

Classification of a class of torsion-free abelian groups

EBRU SOLAK (*)

ABSTRACT – The class of almost completely decomposable groups with a critical typeset of type $(2, 2)$ and a regulator quotient of exponent $\leq p^2$ is shown to have exactly 4 near-isomorphism classes of indecomposable groups. Every group of the class is up to near-isomorphism uniquely a direct sum of these four indecomposable groups.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 20K15, 20K25, 20K35, 15A21, 16G60.

KEYWORDS. Almost completely decomposable group, indecomposable, bounded representation type

1. Introduction

Kaplansky once observed, in essence, that anything can happen in torsion-free abelian groups even if the groups have finite rank in analogy to finite dimensional vector spaces. One way out is the weakening of the isomorphism concept. M. C. R. Butler introduced “quasi-isomorphism” which lead to an extensive theory, see for example [1]. Another possibility is the study of subclasses that are both sufficiently interesting and reasonably accessible. Such a class is the class of almost completely decomposable groups first introduced by E. L. Lady, [7]. Every torsion-free abelian group of finite rank is the direct sum of indecomposable groups. Even in the case of almost completely decomposable groups such decompositions can be very “pathological”. This problem is avoided by restricting the “regulator index” to be a power of a single prime and to employ a modest weakening of isomorphism, also due to E. L. Lady, called near-isomorphism. This way one obtains a Remak–Krull–Schmidt category and a classification up to near-isomorphism as soon as the indecomposable groups in the class are found. As was shown in [2] most of these classes contain indecomposable groups of arbitrarily

(*) *Indirizzo dell’A.*: Ebru Solak, Dept. of Math., Middle East Technical University, Üniversiteler Mah., Dumlupınar Bulvarı No:1, 06800 Ankara, Turkey
E-mail: esolak@metu.edu.tr

large rank in which case it is hopeless to try to describe all near-isomorphism classes of indecomposable groups. This leaves some special subclasses that may have a finite number of near-isomorphism classes of indecomposable groups. The class considered in this paper is shown to be such a class and the indecomposables are determined.

Any torsion-free abelian group G is an additive subgroup of a \mathbb{Q} -vector space V . The \mathbb{Q} -subspace of V generated by G is denoted $\mathbb{Q}G$ and $\dim(\mathbb{Q}G)$ is the *rank* of G . A torsion-free abelian group G of finite rank is *completely decomposable* if G is the direct sum of rank-1 groups and *almost completely decomposable* if G contains a completely decomposable subgroup of finite index. An almost completely decomposable group G contains a well-understood fully invariant completely decomposable subgroup of finite index, the *regulator* $R(G)$, [5].

A *type* $[X]$ is an isomorphism class of a rank-1 group X . The set of types is a poset, where $[X] \leq [Y]$ if X is isomorphic to a subgroup of Y . The *critical typeset* of an almost completely decomposable group G is the finite poset

$$T_{\text{cr}}(G) = \{[X]: X \text{ a rank-1 summand of } R(G)\}.$$

Given a prime p , an almost completely decomposable group G is *p-reduced* if each type $[X] \in T_{\text{cr}}(G)$ is *p-locally free*, i.e., $pX \neq X$.

Given a finite poset S of p -locally free types, an almost completely decomposable group G is an (S, p^k) -group if $S = T_{\text{cr}}(G)$ and the exponent of the *regulator quotient* $G/R(G)$ is p^k , i.e., $\exp(G/R(G)) = p^k$. Two (S, p^k) -groups G and H are *nearly isomorphic* if there is an integer n relatively prime to p and homomorphisms $f: G \rightarrow H$ and $g: H \rightarrow G$ with $fg = n$ and $gf = n$. The group G is indecomposable if and only if G is nearly isomorphic to an indecomposable group, [1]. Moreover, an almost completely decomposable G with $G/R(G)$ p -primary is, up to near-isomorphism, uniquely a direct sum of indecomposable groups, [6], [8, Corollary 10.4.6]. Consequently, a classification of all indecomposable (S, p^k) -groups up to near isomorphism results in a classification of all (S, p^k) -groups up to near isomorphism. Hence, for almost completely decomposable groups G with $G/R(G)$ p -primary, the main question is to determine the near-isomorphism classes of indecomposable (S, p^k) -groups.

As was shown in [9] and [10] the class of $(S = (1, 2), p^k)$ -groups for $k \leq 4$ contains finite number of near-isomorphism classes of indecomposable groups and it contains indecomposable groups of arbitrarily large rank if $k \geq 6$, cf. [2]. The class of $(S = (1, 3), p^k)$ -groups for $k \leq 3$ contains finite number of near-isomorphism classes of indecomposable groups and it contains indecomposable groups of arbitrarily large rank if $k \geq 4$, cf. [2] and [3]. Moreover, it was shown

in [4] that the class of $(S = (1, 1, 1), p)$ -groups contains finite number of near-isomorphism classes of indecomposable groups.

Let

$$(2, 2) = (\tau_1 \mid \tau_2 \parallel \tau_3 \mid \tau_4)$$

denote a poset of p -locally free types $\tau_1, \tau_2, \tau_3, \tau_4$ such that $\tau_1 < \tau_2, \tau_3 < \tau_4$ and τ_i is incomparable with τ_j for $i \leq 2 < j$. We are concerned with $((2, 2), p^k)$ -groups and simply write $(2, 2)$ -group if the value of the exponent p^k of the regulator quotient is not relevant.

In this paper we show that

- there are no indecomposable $((2, 2), p)$ -groups G ;
- there are four near-isomorphism classes of indecomposable $((2, 2), p^2)$ -groups G . All of these have rank 4, and the regulator quotients are isomorphic to $\mathbb{Z}/p^2\mathbb{Z}, (\mathbb{Z}/p^2\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z}),$ or $(\mathbb{Z}/p^2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z}).$

There exist indecomposable $((2, 2), p^3)$ -groups of rank $4n$ for any integer $n \geq 1$, [2, Proposition 9]. Consequently, by [4, Lemma 4.5], for $m \geq 3$ there exist indecomposable $((2, 2), p^m)$ -groups of arbitrarily large rank. The description of indecomposable groups in this case is hopeless. This result together with our present results settle completely the case of $(2, 2)$ -groups.

Our method consists in turning the decomposition question into an equivalence problem for matrices.

2. Coordinate matrices

We exclusively deal with almost completely decomposable groups G with p -primary regulator quotient $G/R(G)$.

The goal of this section is to describe a p -reduced $((2, 2), p^2)$ -group with p -primary regulator quotient $G/R(G)$ by means of an integer matrix, the “coordinate matrix”. The coordinate matrix is obtained by means of “bases” of $R = R(G)$ and G/R .

Let G be a $((2, 2), p^k)$ -group of rank m with regulator

$$R = R_1 \oplus R_2 \oplus R_3 \oplus R_4,$$

where R_i is homogeneous completely decomposable of rank $r_i \geq 1$ and type τ_i such that $\tau_1 < \tau_2, \tau_3 < \tau_4$ and τ_i incomparable with τ_j for $i \leq 2 < j$. In particular, $m = r_1 + r_2 + r_3 + r_4$. We indicate a purification by the subscript “*”. Then the ordered set

$$(x_{1,1}, \dots, x_{1,r_1}, x_{2,1}, \dots, x_{2,r_2}, x_{3,1}, \dots, x_{3,r_3}, x_{4,1}, \dots, x_{4,r_4}) = (x_1, \dots, x_m),$$

is called a p -basis of R if

$$R = \bigoplus_{i,j} \langle x_{i,j} \rangle_*, \quad R_i = \bigoplus_{j=1}^{r_i} \langle x_{i,j} \rangle_*$$

and $x_i \notin pR$.

DEFINITION 2.1. Let G be a $((2, 2), p^k)$ -group with regulator R and p -basis (x_1, \dots, x_m) of R . An $r \times m$ matrix $\delta = [\delta_{i,j}]$ is a *coordinate matrix* of G modulo R if $\delta_{i,j} \in \mathbb{Z}$, there is a basis $(\epsilon_1, \dots, \epsilon_r)$ of G/R , there are representatives $g_i \in G$ of ϵ_i , and there is a p -basis (x_1, \dots, x_m) of R such that

$$g_i = p^{-k_i} (\sum_{j=1}^m \delta_{i,j} x_j) \quad \text{where } \langle \epsilon_i \rangle \cong \mathbb{Z}_{p^{k_i}} \text{ and } 1 \leq k_i \leq k.$$

We may write the coordinate matrix δ in the form

$$\delta = \left[\begin{array}{ccc|c} \delta_{11} & \cdots & \delta_{1m} & p^{k_1} \\ \vdots & \cdots & \vdots & \vdots \\ \delta_{r1} & \cdots & \delta_{rm} & p^{k_r} \end{array} \right]$$

where $p^{k_i} = \text{ord}(\epsilon_i)$ and $k_1 \geq k_2 \geq \dots \geq k_r$.

The coordinate matrix could be defined with respect to any completely decomposable “base” subgroup of finite index but to be really useful the subgroup must be the regulator. The Regulator Criterion stated below states how it can be seen by inspecting the coordinate matrix that the base group is indeed the regulator.

The choice of the p -basis divides the coordinate matrix in four blocks $\alpha_1, \alpha_2, \beta_1, \beta_2$ of sizes $r \times r_i, i = 1, 2, 3, 4$ and we have $\delta = [\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2]$.

The matrices $\alpha = [\alpha_1 \mid \alpha_2]$ and $\beta = [\beta_1 \mid \beta_2]$ are called the α - and β -part of the coordinate matrix, respectively.

We now state the Regulator Criterion in [3, Lemma 13], in the special case of $(2, 2)$ -groups.

LEMMA 2.2 (REGULATOR CRITERION). *Let G be a $(2, 2)$ -group. The completely decomposable subgroup R of finite index in G is the regulator of G if and only if $R_1 \oplus R_2$ and $R_3 \oplus R_4$ are pure in G , and this holds if and only if α and β of a coordinate matrix $\gamma = [\alpha \mid \beta]$ both have p -rank equal to the number of rows r of δ .*

3. Direct decomposition and coordinate matrices

Two (2, 2)-groups are nearly isomorphic if and only if their coordinate matrices are equivalent via an equivalence relation defined by certain row and column operations listed below. The connection between almost completely decomposable groups with p -primary regulator quotient and integer matrices is explicitly documented in [3].

By Arnold's Theorem, [1, Chapter 2.2, Exercise 1], if G is nearly isomorphic to $H_1 \oplus H_2$, then there exists subgroups G_i nearly isomorphic to H_i such that $G = G_1 \oplus G_2$. Hence, to classify the near-isomorphism classes of indecomposable groups, we start with some coordinate matrix of an indecomposable group G , and simplify the matrix by means of the row and column transformations listed in Remark 3.1 because the group G' belonging to the transformed coordinate matrix is nearly isomorphic to G and is also indecomposable. If we arrive at a specific matrix containing no unknowns, then the matrix describes the near isomorphism class of the indecomposable group G , see Proposition 5.2.

We call transformations of rows and of columns of a coordinate matrix of G *allowed* if the transformed coordinate matrix is the coordinate matrix of the same or a near isomorphic group. These are exactly the transformations listed in Remark 3.1, see [3] for details.

REMARK 3.1. *Let $\delta = [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$ be a coordinate matrix of a $((2, 2), p^k)$ -group. Then the following row and column operations on the coordinate matrix are allowed.*

- (1) *Replace any entry of δ by an integer congruent to it modulo p^k . Consequently, we assume that the entries of our coordinate matrices are reduced modulo p^k .*
- (2) *Any multiple of a row may be added to any row below it.*
- (3) *Any row or column may be multiplied by an integer relatively prime to p .*
- (4) *Any multiple of the $p^{k_{i_1} - k_{i_2}}$ -fold of row i_2 may be added to a row $i_1 < i_2$.*
- (5) *Any multiple of a column of α_1 may be added to another column of $[\alpha_1 \mid \alpha_2]$ and any multiple of a column of α_2 may be added to another column of α_2 .*
- (6) *Any multiple of a column of β_1 may be added to another column of $[\beta_1 \mid \beta_2]$ and any multiple of a column of β_2 may be added to another column of β_2 .*

4. Standard coordinate matrices

For the convenience of the reader we collect techniques, language conventions and standard conclusions.

- The term *line* means a row or a column. A line is called a *p-line* if all its entries are in $p\mathbb{Z}$.
- The Smith Normal Form of a square integral matrix M is a diagonal matrix $\text{diag}(a_1, a_2, \dots, a_r)$ such that a_i divides a_{i+1} for $1 \leq i \leq r - 1$. The Smith Normal Form of M can be produced by applying arbitrary elementary row and column transformations, or, equivalently, by left and right multiplication by invertible integral matrices. If M is a submatrix, not necessarily square, of a coordinate matrix of a $((2, 2), p^2)$ -group, then the Smith Normal Form of M can be further simplified by reducing modulo p^2 and by multiplying rows and columns by units modulo p . Consequently, we can achieve the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & pI & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} I & 0 \\ 0 & pI \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} I & 0 & 0 \\ 0 & pI & 0 \end{bmatrix},$$

where the I 's are identity matrices of possibly different sizes. The phrase “we produce the Smith Normal Form of some matrix block A ” means that there are allowed matrix transformations that turn the block into Smith Normal Form in such a way that it is also possible to reestablish submatrices that were zero or of the form $p^h I$ and that were affected by these transformations.

- Let M be a submatrix of a coordinate matrix of a $((2, 2), p^2)$ -group. By *we annihilate M* we mean that by applying allowed row and columns operations on M , the submatrix M changes to the 0-matrix.
- Let M and N be submatrices of a coordinate matrix of a $((2, 2), p^2)$ -group. If M is in Smith Normal Form, then by *we annihilate with the part I of M in N* we mean that the I in the Smith Normal Form of M can be used to annihilate columns or rows of N .

DEFINITION 4.1. Let A be a completely decomposable group and let e be a positive integer. Let $\bar{}: A \rightarrow A/eA$ denote the natural epimorphism. So, in particular, $\bar{A} = A/eA$. Furthermore, $\bar{}$ will be used to denote as well the induced homomorphisms $\bar{}: \text{Aut } A \rightarrow \text{Aut } \bar{A}$. The images $\overline{A(\tau)}$, $\overline{A^*(\tau)} = \overline{A^\#(\tau)}$ of the type subgroups of A form a distinguished family of subgroups of \bar{A} which will be called the *type subgroups of \bar{A}* . A *type automorphism* is an automorphism ψ

of \bar{A} such that $\psi(\overline{A(\tau)}) = \overline{A(\tau)}$ for all critical types of A . The group of type automorphisms is denoted by $\text{TypAut } \bar{A}$. Let

$$\text{RFEE}(A, e) = \{X \subset \mathbb{Q} A \mid A = R(X) \text{ and } eX \subset A\}$$

denote the *regulated extensions of A with e -bounded regulator quotient*. The groups $G, H \in \text{RFEE}(A, e)$ are said to be *type-isomorphic* if there is a $\phi \in \text{TypAut}(A, e)$ such that $\phi(eG) = eH$, and we write $G =_{\text{tp}} H$.

LEMMA 4.2. *Let G, H be $((2, 2), p^2)$ -groups with common regulator $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4$, as above, i.e., $G, H \in \mathbb{Q} R$. If $G =_{\text{nr}} H$, then*

$$\frac{G + p^{-1}(R_2 + R_3 + R_4)}{R_1 \oplus p^{-1}(R_2 + R_3 + R_4)} \cong \frac{H + p^{-1}(R_2 + R_3 + R_4)}{R_1 \oplus p^{-1}(R_2 + R_3 + R_4)},$$

and

$$\frac{G + \langle R_2 + R_3 + p^{-1}R_4 \rangle_*}{R_1 \oplus \langle R_2 + R_3 + p^{-1}R_4 \rangle_*} \cong \frac{H + \langle R_2 + R_3 + p^{-1}R_4 \rangle_*}{R_1 \oplus \langle R_2 + R_3 + p^{-1}R_4 \rangle_*}.$$

PROOF. By [8, 9.2.4] $G =_{\text{tp}} H$ if and only if $G =_{\text{nr}} H$. So near-isomorphism of G, H induces a type-automorphism φ of p^2R/R such that $\varphi(G/R) = H/R$. Since $(p^{-1}(R_2 + R_3 + R_4) + R)/R$ and $(\langle R_2 + R_3 + p^{-1}R_4 \rangle_* + R)/R$ are φ -invariant we obtain the above isomorphisms. \square

We establish a kind of standard form for coordinate matrices of $((2, 2), p^2)$ -groups.

An almost completely decomposable group is called *clipped* if it has no summand of rank 1.

PROPOSITION 4.3. *A clipped $(2, 2)$ -group G with regulator quotient isomorphic to $\mathbb{Z}_{p^2}^{l_1} \oplus \mathbb{Z}_p^{l_2}$, $l_1 \geq 1, l_2 \geq 0$ has a coordinate matrix*

(1)

$$[\alpha_1, \alpha_2 \parallel \beta_1, \beta_2] = \left[\begin{array}{ccc|ccc} I_{s_1} & 0 & 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & pI_{s_2} & 0 & I_{s_2} & 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 & I_{s_3} & 0 & C_1 & C_2 \\ \hline 0 & 0 & I_{s_4} & 0 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & 0 & 0 & I_{s_5} & E_1 & E_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p^2 \\ p \\ p \end{array} .$$

The non-negative integers s_i all are near-isomorphism invariants of G . Also, $l_1 = s_1 + s_2 + s_3$, $l_2 = s_4 + s_5$, and the last two block columns forming β both are present.

PROOF. Let G be a clipped $(2, 2)$ -group with regulator quotient isomorphic to $\mathbb{Z}_p^{l_1} \oplus \mathbb{Z}_p^{l_2}$, $l_1 \geq 1, l_2 \geq 0$ and let

$$(2) \quad [\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2] = \left[\begin{array}{cc|cc} X_1 & X_2 & A_1 & A_2 \\ Y_1 & Y_2 & D_1 & D_2 \end{array} \right] \begin{array}{l} p^2 \\ p \end{array}$$

be its coordinate matrix. As G is clipped neither α nor β can contain a 0-column. The Regulator Criterion implies that neither α nor β can have a 0-row.

Starting with Equation 2 we first form the (partial) Smith Normal Form for X_1 to get

$$[\alpha_1 \mid \alpha_2 \mid v\beta_1 \mid \beta_2] = \left[\begin{array}{cc|cc} I & 0 & X_{21} & A_1 & A_2 \\ 0 & pX_1 & X_{22} & B_1 & B_2 \\ Y_{11} & Y_{12} & Y_2 & D_1 & D_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p \end{array}.$$

We annihilate with I downward Y_{11} and then we annihilate X_{21} . Hence we get

$$[\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2] = \left[\begin{array}{cc|cc} I & 0 & 0 & A_1 & A_2 \\ 0 & pX_1 & X_2 & B_1 & B_2 \\ 0 & Y_1 & Y_2 & D_1 & D_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p \end{array}.$$

The Smith Normal Form of Y_1 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. We annihilate with the part I in pX_1 and in Y_2 . Hence we get

$$[\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2] = \left[\begin{array}{ccc|cc} I & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & pX_1 & X_2 & B_1 & B_2 \\ 0 & I & 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & Y_2 & E_1 & E_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p \\ p \end{array}.$$

Then the Smith Normal Form of pX_1 is $\begin{bmatrix} pI \\ 0 \end{bmatrix}$ and hence

$$[\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2] = \left[\begin{array}{ccc|cc} I & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & pI & X_2 & B_1 & B_2 \\ 0 & 0 & 0 & X_3 & C_1 & C_2 \\ 0 & I & 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & Y_2 & E_1 & E_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p^2 \\ p \\ p \end{array}.$$

The Regulator Criterion requires that the Smith Normal Form of X_3 is $[I \ 0]$. We produce the Smith Normal Form of X_3 to get

$$[\alpha_1 \mid \alpha_2 \mid \beta_1 \mid \beta_2] = \left[\begin{array}{ccc|cc} I & 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & pI & X_{21} & X_{22} & B_1 & B_2 \\ 0 & 0 & 0 & I & 0 & C_1 & C_2 \\ 0 & I & 0 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & 0 & Y_{21} & Y_{22} & E_1 & E_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p^2 \\ p \\ p \end{array}.$$

The submatrices X_{21} and Y_{21} can be annihilated due to the presence of I in α_2 , i.e., $X_{21} = 0$ and $Y_{21} = 0$. Then the Smith Normal Form of X_{22} is $[I \ 0]$ due to the Regulator Criterion. We use the part I in this Smith Normal Form to annihilate in Y_{22} and then forming the Smith Normal Form of the nonzero rest of Y_{22} we get

$$[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{ccc|ccc||cc} I & 0 & 0 & 0 & 0 & 0 & A_1 & A_2 & p^2 \\ 0 & 0 & pI & 0 & I & 0 & B_1 & B_2 & p^2 \\ 0 & 0 & 0 & I & 0 & 0 & C_1 & C_2 & p^2 \\ \hline 0 & I & 0 & 0 & 0 & 0 & D_1 & D_2 & p \\ 0 & 0 & 0 & 0 & 0 & I & E_1 & E_2 & p \end{array} \right].$$

Interchanging the second and the third column in α_1 and the first and the second column in α_2 produces the claimed result.

Now we prove that the numbers s_i are near-isomorphism invariants. A first observation is that the numbers

$$l_1 = s_1 + s_2 + s_3 \quad \text{and} \quad l_2 = s_4 + s_5$$

describe the regulator quotient, so they are near isomorphism invariants. Secondly,

$$\text{rank } R_1 = s_1 + s_2 + s_4 \quad \text{and} \quad \text{rank } R_2 = s_2 + s_3 + s_5,$$

again invariants.

Moreover, if G, H are near-isomorphic, we may assume that $G, H \in \mathbb{Q}R$ where R is the regulator of both.

By Lemma 4.2 and using the coordinate matrix following the block rows A, B, C, D, E we may read off the isomorphism types of those groups

$$\begin{aligned} \frac{G + p^{-1}(R_2 + R_3 + R_4)}{R_1 \oplus p^{-1}(R_2 + R_3 + R_4)} &\cong \frac{H + p^{-1}(R_2 + R_3 + R_4)}{R_1 \oplus p^{-1}(R_2 + R_3 + R_4)} \\ &\cong \mathbb{Z}_p^{s_1} \oplus \mathbb{Z}_{p^2}^{s_2} \oplus \mathbb{Z}_p^{s_3} \oplus \mathbb{Z}_p^{s_4}, \\ \frac{G + \langle R_2 + R_3 + p^{-1}R_4 \rangle_*}{R_1 \oplus \langle R_2 + R_3 + p^{-1}R_4 \rangle_*} &\cong \frac{H + \langle R_2 + R_3 + p^{-1}R_4 \rangle_*}{R_1 \oplus \langle R_2 + R_3 + p^{-1}R_4 \rangle_*} \\ &\cong \mathbb{Z}_p^{s_1} \oplus \mathbb{Z}_{p^2}^{s_2} \oplus \mathbb{Z}_p^{s_4}. \end{aligned}$$

So also $s_1 + s_3 + s_4, s_2$ and $s_1 + s_4$ are near-isomorphism invariants. Linearly combining those invariants we get that all s_i are near-isomorphism invariants.

Eventually, if one of the last two block columns forming β is not present, then G is not a (2, 2)-group. \square

A coordinate matrix of a $((2, 2), p^2)$ -group as in Proposition 4.3 is called *standard*.

REMARK 4.4. We will usually use the following abbreviated form where the first two columns must be interpreted correctly, the sizes of the identity matrices I need not be the same, and block lines may even be absent altogether.

$$\left[\begin{array}{c|c|c|c} I & 0 & A_1 & A_2 \\ pI & I & B_1 & B_2 \\ 0 & I & C_1 & C_2 \\ \hline I & 0 & D_1 & D_2 \\ 0 & I & E_1 & E_2 \end{array} \right] \begin{array}{l} p^2 \\ p^2 \\ p^2 \\ p \\ p \end{array}.$$

A matrix is *decomposed* if it is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Here either one of the matrices A, B is allowed to have no rows or no columns, i.e., the decomposed matrices include the special cases $[0 \ B]$, $\begin{bmatrix} 0 \\ B \end{bmatrix}$, $[A \ 0]$, $\begin{bmatrix} A \\ 0 \end{bmatrix}$.

A matrix A is called *decomposable* if there are row and column permutations that transform it to a decomposed form, i.e., there are permutation matrices P, Q such that PAQ is decomposed.

Lemma 4.5 shows that indecomposability of a $(2, 2)$ -group with coordinate matrix $[\alpha \parallel \beta]$ can be decided by just studying β .

LEMMA 4.5. *A $(2, 2)$ -group is decomposable if and only if there is a standard coordinate matrix $[\alpha \parallel \beta]$ with decomposable β .*

PROOF. For nearly isomorphic groups the α -part of all standard coordinate matrices is the same, cf. Proposition 4.3. If $G = H \oplus L$ is a decomposable $(2, 2)$ -group, then $R(G) = R(H) \oplus R(L)$, cf. [4, Lemma 3.1]. So, if $G = \bigoplus G_l$, then all summands G_l can be assumed to be given by standard coordinate matrices $[\alpha_l \mid \beta_l]$ and the direct sum $[\bigoplus \alpha_l \mid \bigoplus \beta_l]$ is a coordinate matrix of G . We can rearrange the lines of $[\bigoplus \alpha_l \mid \bigoplus \beta_l]$ and obtain a standard coordinate matrix $[\alpha \mid \beta]$ of the whole group G and β is permutation equivalent to a direct sum of matrices and hence decomposable.

Conversely, a $(2, 2)$ -group which has a standard coordinate matrix $[\alpha \parallel \beta]$ with decomposable β is decomposable. \square

So, if G is decomposable and a coordinate matrix of G is given in standard form, then there are allowed transformations that produce a coordinate matrix of G with decomposable β . Thus, for a proof of indecomposability, we only have to check that with allowed transformations that maintain the part α of a standard form it is impossible to change β into a decomposable form.

5. Indecomposable (2,2)-groups with regulator quotient of exponent p^k

Our first main result says that sometimes there are no indecomposable groups.

THEOREM 5.1. *$((2, 2), p)$ -groups decompose and the summands are of rank ≤ 2 .*

PROOF. Let G be a clipped $((2, 2), p)$ -group with coordinate matrix

$$\delta = [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$$

whose entries are necessarily either zeros or units. We apply transformations to annihilate entries. While doing this, some other entries that were originally zero may become nonzero; those entries are called *fill-ins*.

By Proposition 4.3 we may assume that

$$\delta = \left[\begin{array}{c|c} I_a & 0 \\ \hline 0 & I_b \end{array} \parallel \left[\begin{array}{c|c} A_1 & A_2 \\ \hline B_1 & B_2 \end{array} \right] \right].$$

The Smith Normal Form for B_1 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. We annihilate with I in A_1 and in B_2 . The fill-ins in α_2 can be annihilated by I_a . Hence we obtain

$$\delta = \left[\begin{array}{c|cc} I_a & 0 & 0 \\ \hline 0 & I_b & 0 \\ \hline 0 & 0 & I_b \end{array} \parallel \left[\begin{array}{c|c|c} 0 & A_1 & A_2 \\ \hline I & 0 & 0 \\ \hline 0 & 0 & B_2 \end{array} \right] \right].$$

By the Regulator Criterion the Smith Normal Form for B_2 is $[I \ 0]$. A 0-column in A_1 displays a direct summand of rank 1. Since G is clipped the Smith Normal Form of A_1 is $\begin{bmatrix} I \\ 0 \end{bmatrix}$. We may annihilate with I in A_2 . Hence we obtain

$$\delta = \left[\begin{array}{c|cc} I_a & 0 & 0 \\ \hline 0 & I_a & 0 \\ \hline 0 & 0 & I_b \\ \hline 0 & 0 & 0 \end{array} \parallel \left[\begin{array}{c|c|cc} 0 & I & 0 & 0 \\ \hline 0 & 0 & A_2 & A'_2 \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \end{array} \right] \right].$$

We may annihilate with I upward in A_2 . The fill-ins in α_2 can be annihilated by I_a . The Smith Normal Form of A'_2 is I by clipped and by the Regulator Criterion. Hence we get

$$\delta = \left[\begin{array}{c|cc} I_a & 0 & 0 \\ \hline 0 & I_a & 0 \\ \hline 0 & 0 & I_b \\ \hline 0 & 0 & 0 \end{array} \parallel \left[\begin{array}{c|c|cc} 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \end{array} \right] \right].$$

Now we can read off all summands, and all are of rank 2. \square

Next we produce examples of indecomposable $((2, 2), p^2)$ -groups.

PROPOSITION 5.2. *The following four $((2, 2), p^2)$ -groups, all of rank 4 with fixed isomorphism type of the regulator, given by their regulator quotient and a coordinate matrix are indecomposable and pairwise not near-isomorphic.*

$$(1) \quad [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = [p \mid 1 \parallel p \mid 1]$$

with regulator quotient $\cong \mathbb{Z}_{p^2}$;

$$(2) \quad [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \parallel \begin{array}{c|c} 1 & 0 \\ \hline p & 1 \end{array} \right]$$

with regulator quotient $\cong (\mathbb{Z}_{p^2})^2$.

$$(3) \quad [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \parallel \begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array} \right]$$

with regulator quotient $\cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$.

$$(4) \quad [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \parallel \begin{array}{c|c} p & 1 \\ \hline 1 & 0 \end{array} \right]$$

with regulator quotient $\cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$.

PROOF. A $(2, 2)$ -group has a standard coordinate matrix

$$[\alpha \parallel \beta] = [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$$

by Proposition 4.3. In order to show that the groups (1) through (4) are indecomposable we must show that there is no allowed transformation that changes the coordinate matrix to a “decomposable form.” By Lemma 4.5 it is enough to exclude that a standard coordinate matrix can be obtained where β is decomposable. By Lemma 4.5 left multiplication is allowed by an invertible matrix U such that there is an upper triangular invertible matrix Z with $U\alpha Z = \alpha$, and there is an invertible upper triangular matrix Y such that $\beta' = U\beta Y$ is decomposable. Note that U describes row transformations and Z describes column transformations, and that there is the restriction in the cases (3) and (4) that only p -multiples of the second row can be added to the first row. Note that the only decomposed 1×4 matrices are $[\ast \ 0]$ and $[0 \ \ast]$, and the only decomposed 2×2 matrices, without 0-lines, are $\begin{bmatrix} \ast & 0 \\ 0 & \ast \end{bmatrix}$ and $\begin{bmatrix} 0 & \ast \\ \ast & 0 \end{bmatrix}$.

(1) With $U = u$ and $Y = \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix}$ where u, y, y' are units. So

$$u \cdot \beta' = u \cdot \beta Y = u \cdot [p \ 1] \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix} = [uyp \ u(y' + pb)]$$

that is not decomposable for any choice of u, y, y', b .

(2) We get $U = \begin{bmatrix} u & a \\ 0 & u' \end{bmatrix}$ and $Y = \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix}$ where u, u', y, y' are units. So

$$\beta' = U\beta Y = \begin{bmatrix} u & a \\ 0 & u' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix} = \begin{bmatrix} y(u + pa) & y'a + b(u + pa) \\ u'yp & u'(y' + pb) \end{bmatrix}$$

that is not decomposable.

(3) We get $U = \begin{bmatrix} u & pa \\ 0 & u' \end{bmatrix}$ and $Y = \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix}$ where u, u', y, y' are units. So

$$\beta' = U\beta Y = \begin{bmatrix} u & pa \\ 0 & u' \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix} = \begin{bmatrix} y(u + pa) & uy' + b(u + pa) \\ u'y & u'b \end{bmatrix}$$

that is not decomposable for any Y .

(4) In general $U = \begin{bmatrix} u & pa \\ x & u' \end{bmatrix}$ and $Y = \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix}$ where u, u', y, y' are units. Now as $\alpha = U\alpha Z$ for some invertible upper triangular $Z = \begin{bmatrix} z & c \\ 0 & z' \end{bmatrix}$, i.e., z, z' are units, we have

$$\begin{bmatrix} u & pa \\ x & u' \end{bmatrix} = U = \alpha Z^{-1} \alpha^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z & c \\ 0 & z' \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} z' & 0 \\ c & z \end{bmatrix}.$$

Thus $U = \begin{bmatrix} u & 0 \\ x & u' \end{bmatrix}$ and we get

$$\beta' = U\beta Y = \begin{bmatrix} u & 0 \\ x & u' \end{bmatrix} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y & b \\ 0 & y' \end{bmatrix} = \begin{bmatrix} uyp & u(y' + pb) \\ y(u + xp) & y'x + b(u' + xp) \end{bmatrix}$$

that is not decomposable.

The four groups differ either in α or in the regulator quotient. So they are pairwise not nearly isomorphic because by Proposition 4.3 the α -parts are near isomorphism invariants and so are the regulator quotients. \square

EXAMPLE 5.3. (1) Let $G = \langle R, g \rangle$ be a $((2, 2), p^2)$ -group with regulator

$$R = \mathbb{Z}[2^{-1}]x_1 \oplus \mathbb{Z}[(2 \cdot 3)^{-1}]x_2 \oplus \mathbb{Z}[5^{-1}]x_3 \oplus \mathbb{Z}[(5 \cdot 7)^{-1}]x_4$$

where $x_i \in R$ and with $g = 11^{-2}(11x_1 + x_2 + 11x_3 + x_4)$. Then G is indecomposable by Proposition 5.2(1).

(2) Let $G = \langle R, g_1, g_2 \rangle$ be a $((2, 2), p^2)$ -group with regulator

$$R = \mathbb{Z}[3^{-1}]x_1 \oplus \mathbb{Z}[(3 \cdot 5)^{-1}]x_2 \oplus \mathbb{Z}[7^{-1}]x_3 \oplus \mathbb{Z}[(7 \cdot 11)^{-1}]x_4$$

where $x_i \in R$ and with $g_1 = 13^{-2}(x_1 + x_3)$ and $g_2 = 13^{-2}(x_2 + 13x_3 + x_4)$. Then G is indecomposable by Proposition 5.2(2).

(3) Let $G = \langle R, g_1, g_2 \rangle$ be a $((2, 2), p^2)$ -group with regulator

$$R = \mathbb{Z}[2^{-1}]x_1 \oplus \mathbb{Z}[(2 \cdot 3)^{-1}]x_2 \oplus \mathbb{Z}[5^{-1}]x_3 \oplus \mathbb{Z}[(5 \cdot 7)^{-1}]x_4$$

where $x_i \in R$ and with $g_1 = 11^{-2}(x_1 + x_3 + x_4)$ and $g_2 = 11^{-2}(x_2 + x_3)$. Then G is indecomposable by Proposition 5.2(3).

(4) Let $G = \langle R, g_1, g_2 \rangle$ be a $((2, 2), p^2)$ -group with regulator

$$R = \mathbb{Z}[3^{-1}]x_1 \oplus \mathbb{Z}[(3 \cdot 5)^{-1}]x_2 \oplus \mathbb{Z}[7^{-1}]x_3 \oplus \mathbb{Z}[(7 \cdot 11)^{-1}]x_4$$

where $x_i \in R$ and with $g_1 = 13^{-2}(x_2 + 13^{-1}x_3 + x_4)$ and $g_2 = 13^{-1}(x_1 + x_3)$. Then G is indecomposable by Proposition 5.2(4).

By Proposition 5.2 there are at least four near-isomorphism classes of indecomposable $((2, 2), p^2)$ -groups. The next theorem shows that there are no others.

THEOREM 5.4. *There are precisely the four near-isomorphism classes of indecomposable $((2, 2), p^2)$ -groups listed in Proposition 5.2.*

PROOF. Let G be a $((2, 2), p^2)$ -group without summands of rank ≤ 3 . This is no restriction, because every indecomposable $(2, 2)$ -group has rank ≥ 4 . Without loss of generality the group G has a standard coordinate matrix. This matrix incorporates all possibilities where block rows as well as block columns may be absent.

The matrix $X = [x_{i,j}]$ has a *cross* at (i_0, j_0) if $x_{i_0, j_0} \neq 0$ and $a_{i_0, j} = 0$, $a_{i, j_0} = 0$ for all $i \neq i_0$ and $j \neq j_0$. Crosses display possible or impossible summands and means that certain rows and columns of the coordinate matrix may or must be omitted. By *An entry x leads to a cross in $[\beta_1 \mid \beta_2]$* we mean that

this entry x can be used to annihilate by allowed line transformations in its row and its column to produce a cross at x .

Mostly we want to change certain submatrices either to a 0-matrix or to a matrix of the form $p^h I$, $h \geq 0$. While executing matrix transformations with this goal, fill-ins may occur. The phrase *we can annihilate* tacitly includes that the occurring fill-ins can be removed by subsequent transformations so that already obtained blocks 0 or $p^f I$ can be reestablished.

We briefly say “in A ” instead of “in the A -row,” etc.

We progressively simplify the coordinate matrix and make it more concrete. This must be done without changing the special α -part of the matrix. This restricts the use of the allowed transformations. Note that in $[\alpha_1 | \alpha_2]$ only the fill-ins due to the following row transformations can be reversed by column transformations:

- with pivots in B we can annihilate in A, D, E ;
- with pivots in A we can annihilate in D and also we can annihilate the entries in B that are in $p\mathbb{Z}_{p^2}$;
- with pivots in C we can annihilate in A, B, D, E ;
- with pivots in D we can annihilate in A and also we can annihilate the entries in B that are in $p\mathbb{Z}_{p^2}$;
- with pivots in E we can annihilate in D , and entries in A, B, C that are in $p\mathbb{Z}_{p^2}$.

Only the row transformations above are allowed to β .

(a1) *Smith normal forms for*

$$\begin{bmatrix} B \\ C \\ E \end{bmatrix}.$$

Next, successively simplifying, we show that

$$(3) \quad \left[\begin{array}{c|c} B_1 & B_2 \\ \hline C_1 & C_2 \\ \hline E_1 & E_2 \end{array} \right] = \left[\begin{array}{cccc|cccc} 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{array} \right] \begin{array}{l} p^2, B \\ p^2, C \\ p, E \\ p, E \end{array}.$$

There is no unit in C_1 , because otherwise there is a cross in β indicating the existence of a rank-2 summand. Then in turn there is no unit in B_1 , because a unit

leads to a cross in β . Now we form the Smith Normal Form for E_1 and get $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Thus we obtain

$$(4) \quad \left[\begin{array}{c|c} B_1 & B_2 \\ \hline C_1 & C_2 \\ \hline E_1 & E_2 \end{array} \right] = \left[\begin{array}{cc|c} pB_1 & pB'_1 & B_2 \\ pC_1 & pC'_1 & C_2 \\ \hline I & 0 & E_2 \\ \hline 0 & 0 & E'_2 \end{array} \right] \begin{array}{l} p^2, B \\ p^2, C \\ p, E \\ p, E \end{array}.$$

The submatrices pB_1 , pC_1 and E_2 can be annihilated due to the presence of I in E below, so without loss of generality $pB_1 = 0$, $pC_1 = 0$ and $E_2 = 0$. Thus we have

$$(5) \quad \left[\begin{array}{c|c} B_1 & B_2 \\ \hline C_1 & C_2 \\ \hline E_1 & E_2 \end{array} \right] = \left[\begin{array}{cc|c} 0 & pB'_1 & B_2 \\ 0 & pC'_1 & C_2 \\ \hline I & 0 & 0 \\ \hline 0 & 0 & E'_2 \end{array} \right] \begin{array}{l} p^2, B \\ p^2, C \\ p, E \\ p, E \end{array}.$$

The Regulator Criterion requires that the submatrices B_2 , C_2 and E'_2 have units in each row. So the Smith Normal Form of C_2 is $\begin{bmatrix} I & 0 \end{bmatrix}$. We may annihilate all other entries in the block column of this I in

$$\left[\begin{array}{c} B \\ C \\ E \end{array} \right].$$

Hence we get

$$\left[\begin{array}{c|c} B_1 & B_2 \\ \hline C_1 & C_2 \\ \hline E_1 & E_2 \end{array} \right] = \left[\begin{array}{cc|cc} 0 & pB'_1 & 0 & B_2 \\ 0 & pC'_1 & I & 0 \\ \hline I & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E'_2 \end{array} \right].$$

Due to the Regulator Criterion the Smith Normal Form of B_2 is $\begin{bmatrix} I & 0 \end{bmatrix}$. We may annihilate with the part I of B_2 in E'_2 and then we form the Smith Normal Form of the non-zero rest of E'_2 and get $\begin{bmatrix} I & 0 \end{bmatrix}$. Thus we obtain

$$(6) \quad \left[\begin{array}{c|c} B_1 & B_2 \\ \hline C_1 & C_2 \\ \hline E_1 & E_2 \end{array} \right] = \left[\begin{array}{cccc|cccc} 0 & pB'_1 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & pC'_1 & I & 0 & 0 & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array} \right] \begin{array}{l} p^2, B \\ p^2, C \\ p, E \\ p, E \end{array}.$$

If there is a 0-row in pC'_1 then we can annihilate with I in C_2 in the rows A , B , D and we get a cross. Thus there is no 0-row in pC'_1 , to avoid a cross. Hence

the Smith Normal Form of pC'_1 is $[pI \ 0]$. We annihilate with pI in pB'_1 and get

$$(7) \quad \left[\begin{array}{cc|cc} B_1 & B_2 & & \\ C_1 & C_2 & & \\ \hline E_1 & E_2 & & \end{array} \right] = \left[\begin{array}{ccc|cccc} 0 & 0 & pB'_1 & 0 & I & 0 & 0 \\ 0 & pI & 0 & I & 0 & 0 & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \end{array} \right] \begin{array}{l} p^2, B \\ p^2, C \\ \hline p, E \\ p, E \end{array}.$$

If there is a 0-row in pB'_1 then we can annihilate with I in B_2 in the rows A, B, D and this leads to a cross. Hence there is no 0-row in pB'_1 . Thus the Smith Normal Form of pB'_1 is $[pI \ 0]$. Hence we obtain the claimed form of

$$\left[\begin{array}{c} B \\ C \\ \hline E \end{array} \right]$$

as in Equation (3).

figureversiontab(a2) *Block form for β .*

By part (a1) we can write β as

$$(8) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{cccc|cccc} A_1^1 & A_1^2 & A_1^3 & A_1^4 & A_2^1 & A_2^2 & A_2^3 & A_2^4 \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ \hline D_1^1 & D_1^2 & D_1^3 & D_1^4 & D_2^1 & D_2^2 & D_2^3 & D_2^4 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{array} \right] \begin{array}{l} p^2, A \\ p^2, B \\ p^2, C \\ \hline p, D \\ p, E^1 \\ p, E^2 \end{array}.$$

The submatrices A_2^1 and D_2^1 can be annihilated due to the presence of I in C and the submatrices A_2^2 and D_2^2 can be annihilated with I above in B . Moreover, the submatrices D_1^1 and D_2^3 can be annihilated by I in E^1, E^2 , respectively. Hence we get

$$(9) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{cccc|cccc} A_1^1 & A_1^2 & A_1^3 & A_1^4 & 0 & 0 & A_2^3 & A_2^4 \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ \hline 0 & D_1^2 & D_1^3 & D_1^4 & 0 & 0 & 0 & D_2^4 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{array} \right] \begin{array}{l} p^2, A \\ p^2, B \\ p^2, C \\ \hline p, D \\ p, E^1 \\ p, E^2 \end{array}.$$

A unit in A_1^4 allows to annihilate in D_1^4 and in its row, leading to a cross. So we replace A_1^4 by pA_1^4 . Then in turn there is no unit in A_1^3 by the same arguments, and we write pA_1^3 . A unit in A_1^2 allows first to annihilate in D_1^2 and then in its row. This

causes fill-ins in D and C . The fill-ins in D can be annihilated by I in E^1, E^2 . The fill-ins in C are either in $p^2\mathbb{Z}$ and can be ignored or they can be annihilated by I in E^1 . This leads to a summand with coordinate matrix

$$\beta = \left[\begin{array}{c|c} 1 & 0 \\ p & 1 \end{array} \right],$$

where the rows belong to A, C , respectively. This is a summand of type (2). Hence omitting all those summands we may assume that the entries of A_1^2 are in $p\mathbb{Z}$. Thus we get

$$(10) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{cccc|cccc} A_1^1 & pA_1^2 & pA_1^3 & pA_1^4 & 0 & 0 & A_2^3 & A_2^4 & p^2, A \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & p^2, B \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & p^2, C \\ \hline 0 & D_1^2 & D_1^3 & D_1^4 & 0 & 0 & 0 & D_2^4 & p, D \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p, E^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & p, E^2 \end{array} \right].$$

Now a unit in D_1^4 allows to annihilate in its row and then in pA_1^4 , creating a cross. So we may assume that $D_1^4 = 0$. A unit in D_1^3 allows to annihilate in its row, then in pI and in pA_1^3 above. But this leads again to a cross. Hence we may assume that $D_1^3 = 0$.

A unit in D_1^2 allows to annihilate in pA_1^2 and in D_2^4 . This leads to a summand with coordinate matrix

$$\beta = \left[\begin{array}{c|c} p & 1 \\ 1 & 0 \end{array} \right],$$

where the rows belong to C, D , respectively, and a summand of type (4). Omitting those summands we get $D_1^2 = 0$. Hence we obtain

$$(11) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{cccc|cccc} A_1^1 & pA_1^2 & pA_1^3 & pA_1^4 & 0 & 0 & A_2^3 & A_2^4 & p^2, A \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & p^2, B \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & p^2, C \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_2^4 & p, D \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p, E^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & p, E^2 \end{array} \right].$$

A unit in A_2^4 allows to annihilate in D_2^4 creating a 0-row in β , so the entries in the column in A_2^4 above a unit in D_2^4 are all in $p\mathbb{Z}$. But then this column can be annihilated by the unit in D_2^4 . This leads to a cross. Thus the D_2^4 -row is not present. Due to the I in E^1 , the entries in A_1^1 are either units or 0. Note that A_1^1

has no 0-column to avoid a cross. Hence the Smith Normal Form of A_1^1 is $\begin{bmatrix} I \\ 0 \end{bmatrix}$ and we get

$$(12) \quad [\beta_1 | \beta_2] = \left[\begin{array}{cccc|cccc} I & pA_1^{21} & pA_1^{31} & pA_1^{41} & 0 & 0 & A_2^{31} & A_2^{41} & p^2, A^1 \\ 0 & pA_1^{22} & pA_1^{32} & pA_1^{42} & 0 & 0 & A_2^{32} & A_2^{42} & p^2, A^2 \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & p^2, B \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & p^2, C \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p, E^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & p, E^2 \end{array} \right].$$

The submatrices pA_1^{21} , pA_1^{31} , pA_1^{41} and A_2^{31} can be annihilated by using I in A^1 . The fill-ins in E are either in $p\mathbb{Z}$ and can be ignored or they can be annihilated by I in E^2 . Hence we get

$$(13) \quad [\beta_1 | \beta_2] = \left[\begin{array}{cccc|cccc} I & 0 & 0 & 0 & 0 & 0 & 0 & A_2^{41} & p^2, A^1 \\ 0 & pA_1^2 & pA_1^3 & pA_1^4 & 0 & 0 & A_2^3 & A_2^{42} & p^2, A^2 \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 & p^2, B \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 & p^2, C \\ \hline I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p, E^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & p, E^2 \end{array} \right].$$

A unit in A_2^{42} allows to annihilate in A_2^{41} . The fill-ins in A^1 can be annihilated by I in the A -row and then the resulting fill-ins in E^1 can be annihilated due to the presence of I in E^2 . Thus below a unit in A_2^{41} there is no unit in A_2^{42} , i.e., the entries in A_2^{42} which are below a unit in A_2^{41} are either 0 or in $p\mathbb{Z}$. Hence we may annihilate with a unit in A_2^{41} in its column. So a unit in A_2^{41} leads to a summand with coordinate matrix

$$\left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array} \right]$$

where the rows belong to A and E , respectively. This is a summand of type (3). Omitting all those summands we may assume that A_2^{41} has no unit, i.e., we may write pA_2^{41} . But then pA_2^{41} can be annihilated by I in A -row and this leads to a summand with coordinate matrix with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where the rows belong to A , E , respectively, and a summand of rank 3. So the A^1 -row, the E^1 -row and the first block column of β do not exist. Hence we get

$$(14) \quad [\beta_1 | \beta_2] = \left[\begin{array}{ccc|cccc} pA_1^2 & pA_1^3 & pA_1^4 & 0 & 0 & A_2^3 & A_2^4 & p^2, A \\ 0 & pI & 0 & 0 & I & 0 & 0 & p^2, B \\ pI & 0 & 0 & I & 0 & 0 & 0 & p^2, C \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 & p, E \end{array} \right].$$

We form the Smith Normal Form for A_2^4 and get $\begin{bmatrix} I & 0 \\ 0 & pA_2^4 \end{bmatrix}$. We use the part I to annihilate in A_2^3 and form the Smith Normal Form of the nonzero rest of A_2^3 . Note that the nonzero rest of A_2^3 has no 0-row due to the Regulator Criterion and has no 0-column to avoid a cross. Thus,

$$[A_2^3 \mid A_2^4] = \left[\begin{array}{ccc|cc} 0 & & & I & 0 \\ I & & & 0 & pA_2^4 \end{array} \right].$$

Hence we get

$$(15) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{ccc|cccc} pA_1^{21} & pA_1^{31} & pA_1^{41} & 0 & 0 & 0 & I & 0 \\ pA_1^{22} & pA_1^{32} & pA_1^{42} & 0 & 0 & I & 0 & pA_2^4 \\ 0 & pI & 0 & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array} \right] \begin{array}{l} p^2, A^1 \\ p^2, A^2 \\ p^2, B \\ p^2, C \\ \hline p, E \end{array} .$$

A unit in A^2 left to pA_2^4 allows to annihilate in E leading to a 0-row in E . Thus the block row of A_2^4 is not present and we get

$$(16) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{ccc|ccc} pA_1^2 & pA_1^3 & pA_1^4 & 0 & 0 & I \\ 0 & pI & 0 & 0 & I & 0 \\ pI & 0 & 0 & I & 0 & 0 \end{array} \right] \begin{array}{l} p^2, A \\ p^2, B \\ p^2, C \end{array} .$$

The submatrix pA_1^2 can be annihilated due to the presence of pI in C and similarly pA_1^3 can be annihilated due to the presence of pI in B . Hence we may assume that $pA_1^2 = 0$ and $pA_1^3 = 0$. Then the submatrix pA_1^4 has no 0-line to avoid a cross and hence the Smith Normal Form of pA_1^4 is pI . Thus we get

$$(17) \quad [\beta_1 \mid \beta_2] = \left[\begin{array}{ccc|ccc} 0 & 0 & pI & 0 & 0 & I \\ 0 & pI & 0 & 0 & I & 0 \\ pI & 0 & 0 & I & 0 & 0 \end{array} \right] \begin{array}{l} p^2, A \\ p^2, B \\ p^2, C \end{array} .$$

Now we can read off all summands, and all are of rank ≤ 4 and known. This ends the proof. \square

REFERENCES

- [1] D. M. ARNOLD, *Abelian groups and representations of finite partially ordered sets*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 2. Springer-Verlag, New York, 2000.

- [2] D. M. ARNOLD – A. MADER – O. MUTZBAUER – E. SOLAK, *Almost completely decomposable groups and unbounded representation type*, J. Algebra **349** (2012), pp. 50–62.
- [3] D. M. ARNOLD – A. MADER – O. MUTZBAUER – E. SOLAK, *Indecomposable (1, 3)-groups and a matrix problem*, Czechoslovak Math. J. **63**(138) (2013), no. 2, pp. 307–355.
- [4] D. M. ARNOLD – A. MADER – O. MUTZBAUER – E. SOLAK, *Representations of posets and decompositions of rigid almost completely decomposable groups*, Palest. J. Math. **3** (2014), special issue, Proceedings of the International Conference on Algebra in Honor of P. Smith and J. Clark (Balikesir, Turkey, 2013), pp. 320–342.
- [5] R. BURKHARDT, *On a special class of almost completely decomposable torsion free abelian groups I*, Abelian groups and modules (Udine, 1984), CISM Courses and Lectures, 287, Springer, Vienna, 1984, pp. 141–150.
- [6] T. FATICONI – P. SCHULTZ, *Direct decompositions of ACD groups with primary regulating index*, Abelian groups and modules (Colorado Springs, CO, 1995), Lecture Notes in Pure and Appl. Math., 182, Dekker, New York, 1996, pp. 233–241.
- [7] E. L. LADY, *Almost completely decomposable torsion free abelian groups*, Proc. Amer. Math. Soc. **45** (1974), pp. 41–47.
- [8] A. MADER, *Almost completely decomposable groups*, Algebra, Logic and Applications, 13. Gordon and Breach Science Publishers, Amsterdam, 2000.
- [9] O. MUTZBAUER – E. SOLAK, *(1, 2)-groups with p^3 -regulator quotient*, J. Algebra **320** (2008), no. 10, pp. 3821–3831.
- [10] O. MUTZBAUER – E. SOLAK, *(1, 2)-groups with a regulator quotient of exponent p^4* , Groups and Model Theory, Contemporary Mathematics, 576, American Mathematical Society, Providence, R.I., 2012, pp. 269–285.

Manoscritto pervenuto in redazione il 7 marzo 2014.