theorem 2.1]. Hence F is a strongly flat functor in $((\text{mod-}R)^{\text{op}}, \text{Ab})$. □

As a consequence of Propositions 2.4 and 2.5, we have the following result.

Corollary 2.6 (cf. [7], Theorem 3.2). *The class of strongly flat functors in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is closed under extensions, direct limits, pure subfunctors and pure quotients.*

Proof. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with A and C strongly flat. Then we obtain the exact sequence

$$0 \longrightarrow C^+ \longrightarrow B^+ \longrightarrow A^+ \longrightarrow 0$$

in $(\text{mod-}R, \text{Ab})$. Note that A^+ and C^+ are strongly FP-injective by Proposition 2.5. So B^+ is strongly FP-injective by Proposition 2.4. Thus B is strongly flat.

Let $((-, M_i))_{i \in I}$ be any direct system in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with each $(-, M_i)$ a strongly flat functor. Then $\varinjlim(-, M_i) \cong (-, \varinjlim M_i)$. Since $\varinjlim M_i$ is a flat right R-module, $\varinjlim(-, M_i)$ is a strongly flat functor.

Let

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

be a pure exact sequence in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ with G strongly flat. Then we obtain the split exact sequence

$$0 \longrightarrow H^+ \longrightarrow G^+ \longrightarrow F^+ \longrightarrow 0$$

in $(\text{mod-}R, \text{Ab})$ by [23, Theorem 2]. Since G^+ is strongly FP-injective by Proposition 2.5, F^+ and H^+ are strongly FP-injective by Proposition 2.4. Thus F and H are strongly flat. □

Next we characterize several important rings in terms of strongly FP-injective and strongly flat functors.

Theorem 2.7. *The following conditions are equivalent for a ring* R:

(1) *R is a left coherent ring*;

(2) *the class of strongly FP-injective functors in* $(\text{mod-}R, \text{Ab})$ *is closed under direct limits*;

(3) *the class of strongly FP-injective functors in* $(\text{mod-}R, \text{Ab})$ *is closed under pure quotients*;

(4) *a functor F of* $(\text{mod-}R, \text{Ab})$ *is strongly FP-injective if and only if F^+ is strongly flat in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$;

(5) *a functor F of* $(\text{mod-}R, \text{Ab})$ *is strongly FP-injective if and only if F^{++} is strongly FP-injective in* $(\text{mod-}R, \text{Ab})$;

(6) *a functor F of* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is strongly flat if and only if F^{++} is strongly flat in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$;

(7) *the class of strongly flat functors in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is closed under direct products.*

Proof. (1) $\Longrightarrow$ (2). Let $(- \otimes M_i)_{i \in I}$ be any direct system in $(\text{mod-}R, \text{Ab})$ with each $- \otimes M_i$ a strongly FP-injective functor. Then $\varinjlim(- \otimes M_i) \cong - \otimes \varinjlim M_i$. By [24, Theorem 3.2], $\varinjlim M_i$ is FP-injective. So $\varinjlim(- \otimes M_i)$ is a strongly FP-injective functor.

(2) $\Longrightarrow$ (1). Let $(M_i)_{i\in I}$ be any direct system in R-Mod with each M_i an FP-injective left R-module. Then $-\otimes \varinjlim M_i \cong \varinjlim(-\otimes M_i)$ is strongly FP-injective. Thus $\varinjlim M_i$ is FP-injective and so R is a left coherent ring by [24, Theorem 3.2].

(1) $\Longrightarrow$ (3). Let $0 \to A \to -\otimes M \to C \to 0$ be any pure exact sequence in (mod-R, Ab) with $-\otimes M$ a strongly FP-injective functor. Then A is a strongly FP-injective functor by Proposition 2.4. Let $A = -\otimes N$. By [18, Proposition 2.3(1)], C is an FP-injective functor. Let $C = -\otimes L$. Then we get the pure exact sequence $0 \to N \to M \to L \to 0$ of left R-modules. By [26, 35.9], L is an FP-injective left R-module. So C is a strongly FP-injective functor.

(3) $\Longrightarrow$ (1). Let $0 \to K \to M \to N \to 0$ be any pure exact sequence of left R-modules with M an FP-injective left R-module. Then we get the pure exact sequence $0 \to -\otimes K \to -\otimes M \to -\otimes N \to 0$ in (mod-R, Ab). Thus $-\otimes N$ is a strongly FP-injective functor and so N is an FP-injective left R-module. Thus R is a left coherent ring by [26, 35.9].

Conversely, if $F = -\otimes M$ is a strongly FP-injective functor in (mod-R, Ab), then $(-\otimes M)^+ \cong (-, M^+)$. Note that M^+ is a flat right R-module by [26, 35.9]. So $(-\otimes M)^+$ is a strongly flat functor.

(4) $\Longrightarrow$ (1). If F^+ is a strongly flat functor, then F is an FP-injective functor by [18, Proposition 2.9(1)]. Let $F = -\otimes N$. Then $(-\otimes N)^+ \cong (-, N^+)$. So N^+ is a flat right R-module and hence N is an FP-injective left R-module. Therefore F is strongly FP-injective.

(4) $\Longrightarrow$ (5). If F is strongly FP-injective, then F^+ is strongly flat. So F^{++} is strongly FP-injective by Proposition 2.5.

Conversely, if F^{++} is strongly FP-injective, then, by Proposition 2.5, also F^+ is strongly flat. So F is strongly FP-injective by (4).

(5) $\Longrightarrow$ (6). If $F \in (\text{mod} - R)^{\text{op}}$ is strongly flat, then F^+ is strongly FP-injective by Proposition 2.5. Thus F^{+++} is strongly FP-injective by (5). So F^{++} is strongly flat by Proposition 2.5.

Conversely, if F^{++} is strongly flat, then F^{+++} is strongly FP-injective by Proposition 2.5. So F^+ is strongly FP-injective by (5). Thus F is strongly flat by Proposition 2.5.

(6) $\Longrightarrow$ (7). Let $\{(-, M_i)\}$ be a family of strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$. Then $\coprod(-, M_i)$ is strongly flat by Corollary 2.6. So $(\coprod(-, M_i))^{++}$ is strongly flat by (6). Note that $\big(\coprod(-, M_i)\big)^{++} \cong \big(\prod(-, M_i)^{+}\big)^{+}$. Thus $(\prod(-, M_i)^{+})^{+}$ is strongly flat. Since $\coprod(-, M_i)^{+}$ is a pure subfunctor of $\prod(-, M_i)^{+}$ by [23, p.354], the exact sequence

$$\Big(\prod(-, M_i)^{+}\Big)^{+} \longrightarrow \Big(\coprod(-, M_i)^{+}\Big)^{+} \longrightarrow 0$$

is split by [23, Theorem 2]. Hence $\prod(-, M_i)^{++} \cong (\coprod(-, M_i)^{+})^{+}$ is strongly flat by Corollary 2.6. Since M_i is a pure submodule of M_i^{++}, $(-, M_i)$ is a pure subfunctor of $(-, M_i^{++})$. Note that $(-, M_i^{++}) \cong (-, M_i)^{++}$. Thus $\prod(-, M_i)$ is a pure subfunctor of $\prod(-, M_i)^{++}$, and so $\prod(-, M_i)$ is strongly flat by Corollary 2.6.

(7) $\Longrightarrow$ (1). Let $\{M_i\}$ be a family of flat right R-modules. Then $\prod(-, M_i)$ is strongly flat by (7). Since $(-, \prod M_i) \cong \prod(-, M_i)$, $(-, \prod M_i)$ is strongly flat. Thus $\prod M_i$ is a flat right R-module. So R is a left coherent ring by [26, 26.6]. ☐

Gruson and Jensen [14, 1.2] characterized the injective objects of $(\text{mod-}R, \text{Ab})$ as the functors isomorphic to some $-\otimes M$ with M a pure-injective left R-module. Thus a left R-module M is injective if and only if the functor $-\otimes M$ in $(\text{mod-}R, \text{Ab})$ is both injective and strongly FP-injective.

Although an injective functor in $(\text{mod-}R, \text{Ab})$ is always FP-injective, an injective functor in $(\text{mod-}R, \text{Ab})$ need not be strongly FP-injective in general. For example, since $\mathbb{Q}/\mathbb{Z}$ is not a flat $\mathbb{Z}$-module, $(\mathbb{Q}/\mathbb{Z})^{+}$ is not an FP-injective $\mathbb{Z}$-module. So the injective functor $-\otimes(\mathbb{Q}/\mathbb{Z})^{+}$ in $(\text{mod-}\mathbb{Z}, \text{Ab})$ is not strongly FP-injective.

Conversely, a strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$ need not be injective in general. For instance, let R be the endomorphism ring of a right infinite-dimensional vector space over a division ring. Then R is a von Neumann regular ring but is not left self-injective. So the functor $-\otimes R$ in $(\text{mod-}R, \text{Ab})$ is strongly FP-injective but is not injective since ${}_RR$ is not a pure-injective left R-module.

The following results further clarify the relationship among these functors.

Proposition 2.8. *The following conditions are equivalent for a ring R:*

(1) *R is a von Neumann regular ring;*
(2) *every functor in* $(\text{mod-}R, \text{Ab})$ *is strongly FP-injective;*
(3) *every FP-injective functor in* $(\text{mod-}R, \text{Ab})$ *is strongly FP-injective;*
(4) *every injective functor in* $(\text{mod-}R, \text{Ab})$ *is strongly FP-injective;*
(5) *every functor in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is strongly flat;*
(6) *every flat functor in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is strongly flat;*
(7) *every projective functor in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is strongly flat.*

Proof. (1) $\Longrightarrow$ (2). Let $F \in (\text{mod-}R, \text{Ab})$. Then $F = - \otimes M$ by [18, Theorem 2.17]. Note that any left R-module is FP-injective. So F is strongly FP-injective.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (5) $\Longrightarrow$ (6) $\Longrightarrow$ (7) are trivial.

(4) $\Longrightarrow$ (1). Let M be any right R-module. Then $- \otimes M^{+}$ is an injective functor in $(\text{mod-}R, \text{Ab})$ and so is strongly FP-injective. Thus M^{+} is FP-injective. So M is flat. Hence R is a von Neumann regular ring.

(1) $\Longrightarrow$ (5). Let $F \in ((\text{mod-}R)^{\text{op}}, \text{Ab})$. Then $F = (-, N)$ by [4, Proposition 4.1]. Since N is flat, F is strongly flat.

(7) $\Longrightarrow$ (1). Let M be any finitely presented right R-module. Then $(-, M)$ is a projective functor in $(\text{mod-}R, \text{Ab})$ and so is strongly flat. Thus M is flat. Hence R is a von Neumann regular ring. □

Proposition 2.9. *Let R be a ring. Then*

(1) *R is a left Noetherian ring if and only if every strongly FP-injective functor in* $(\text{mod-}R, \text{Ab})$ *is injective*;

(2) *R is a right perfect ring if and only if every strongly flat functor in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is projective.*

Proof. (1) R is left Noetherian if and only if every FP-injective left R-module is (pure-)injective if and only if every strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$ is injective.

(2) R is right perfect if and only if every flat right R-module is (pure-)projective if and only if every strongly flat functor in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is projective. □

3. Precovers and preenvelopes by strongly FP-injective and strongly flat functors

Let $\mathcal{A}$ be any category and $\mathcal{B}$ a class of objects in $\mathcal{A}$. Recall from [9, 11] that a morphism $\phi: B \to A$ in $\mathcal{A}$ is a $\mathcal{B}$-*precover* of A if $B \in \mathcal{B}$ and, for any morphism $f: B' \to A$ with $B' \in \mathcal{B}$, there is a morphism $g: B' \to B$ such that $\phi g = f$. A $\mathcal{B}$-precover $\phi: B \to A$ is called a $\mathcal{B}$-*cover of* A if every endomorphism $g: B \to B$ such that $\phi g = \phi$ is an isomorphism. The class $\mathcal{B}$ is called a *precovering class* (resp. *covering class*) if every object in $\mathcal{A}$ has a $\mathcal{B}$-precover (resp. $\mathcal{B}$-cover). Dually we have the definitions of $\mathcal{B}$-(pre)envelopes and (pre)enveloping classes. $\mathcal{B}$-covers ($\mathcal{B}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

It is known that the class of strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is a covering class by the proof of [7, Corollary 4]. Here we have

Proposition 3.1. *Let R be a ring.*

(1) *The class of strongly FP-injective functors in* $(\text{mod-}R, \text{Ab})$ *is a preenveloping class.*

(2) *If R is a left coherent ring, then the class of strongly FP-injective functors in* $(\text{mod-}R, \text{Ab})$ *is a covering class.*

(3) *R is a left coherent ring if and only if the class of strongly flat functors in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$ *is a preenveloping class.*

Proof. (1) Since the class of strongly FP-injective functors in $(\text{mod-}R, \text{Ab})$ is closed under direct products and pure subfunctors by Proposition 2.4, it is a preenveloping class by [8, Theorem 4.1].

(2) Since the class of strongly FP-injective functors in $(\text{mod-}R, \text{Ab})$ is closed under direct limits and pure quotients by Theorem 2.7, it is a covering class by [8, Theorem 2.4].

(3) $\Longrightarrow$. Since the class of strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ is closed under direct products by Theorem 2.7 and closed under pure subfunctors by Corollary 2.6, it is a preenveloping class by [8, Theorem 4.1].

$\Longleftarrow$. Let $\{(-, M_i)\}$ be a family of strongly flat functors in $((\text{mod-}R)^{\text{op}}, \text{Ab})$. By hypothesis, $\prod(-, M_i)$ has a strongly flat preenvelope $\beta\colon \prod(-, M_i) \to (-, N)$ in $((\text{mod-}R)^{\text{op}}, \text{Ab})$. Let $\pi_i\colon \prod(-, M_i) \to (-, M_i)$ be the ith projection. Then there are $\psi_i\colon (-, N) \to (-, M_i)$ such that $\pi_i = \psi_i\beta$. Hence there is a map $\eta\colon (-, N) \to \prod(-, M_i)$ such that $\pi_i\eta = \psi_i$. So $\pi_i = \psi_i\beta = \pi_i\eta\beta$, and whence $1 = \eta\beta$. Thus $\prod(-, M_i)$ is strongly flat in $((\text{mod-}R)^{\text{op}}, \text{Ab})$. So R is a left coherent ring by Theorem 2.7. □

The following theorem has appeared implicitly in several papers, and the proof is immediate.

Theorem 3.2. *Let $H\colon \mathcal{A} \to \mathcal{B}$ be a fully faithful covariant functor between additive categories, and let $\mathcal{C}$ be a class of objects of $\mathcal{A}$. Then*

(1) *$M \xrightarrow{f} N$ is an $\mathcal{C}$-precover (resp. cover) of N in $\mathcal{A}$ if and only if*

$$H(M) \xrightarrow{H(f)} H(N)$$

is an $H(\mathcal{C})$-precover (resp. cover) of $H(N)$ in $\mathcal{B}$.

(2) $M \xrightarrow{f} N$ *is an* $\mathcal{C}$*-preenvelope* (*resp. envelope*) *of* M *in* $\mathcal{A}$ *if and only if*

$$H(M) \xrightarrow{H(f)} H(N)$$

is an $H(\mathcal{C})$*-preenvelope* (*resp. envelope*) *of* $H(M)$ *in* $\mathcal{B}$.

Corollary 3.3. *Let* $A \xrightarrow{f} B$ *be a right* R*-module homomorphism. Then*

(1) $A \xrightarrow{f} B$ *is a flat precover* (*resp. cover*) *of* B *in* Mod-R *if and only if*

$$(-, A) \xrightarrow{(-,f)} (-, B)$$

is a strongly flat precover (*resp. cover*) *of* $(-, B)$ *in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$.

(2) $A \xrightarrow{f} B$ *is a flat preenvelope* (*resp. envelope*) *of* A *in* Mod-R *if and only if*

$$(-, A) \xrightarrow{(-,f)} (-, B)$$

is a strongly flat preenvelope (*resp. envelope*) *of* $(-, A)$ *in* $((\text{mod-}R)^{\text{op}}, \text{Ab})$.

Proof. It follows from Theorem 3.2 by letting $H\colon \text{Mod-}R \to ((\text{mod-}R)^{\text{op}}, \text{Ab})$ be the covariant functor given by $M \to (-, M)$ and $\mathcal{C}$ = the class of all flat right R-modules. □

Corollary 3.4. *Let* $A \xrightarrow{f} B$ *be a left* R*-module homomorphism. Then*

(1) $A \xrightarrow{f} B$ *is an FP-injective precover* (*resp. cover*) *of* B *in* R-Mod *if and only if* $-\otimes A \xrightarrow{-\otimes f} -\otimes B$ *is a strongly FP-injective precover* (*resp. cover*) *of* $-\otimes B$ *in* (mod-R, Ab).

(2) $A \xrightarrow{f} B$ *is an FP-injective preenvelope* (*resp. envelope*) *of* A *in* R-Mod *if and only if* $-\otimes A \xrightarrow{-\otimes f} -\otimes B$ *is a strongly FP-injective preenvelope* (*resp. envelope*) *of* $-\otimes A$ *in* (mod-R, Ab).

Proof. It follows from Theorem 3.2 by letting $H\colon R\text{-Mod} \to (\text{mod-}R, \text{Ab})$ be the covariant functor given by $H(M) = -\otimes M$ and $\mathcal{C}$ = the class of all FP-injective left R-modules. □

Remark 3.5. Unlike the FP-injective preenvelope of a functor in (mod-R, Ab), a strongly FP-injective preenvelope of a functor in (mod-R, Ab) need not be a monomorphism. For example, let $R = \mathbb{Z}/4\mathbb{Z}$ and $M = 2R$. Then M is not an FP-injective R-module. If $M \xrightarrow{f} N$ is an FP-injective preenvelope, then $-\otimes M \xrightarrow{-\otimes f} -\otimes N$ is a strongly FP-injective preenvelope of $-\otimes M$ in (mod-R, Ab) by Corollary 3.4(2). Since M is not an FP-injective R-module, $- \otimes f$ is not a monomorphism.

However we have the following result.

Proposition 3.6. *The following conditions are equivalent for a ring R:*

(1) *R is a von Neumann regular ring*;

(2) *every functor in* (mod-R, Ab) *has a monic strongly FP-injective* (*pre*)*envelope*;

(3) *every functor in* ((mod-R)$^{\mathrm{op}}$, Ab) *has an epic strongly flat* (*pre*)*cover.*

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) are clear by Proposition 2.8.

(2) $\Longrightarrow$ (1). By (2), every injective functor in (mod-R, Ab) is strongly FP-injective. So R is a von Neumann regular ring by Proposition 2.8.

(3) $\Longrightarrow$ (1). By (3), every projective functor in ((mod-R)$^{\mathrm{op}}$, Ab) is strongly flat. Thus R is a von Neumann regular ring by Proposition 2.8. □

Finally, we point out that there exists some kind of duality between strongly FP-injective (resp. strongly flat) preenvelopes and strongly flat (resp. strongly FP-injective) precovers of functors.

For a covariant additive functor F from an additive category $\mathcal{C}$ to Ab and a contravariant additive functor G from $\mathcal{C}$ to Ab, there is a natural isomorphism $\tau_{F,G}\colon [F, G^+] \to [G, F^+]$ defined by $\tau_{F,G}(\omega)_M(x)(y) = \omega_M(y)(x)$ for any $\omega \in [F, G^+]$, $M \in \mathcal{C}$, $x \in G(M)$ and $y \in F(M)$.

Theorem 3.7. *Let R be a left coherent ring.*

(1) *If $f\colon A \to B$ is a strongly FP-injective preenvelope of a functor A in* (mod-R, Ab), *then the map $f^+\colon B^+ \to A^+$ is a strongly flat precover of A^+ in* ((mod-R)$^{\mathrm{op}}$, Ab). *The converse holds if R is a left Noetherian ring.*

(2) *If $g\colon C \to D$ is a strongly flat preenvelope of a functor C in* ((mod-R)$^{\mathrm{op}}$, Ab), *then the map $g^+\colon D^+ \to C^+$ is a strongly FP-injective precover of C^+ in* (mod-R, Ab). *The converse holds if R is a right perfect ring.*

Proof. (1) If $f: A \to B$ is a strongly FP-injective preenvelope of a functor A in (mod-R, Ab), then B^+ is strongly flat in ((mod-R)$^{\mathrm{op}}$, Ab) by Theorem 2.7. For any strongly flat functor F in ((mod-R)$^{\mathrm{op}}$, Ab), F^+ is a strongly FP-injective functor in (mod-R, Ab) by Proposition 2.5. So $f^*: [B, F^+] \to [A, F^+]$ is an epimorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} [B, F^+] & \xrightarrow{f^*} & [A, F^+] \\ {\scriptstyle \tau_{B,F}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{A,F}} \\ [F, B^+] & \xrightarrow{(f^+)_*} & [F, A^+]. \end{array}$$

Since $\tau_{A,F}$ and $\tau_{B,F}$ are isomorphisms, $(f^+)_*: [F, B^+] \to [F, A^+]$ is an epimorphism. So $f^+: B^+ \to A^+$ is a strongly flat precover of A^+ in ((mod-R)$^{\mathrm{op}}$, Ab).

Conversely we assume that R is a left Noetherian ring and $f^+: B^+ \to A^+$ is a strongly flat precover of A^+ in ((mod-R)$^{\mathrm{op}}$, Ab). For any strongly FP-injective functor $- \otimes M$ in (mod-R, Ab) and any morphism $\alpha: A \to - \otimes M$, $(- \otimes M)^+$ is a strongly flat functor in ((mod-R)$^{\mathrm{op}}$, Ab) by Theorem 2.7. Let

$$\phi: (- \otimes M)^{++} \longrightarrow (-, M^+)^+ \longrightarrow - \otimes M^{++}$$

be the natural isomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} & & - \otimes M & \xleftarrow{\alpha} & A & \xrightarrow{f} & B \\ & {\scriptstyle -\otimes\delta_M}\swarrow & \big\downarrow{\scriptstyle \delta_{-\otimes M}} & & \big\downarrow{\scriptstyle \delta_A} & & \big\downarrow{\scriptstyle \delta_B} \\ - \otimes M^{++} & \xleftarrow{\phi} & (- \otimes M)^{++} & \xleftarrow{\alpha^{++}} & A^{++} & \xrightarrow{f^{++}} & B^{++}. \end{array}$$

By hypothesis, there exists $\beta: (- \otimes M)^+ \to B^+$ such that $f^+\beta = \alpha^+$. Note that M is injective and so the monomorphism $M \xrightarrow{\delta_M} M^{++}$ is split, i.e., there exists $\gamma: M^{++} \to M$ such that $\gamma\delta_M = 1$. Thus

$$\begin{aligned} ((- \otimes \gamma)\phi\beta^+\delta_B)f &= (- \otimes \gamma)\phi\beta^+ f^{++}\delta_A \\ &= (- \otimes \gamma)\phi\alpha^{++}\delta_A \\ &= (- \otimes \gamma)\phi\delta_{-\otimes M}\alpha \\ &= (- \otimes \gamma)(- \otimes \delta_M)\alpha \\ &= (- \otimes (\gamma\delta_M))\alpha = \alpha. \end{aligned}$$

Therefore $f: A \to B$ is a strongly FP-injective preenvelope of A in (mod-R, Ab).

(2) If $g\colon C \to D$ is a strongly flat preenvelope of a functor C in the category $((\text{mod-}R)^{\text{op}}, \text{Ab})$, then D^+ is a strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$ by Proposition 2.5. For any strongly FP-injective functor G in $(\text{mod-}R, \text{Ab})$, G^+ is strongly flat by Theorem 2.7. Thus $g^*\colon [D, G^+] \to [C, G^+]$ is an epimorphism. Consider the following commutative diagram:

$$\begin{array}{ccc} [G, D^+] & \xrightarrow{(g^+)_*} & [G, C^+] \\ {\scriptstyle \tau_{G,D}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{G,C}} \\ [D, G^+] & \xrightarrow{g^*} & [C, G^+]. \end{array}$$

Since $\tau_{G,C}$ and $\tau_{G,D}$ are isomorphisms, $(g^+)_*\colon [G, D^+] \to [G, C^+]$ is an epimorphism. So $g^+\colon D^+ \to C^+$ is a strongly FP-injective precover of C^+ in the category $(\text{mod-}R, \text{Ab})$.

Conversely we assume that R is a right perfect ring and $g^+\colon D^+ \to C^+$ is a strongly FP-injective precover of C^+ in $(\text{mod-}R, \text{Ab})$. For any strongly flat functor $(-, N)$ in $((\text{mod-}R)^{\text{op}}, \text{Ab})$ and any morphism $\varphi\colon C \to (-, N)$, $(-, N)^+$ is a strongly FP-injective functor in $(\text{mod-}R, \text{Ab})$ by Proposition 2.5. Let

$$\psi\colon (-, N)^{++} \longrightarrow (- \otimes N^+)^+ \longrightarrow (-, N^{++})$$

be the natural isomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} & & (-, N) & \xleftarrow{\ \varphi\ } & C & \xrightarrow{\ g\ } & D \\ & {\scriptstyle (-,\delta_N)}\swarrow & \big\downarrow{\scriptstyle \delta_{(-,N)}} & & \big\downarrow{\scriptstyle \delta_C} & & \big\downarrow{\scriptstyle \delta_D} \\ (-, N^{++}) & \xleftarrow{\ \psi\ } & (-, N)^{++} & \xleftarrow{\ \varphi^{++}\ } & C^{++} & \xrightarrow{\ g^{++}\ } & D^{++}. \end{array}$$

By hypothesis, there exists $\theta\colon (-, N)^+ \to D^+$ such that $g^+\theta = \varphi^+$. Note that N is pure-injective by [17, Example 3.19] and so the pure monomorphism $N \xrightarrow{\delta_N} N^{++}$ is split, i.e., there exists $\xi\colon N^{++} \to N$ such that $\xi\delta_N = 1$. Hence

$$\begin{aligned} ((-, \xi)\psi\theta^+\delta_D)g &= (-, \xi)\psi\theta^+ g^{++}\delta_C \\ &= (-, \xi)\psi\varphi^{++}\delta_C \\ &= (-, \xi)\psi\delta_{(-,N)}\varphi \\ &= (-, \xi)(-, \delta_N)\varphi \\ &= (-, \xi\delta_N)\varphi = \varphi. \end{aligned}$$

This implies that that $g\colon C \to D$ is a strongly flat preenvelope of C. $\square$

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References

[1] M. Auslander, *Coherent functors,* Proceedings of the Conference on Categorical Algebra (La Jolla, 1965), Springer, New York, 1966, pp. 189–231.

[2] M. Auslander, *Representation Dimension of Artin Algebras,* Queen Mary College, Mathematics Notes, University of London, London, 1971.

[3] M. Auslander and I. Reiten, *Stable equivalence of Artin algebras,* Proceedings of the Conference on Orders, Groups Rings and Related Topics (Ohio, 1972), Lecture Notes in Mathematics, 353, Springer-Verlag, Berlin etc., 1973, pp. 8–71.

[4] H. Brune, *On the global dimension of the functor category* $((\mathrm{mod}-R)^{\mathrm{op}}, \mathrm{Ab})$ *and a theorem of Kulikov,* J. Pure Appl. Algebra **28** (1983), pp. 31–39.

[5] R. R. Colby, *Rings which have flat injective modules,* J. Algebra **35** (1975), pp. 239–252.

[6] W. W. Crawley-Boevey, *Locally finitely presented additive categories,* Comm. Algebra **22** (1994), pp. 1641–1674.

[7] S. Crivei, *On flat objects of finitely accessible categories,* Scientific World J. **2013**, article id. 451091, 4 pp.

[8] S. Crivei – M. Prest – B. Torrecillas, *Covers in finitely accessible categories,* Proc. Amer. Math. Soc. **138** (2010), pp. 1213–1221.

[9] E. E. Enochs, *Injective and flat covers, envelopes and resolvents,* Israel J. Math. **39** (1981), pp. 189–209.

[10] E. E. Enochs – O. M. G. Jenda, *Relative homological algebra,* Walter de Gruyter, Berlin and New York, 2000.

[11] E. E. Enochs – L. Oyonarte, *Covers, envelopes and cotorsion theories,* Nova Science Publishers, New York, 2002.

[12] D. J. Fieldhouse, *Character modules, dimension and purity,* Glasgow Math. J. **13** (1972), pp. 144–146.

[13] R. Göbel – J. Trlifaj, *Approximations and endomorphism algebras of modules,* de Gruyter Expositions in Mathematics, 41, Walter de Gruyter, Berlin and New York, 2006.

[14] L. Gruson – C. U. Jensen, *Dimensions cohomologique reliées aux foncteurs* $\varprojlim^i$, Séminaire, in d'Algèbre Paul Dubreil et Marie-Paule Malliavin, Paris 1980, Lecture Notes in Mathematics, 867, Springer-Verlag, 1981, Berlin and New York, pp. 234–294.

[15] I. Herzog, *The phantom cover of a module,* Adv. Math. **215** (2007), pp. 220–249.

[16] I. Herzog, *Contravariant functors on the category of finitely presented modules,* Israel J. Math. **167** (2008), pp. 347–410.

[17] H. Krause, *The spectrum of a module category,* Mem. Amer. Math. Soc. **149** (2001), no. 707, x+125 pp.

[18] L. X. Mao, *On covers and envelopes in some functor categories,* Comm. Algebra **41** (2013), pp. 1655–1684.

[19] C. Megibben, *Absolutely pure modules,* Proc. Amer. Math. Soc. **26** (1970), pp. 561–566.

[20] M. Prest, *Model theory and modules,* London Mathematical Society Lecture Note Series, 130, Cambridge University Press, Cambridge, 1988.

[21] J. J. Rotman, *An introduction to homological algebra,* Academic Press, New York, 1979.

[22] M. Saorín – A. Del Valle, *Covers and envelopes in functor categories,* Interactions between Ring Theory and Representations of Algebras (Murcia Conference, January 1998), Lecture Notes in Pure and Applied Mathematics 210, Marcel Dekker, New York, 2000, pp. 321–329.

[23] B. Stenström, *Purity in functor categories,* J. Algebra **8** (1968), pp. 352–361.

[24] B. Stenström, *Coherent rings and FP-injective modules,* J. London Math. Soc. **2** (1970), pp. 323–329.

[25] B. Stenström, *Rings of quotients,* Springer-Verlag, Berlin etc., 1975.

[26] R. Wisbauer, *Foundations of module and ring theory,* Revised and translated from the 1988 German edition, Algebra, Logic and Applications, 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

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