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Finitely generated modules and chain rings

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Abstract – This paper investigated finitely generated singular modules over a right chain ring R. We show that these modules behave similar to those over valuation rings provided R is a right duo ring. We also demonstrate that the duo condition cannot be removed from our discussion.

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1. Introduction

Although many results concerning Abelian groups carry over to modules over integral domains, the situation is more complex in the non-commutative case since there are several non-commutative notions of torsion-freeness. In this paper, we focus on Goodearl's notion of non-singularity [11]: the *singular submodule* of a right R-module M which is defined as

$$Z(M) = \{x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R\}$$

takes the place of the torsion submodule in the commutative setting. The module M is *singular* if $Z(M) = M$, and *non-singular* if $Z(M) = 0$, while R is *right non-singular* if it is non-singular as a right R-module. The discussion of modules over non-singular rings usually concentrates on non-singular modules since finitely generated torsion modules over integral domains already behave different from torsion Abelian groups [9].

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It is a goal of this paper to investigate finitely generated singular modules in the non-commutative setting. Since the structure of torsion modules can be successfully investigated for valuation rings, we consider their non-commutative equivalent: A ring R is a *right* (*left*) *chain ring* if the lattice of its right (left) ideals is linearly ordered. A right and left chain ring is simply called a *chain ring*. It is easy to see that a chain ring R has no zero divisors if and only if $Z(M)$ is a direct summand for every finitely generated right R-module M. Moreover, $M/Z(M)$ is free in this case. This reduces the discussion of finitely generated modules to the case of singular modules.

It is not the goal of this paper simply to extend known results on finitely generated modules over valuation domains to a more general setting. Instead, we are interested in characterizing the right chain rings for which the finitely generated modules behave similar to finitely generated modules over a valuation domain. We immediately realize that some additional restrictions on R are necessary to obtain any meaningful results. For instance, although every finitely generated torsion module over a domain has a non-zero annihilator, Dubrovnin gave an example of a chain domain for which the semi-simple modules are the only ones with a non-zero annihilators (Example 2.1 and [7]). Because of this example, annihilators of finitely generated modules are the focus of Section 2. Theorem 2.3 shows that these annihilators behave similar to the commutative case if and only if R is a right duo ring where R is *right duo* if $Ra \subseteq aR$ for every $a \in R$. Similarly, R is *duo* if $aR = Ra$ for every $a \in R$. Duo rings arise naturally in the discussion of chain rings, in particular when prime and completely prime ideals are considered as can be seen in a series of papers by Bessenroth, Brungs, and Törner ([3], [4], [5], and [6]). These papers also provide examples of such rings. For instance, if G is an ordered group and K is a skew field, then

$$R = \{r = \Sigma g k_g \in K[[G]] \mid \min(\mathrm{supp}(r)) \geq e\}$$

is a duo chain domain [3, Proposition 1.24] where $\mathrm{supp}(r) = \{g \mid k_g \neq 0\}$. Moreover, every right Noetherian right chain domain is right duo. Examples of such rings can also be found in [3]. For further constructions of chain domains, the reader is referred to [2]. Section 3 characterizes right duo chain rings in terms of the existence of RD-composition series for finitely generated modules (Theorem 3.4). This naturally extends the work of Salce and Zanardo in [12] for commutative rings. The paper concludes with some applications of the results of Chapters 2 and 3.

2. Modules and annihilators

Let M be a right R-module, and S a subset of M. The *annihilator of S* is the right ideal $\operatorname{Ann}(S) = \{r \in R \mid Sr = 0\}$ of R which is a two-sided ideal whenever S is a submodule of M. In particular, $M \cong R/\operatorname{Ann}(x)$ whenever M is a cyclic R-module generated by x. If R is not commutative, then $\operatorname{Ann}(xR) \subsetneq \operatorname{Ann}(x)$ is possible in contrast to the commutative setting:

Example 2.1. Let R be a nearly simple chain domain, i.e. $J = J(R)$ is the only proper non-zero two-sided ideal of R. Such a ring was constructed by Dubrovin ([3] and [7]). Since R is a domain, $J^2 = J$, and J is not finitely generated by Nakayama's Lemma. Pick a non-zero $a \in J$ and consider the singular module $M = R/aR$. Since $MJ = J/aR \neq 0$, we obtain $\operatorname{Ann}(M) = 0$, while every generator of M has a non-zero annihilator since M is singular.

To overcome this difficulty, we call an R-module M *finitely annihilated* if there exist $x_1, \dots, x_k$ in M such that $\operatorname{Ann}(M) = \operatorname{Ann}(x_1, \dots, x_k)$ [10]. If $\operatorname{Ann}(M) = \operatorname{Ann}(x)$ for some $x \in M$, then M is *cyclically annihilated*. A finitely generated right R-module M is *strongly cyclically annihilated* if, whenever $M = x_1R + \cdots + x_nR$, then $\operatorname{Ann}(M) = \operatorname{Ann}(x_i)$ for some i.

We begin our discussion with a technical result which will be used frequently throughout this paper:

Lemma 2.2. *Let R be a ring.*

a) *If M is a right R-module, then* $\operatorname{Ann}(a) = \operatorname{Ann}(aR)$ *for every $a \in M$ such that* $\operatorname{Ann}(a)$ *is a two-sided ideal.*

b) *If $M = x_1R + \cdots + x_nR$ such that* $\operatorname{Ann}(x_i)$ *is a two-sided ideal for each $i = 1, \dots, n$, then* $\operatorname{Ann}(M) = \bigcap_{i=1}^n \operatorname{Ann}(x_i)$.

Proof. a) Since $\operatorname{Ann}(a)$ is a two-sided ideal of R,

$$(aR)\operatorname{Ann}(a) = a(R\operatorname{Ann}(a)) \subseteq a\operatorname{Ann}(a) = 0.$$

b) is obvious in view of a). □

A ring R is *strongly right bounded* if every non-zero right ideal of R contains a non-zero two-sided ideal. Obviously, right duo rings are strongly right bounded.

Theorem 2.3. *The following are equivalent for a ring R*:

a) *R is a right duo right chain ring*;

b) *R is a right chain ring such that all cyclic right R-modules are cyclically annihilated*;

c) *every finitely generated right R-module is strongly cyclically annihilated*;

d) *R is a right chain ring such that $R/I \cong R/K$ for right ideals I and K yields $I = K$*;

e) *R is a right chain ring such that every finitely generated right R-module M has the property that* $\mathrm{Ann}(a)$ *is a two-sided ideal for every $a \in M$.*

Proof. a) $\Longrightarrow$ c). Let $M = x_1R + \cdots + x_nR$ be a finitely generated right R-module. By a), $\mathrm{Ann}(x_i)$ is a two-sided ideal. Since R is a right chain ring, we may assume $\mathrm{Ann}(x_1) \subseteq \cdots \subseteq \mathrm{Ann}(x_n)$. By Part b) of Lemma 2.2, $\mathrm{Ann}(M) = \bigcap_{i=1}^{n} \mathrm{Ann}(x_i) = \mathrm{Ann}(x_1)$, and M is strongly cyclically annihilated.

c) $\Longrightarrow$ b). As strongly cyclically annihilated modules are obviously cyclically annihilated, it remains to show that R is a right chain ring. Let $a, b \in R$, and consider $M = R/aR \oplus R/bR$. By c), we may assume $\mathrm{Ann}(M) = \mathrm{Ann}(1+aR) = aR$. On the other hand, $(0,0) = (1 + aR, 1 + bR)x = (x + aR, x + bR)$ yields $x \in aR \cap bR$ for all $x \in \mathrm{Ann}(M)$. Thus, $aR = \mathrm{Ann}(M) \subseteq aR \cap bR \subseteq bR$, and R is a right chain ring.

b) $\Longrightarrow$ a). Suppose that R is a right chain ring such that every cyclic right R-module is cyclically annihilated. If I is a two-sided ideal of R, then every cyclic right R/I-module M can be viewed as a cyclic right R-module via the operation $mr = m(r + I)$. Since M_R is cyclically annihilated, $M_R \,\mathrm{Ann}_R(b) = 0$ for some $b \in M$. But, $bI = 0$ and $\mathrm{Ann}_{R/I}(b) = \mathrm{Ann}_R(b)/I$ show $M_{R/I} \,\mathrm{Ann}_{R/I}(b) = M_R \,\mathrm{Ann}_R(b) = 0$. Thus, M is cyclically annihilated as an R/I-module. Therefore, R/I is a right chain ring such that every cyclic right R/I-module is cyclically annihilated.

We now show that R is a strongly right bounded ring. For this, consider a non-zero $a \in R$. Since R is a right chain ring, aR is essential in R. Then R/aR is a singular R-module, and $\mathrm{Ann}(y + aR)$ is essential in R for all $y \in R$. In particular, $\mathrm{Ann}(y + aR) \neq 0$. Moreover, since R/aR is cyclically annihilated by b), we can find $x \in R$ such that $0 \neq \mathrm{Ann}(x + aR) = \mathrm{Ann}(R/aR) \subseteq \mathrm{Ann}(1 + aR) = aR$. Thus, $\mathrm{Ann}(R/aR)$ is a non-zero two-sided ideal of R contained in aR.

Let K be the largest two-sided ideal contained in aR. If $K \neq aR$, then R/K is a strongly right bounded ring by what has been already shown. Its non-zero right ideal $(a+K)R/K = aR/K$ contains a non-zero two-sided ideal of the form N/K, where N is a two-sided of R such that $K \subsetneq N \subseteq aR$. Since K is the largest two-sided ideal of R contained in aR, we obtain a contradiction. Thus, $K = aR$, and aR is a two-sided ideal of R.

a) $\Longrightarrow$ d). If R is right duo, consider an isomorphism $\phi\colon R/I \to R/K$ for right ideals I and K of R. Observe that I and K are two-sided since R is duo. Select $r \in R$ such that $\phi(1+I) = r + K$. For every $a \in K$, we have $\phi(a+I) = \phi(1+I)a = ra + K = 0$. Since ϕ is one-to-one, $a \in I$, and thus $K \subseteq I$. By symmetry, $I = K$.

Conversely, suppose that d) holds, and let I be any right ideal of R. For a unit $u \in R$, define an isomorphism $\phi\colon R/I \to R/uI$ by $\phi(r+I) = ur + uI$. Hence, $I = uI$ by d). Since R is a right chain ring, I is two-sided.

Since a) $\Longrightarrow$ e) is trivial, it remains to show that e) $\Longrightarrow$ c). Suppose $M = a_1R + \cdots + a_nR$ for $a_1, \ldots, a_n$ in M. We may assume $\mathrm{Ann}(a_1) \subseteq \cdots \subseteq \mathrm{Ann}(a_n)$ because R is a right chain ring. Since $\mathrm{Ann}(a_i)$ is an ideal for every i, we have $\mathrm{Ann}(M) = \bigcap_i \mathrm{Ann}(a_i) = \mathrm{Ann}(a_1)$ as in the proof of Part b) of Lemma 2.2. Thus, M is strongly cyclically annihilated. □

Corollary 2.4. *Let R be a right duo right chain ring. A finitely generated right R-module M is singular if and only if* $\mathrm{Ann}(M) \neq 0$.

Proof. If M is a finitely generated singular right R-module, then we can find $x \in M$ such that $\mathrm{Ann}(M) = \mathrm{Ann}(x)$ by Theorem 2.3. Since M is singular, $\mathrm{Ann}(x)$ is essential, and thus non-zero. Conversely, because M is a finitely generated $R/\mathrm{Ann}(M)$-module, it is an epimorphic image of $[R/\mathrm{Ann}(M)]^n$ for some $n < \omega$. Since R is a right chain domain and $\mathrm{Ann}(M) \neq 0$, it is essential in R. Therefore, $R/\mathrm{Ann}(M)$ is singular, and the same holds for M. □

3. *RD*-submodules and composition series

A submodule N of M is an *RD-submodule* if $Nr = N \cap Mr$ for every $r \in R$, or equivalently if R/rR is projective with respect to $0 \to N \to M \to M/N \to 0$ for all $r \in R$. A submodule N of a right R-module M is *pure* if R/I is projective with respect to $0 \to N \to M \to M/N \to 0$ whenever I is a finitely generated right ideal of R. In particular, $MI \cap N = NI$ for all left ideals I of R if N is pure in M. Since every finitely generated right ideal of a right chain ring is cyclic, relative divisibility and purity are equivalent for right chain rings. In the following, $J = J(R)$ denotes the Jacobson radical of the ring R.

PROPOSITION 3.1. *Let R be a left chain ring. The following are equivalent for a right R-module M and every $a \in M$ such that* $\mathrm{Ann}(a)$ *is a two-sided ideal*:

a) *aR is an RD-submodule of M*;

b) *if $0 \neq ar = zs$ for some $r, s \in R$ and $z \in M$, then $Rr \subseteq Rs$*;

c) $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(a)$ *for every $x \in a + MJ(R)$*.

PROOF. a) $\Longrightarrow$ b). Let aR be an RD-submodule of M, and consider $r, s \in R$ and $z \in M$ with $0 \neq ar = zs$. Since aR is an RD-submodule, $ar = ar's$ for some $r' \in R$. If $Rs \subsetneq Rr$, then $s = jr$ for some $j \in R$ which cannot be a unit of R. Thus, $j \in J(R)$ and $ar = ar's = ar'jr$ so that $a(1 - r'j)r = 0$. Hence, $(1 - r'j)r \in \mathrm{Ann}(a) = \mathrm{Ann}(aR)$ since $\mathrm{Ann}(a)$ is two-sided. But, $u = 1 - r'j$ is a unit of R because $j \in J(R)$. So, $ar = au^{-1}(ur) \in aRur = 0$, a contradiction.

b) $\Longrightarrow$ c). Suppose that $(a + y)r = 0$ but $ar \neq 0$ for some $y \in MJ(R)$. Since R is a chain ring, $y = mj$ for some $m \in M$ and $j \in J$. Then $0 \neq ar = (-m)(jr)$. By b), $Rr \subseteq R(jr)$ which implies that $(1-r'j)r = 0$ for some $r' \in R$. Then $r = 0$ as $1 - r'j$ is a unit, a contradiction.

c) $\Longrightarrow$ a). Suppose $0 \neq ar = ms$ for some $r, s \in R$ and $m \in M$. If $Rr \subseteq Rs$, then $r = ts$ for some $t \in R$. So $ms = ar = ats \in aRs$, and we are done. So, suppose $Rs \subsetneq Rr$. Then $s = jr$ for some $j \in J(R)$, and $0 \neq ar = ms = mjr$. Therefore, $(a - mj)r = 0$. Then, $ar = 0$ by c), a contradiction. □

Let M be a finitely generated right R-module. The smallest number of generators of M is denoted by $\mathrm{gen}(n)$. It is a direct consequence of Nakayama's Lemma that $\mathrm{gen}(M)$ is the R/J-dimension of M/MJ whenever R is a right chain ring. An ascending chain $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ of submodules of a module M is an *RD-composition series* if each M_i is an RD-submodule of M and is cyclic. If $M_i/M_{i-1} \neq 0$ for $i = 1, \ldots, n$, then n is *the length of the series*. The sequence $A_i = \mathrm{Ann}(M_i/M_{i-1})$ of two-sided ideals of R is the *annihilator sequence* of the RD-composition series. It is *non-decreasing* if $A_i \subseteq A_{i+1}$ for $i = 1, 2, \ldots, n-1$. Our next result reduces the discussion of modules with RD-composition series to the case of singular modules:

PROPOSITION 3.2. *Let R be a strongly right bounded right chain domain. If a finitely generated right R-module M has an RD-decomposition series with non-decreasing annihilator sequence, then $M = F \oplus Z(M)$ for some free module F.*

Proof. Suppose that k is the largest index for which $A_k = 0$. For any $m > k$, consider $x \in M_m$. If $0 \neq r \in A_m$, then there is $y \in M_{m-1}$ with $xr = yr$. Since R is a right chain ring, rR is an essential right ideal of R. Hence $x - y \in Z(M)$, and $M_m + Z(M) = M_{m-1} + Z(M)$. Thus, $M = M_k + Z(M)$. It remains to show that M_k is free. Since M_j/M_{j-1} is cyclic, we can find a right ideal I_j of R such that $M_j/M_{j-1} \cong R/I_j$. If $I_j \neq 0$, then it contains a non-zero two-sided ideal K_j, and $K_j \subseteq \operatorname{Ann}(M_j/M_{j-1})$. Thus, $j > k$, and M_k is a free R-module. Since R has no zero-divisors, $M_k \cap Z(M) = 0$. □

As in the commutative case, one can easily show that the length of an RD-composition series of a finitely generated right R-module M is equal to $\operatorname{gen}(M)$ if R is a right duo chain ring [12]. Thus, all RD-composition series have the same length which we denote by $\ell(M)$. Finally, an RD-composition series $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ is *strong* if M_{i+1}/M_i is cyclic and has a generator with two-sided annihilator for each $i = 0, \dots, n-1$.

Remark 3.3. Let R be a nearly simple chain domain. A finitely generated singular module M has a strong RD-composition series if and only if $M \cong \bigoplus_n R/J$, where $n = \ell(M)$.

Proof. We induct on $n = \ell(M)$. Since J is the only non-zero proper two-sided ideal of R, we obtain that $M = xR$ is isomorphic to R/J whenever x has a two-sided annihilator. Suppose that $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ is a strong RD-composition series of M. By the induction hypothesis and by what has already been shown, $M_{n-1} \cong \bigoplus_{n-1} R/J$ and $M/M_{n-1} \cong R/J$. Thus, M can be viewed as a right R/J^2-module. However, $J^2 \neq 0$ since R is a domain. Thus, $J = J^2$ because R is nearly simple. Hence, M is a right R/J-module, and $M \cong M_{n-1} \oplus R/J$. The converse is obvious. □

In particular, if M is any finitely generated singular R-module over a nearly simple chain domain with $MJ \neq 0$, then M does not have a strong RD-composition series.

Theorem 3.4. *The following are equivalent for a chain ring R:*

a) *R is a right duo ring.*

b) *The following hold for every finitely generated right R-module M:*

   i) *M admits a strong RD-composition series $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ with non-decreasing annihilator sequence $A_1, \dots, A_n$.*

ii) *Every RD-composition series $0 = M'_0 \subseteq \cdots \subseteq M'_n = M$ of M with non-decreasing annihilator sequence $B_1, \ldots, B_k$ is strong and $A_k = B_k$ for $k = 1, \ldots, n$.*

PROOF. a) $\Longrightarrow$ b). i) Suppose R is right duo chain ring. Let M be a finitely generated right R-module. Since R/J is a division algebra, M/MJ is a finite dimensional vector space over R/J. If $\{x_1 + MJ, \ldots, x_n + MJ\}$ is a R/J-basis of M/MJ, then the x_i's generate M by Nakayama's Lemma since M is finitely generated. Because R is a right chain ring, we may arrange the x_i's in such a way that $\mathrm{Ann}(x_i) \subseteq \mathrm{Ann}(x_{i+1})$ for all i. Since R is a right duo ring, $\mathrm{Ann}(M) = \mathrm{Ann}(x_1)$ by Lemma 2.2. Let $1 \leq j \leq n$ be chosen maximal with $\mathrm{Ann}(M) = \mathrm{Ann}(x_1) = \cdots = \mathrm{Ann}(x_j)$. We show that there exists $i \in \{1, \ldots, j\}$ such that $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x_i)$ for every $x \in x_i + MJ$.

If, for every $k \leq j$, there exists $x'_k \in x_k + MJ$ such that we have $\mathrm{Ann}(x_k) \subsetneq \mathrm{Ann}(x'_k)$, then M is generated by $\{x'_1, \ldots, x'_j, x_{j+1}, \ldots, x_n\}$ by another application of Nakayama's Lemma. Using Theorem 2.3 once more, we obtain $\mathrm{Ann}(M) = \mathrm{Ann}(x'_\ell)$ for at least one $\ell \leq j$ since $\mathrm{Ann}(M) \neq \mathrm{Ann}(x_k)$ for $k > j$ by the choice of j. But then, $\mathrm{Ann}(x_\ell) = \mathrm{Ann}(M) = \mathrm{Ann}(x'_\ell)$ which contradicts the choice of x'_ℓ. Without loss of generality, we may assume the element obtained in this way is x_1. By Proposition 3.1, $M_1 = x_1 R$ is an RD-submodule of M. Moreover, $\mathrm{Ann}(M_1) = \mathrm{Ann}(x_1) = \mathrm{Ann}(M)$, and M_1 is strongly cyclically annihilated since R is a right duo chain ring.

We construct the strong RD-composition series by induction on the number n of generators of M. By what has just been shown, the result holds for $n = 1$. By induction hypothesis, we can find a chain of submodules $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ of M such that $0 = M_1/M_1 \subseteq M_2/M_1 \subseteq \cdots \subseteq M_n/M_1 = M/M_1$ is a strong RD-composition series of M/M_1 with non-decreasing annihilator sequence because M/M_1 is generated by $x_2 + M_1, \ldots, x_n + M_1$. Since M_1 is an RD-submodule of M, each M_i is an RD-submodule of M, and $M_{i+1}/M_i \cong (M_{i+1}/M_1)/(M_i/M_1)$ is cyclic and strongly cyclically annihilated for each $i > 0$ since R is a right duo chain ring. Thus, $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ is a strong RD-composition series of M. To see that the annihilator sequence is non-decreasing, set $A_i = \mathrm{Ann}(M_i/M_{i-1})$ for $i = 1, \ldots, n$ and note

$$\begin{aligned} A_i &= \mathrm{Ann}(M_i/M_{i-1}) \\ &= \mathrm{Ann}((M_i/M_1)/(M_{i-1}/M_1)) \\ &\subseteq \mathrm{Ann}((M_{i+1}/M_1)/(M_i/M_1)) \\ &= \mathrm{Ann}(M_{i+1}/M_i) = A_{i+1} \end{aligned}$$

for all $i > 1$. Moreover, $A_1 = \mathrm{Ann}(M_1) = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(M_2/M_1) = A_2$.

ii) Consider an RD-composition series $0 = M'_0 \subseteq \cdots \subseteq M'_n = M$ of M with non-descending annihilator sequence. By Theorem 2.3, the composition series is strong.

We show $B_1 = \mathrm{Ann}(M'_1) = \mathrm{Ann}(M) = \mathrm{Ann}(M_1) = A_1$. If this is not the case, then $\mathrm{Ann}(M) \subsetneq \mathrm{Ann}(M'_1)$. Hence, we can find $r \in R$ with $M'_1 r = 0$ but $Mr \neq 0$. Let $k > 1$ be chosen minimal such that $M'_k r \neq 0$, and pick $y \in M'_k$ with $yr \neq 0$. Since $\mathrm{Ann}(M'_1) \subseteq \mathrm{Ann}(M'_k/M'_{k-1})$, we have $yr \in M'_{k-1}$. Since M'_{k-1} is an RD-submodule of M, there is $z \in M'_{k-1}$ such that $0 \neq yr = zr$ contradicting the minimality of k. Hence, $\mathrm{Ann}(M'_1) = \mathrm{Ann}(M)$. Finally, observe $M'_1 = dR$. Since R is right duo, $\mathrm{Ann}(d) = \mathrm{Ann}(dR) = \mathrm{Ann}(M'_1) = \mathrm{Ann}(M)$.

Consider an arbitrary RD-submodule U of M, and $a \in R$. To see that UaR is an RD-submodule of MaR, observe that aR is a two-sided ideal of R since R is a right duo ring. If s is any element of R, then $I = aRs$ is a left ideal of R. However, since R is right chain ring, every RD-submodule of M is pure. Hence, $UaRs \subseteq MaRs \cap UaR \subseteq MaRs \cap U = UaRs$, and UaR is an RD-submodule of MaR. Now assume in addition that M/U is cyclic, say $M = xR + U$. Then, $MaR = xRaR + UaR \subseteq (xa)R + UaR \subseteq MaR$ as $Ra \subseteq aR$ since R is right duo. Hence, $0 = M_0 aR \subseteq \cdots \subseteq M_n aR = MaR$ and $0 = M'_0 aR \subseteq \cdots \subseteq M'_n aR = MaR$ are two RD-composition series of MaR.

Suppose that we have shown that $A_1 = B_1, \ldots, A_k = B_k$ for some $k < n$, and assume that $A_{k+1} \neq B_{k+1}$. Without loss of generality, we may assume $B_{k+1} \subsetneq A_{k+1}$. Pick $a \in A_{k+1} \setminus B_{k+1}$. Since $a \in A_{k+1} \subseteq \cdots \subseteq A_n$, we have $Ma = M_k a$. Moreover, $(M_i/M_{i-1})a \neq 0$ for $i = 1, \ldots, k$. By the last paragraph, $\ell(MaR) = k$. If $a \notin B_i$ for any i, then $0 = M'_0 aR \subseteq \cdots \subseteq M'_n aR = MaR$ is a composition series of MaR of length n, which contradicts the choice $k < n$. Thus, $a \in B_m$ for some smallest $m > k + 1$. Thus, $MaR = M'_m aR$, and MaR has a decomposition series of length $m - 1 > k$, a contradiction. Therefore, $A_{k+1} = B_{k+1}$.

b) $\Longrightarrow$ a). Consider a proper right ideal I of R, and the right R-module $M = R/I$. Clearly, $y = 1 + I \in M$ is a generator of M with $\mathrm{Ann}(y) = I$. Since $0 \subseteq R/I$ is an RD-composition series of $M = R/I$ with non-decreasing annihilator sequence, it is strong. Thus, $M = xR$ such that $\mathrm{Ann}(x)$ is a two-sided ideal, and there are $r, s \in R$ with $x = yr$ and $y = xs$. Then, $x = xsr$. If r or s are in $J = J(R)$, then we obtain $xR \supseteq xRJ \supseteq xsrR = xR$, and $xRJ = xR$. By Nakayama's Lemma, $xR = 0$, a contradiction. Thus, r and s are units of R. If $t \in \mathrm{Ann}(y)$, then $0 = yt = x(st)$, and $s\,\mathrm{Ann}(y) \subseteq \mathrm{Ann}(x)$. On the other hand, if $t' \in \mathrm{Ann}(x)$, then $0 = xt' = y(rt')$, and $r\,\mathrm{Ann}(x) \subseteq \mathrm{Ann}(y)$. Since $\mathrm{Ann}(x)$ is a two-sided ideal and r and s are units of R, we have $\mathrm{Ann}(y) \subseteq s^{-1}\,\mathrm{Ann}(x) = \mathrm{Ann}(x) = r\,\mathrm{Ann}(x) \subseteq \mathrm{Ann}(y)$. Thus, $I = \mathrm{Ann}(y) = \mathrm{Ann}(x)$ is two-sided, and R is duo. □

We now address the question whether the RD-composition series in Theorem 2.3 is unique if R is a right and left duo chain ring thus extending results by Fuchs, Salce and Zanardo in ([9] and [12]) to the non-commutative setting.

Proposition 3.5. *Let R be a right duo, right chain ring and M a right R-module. If N is an RD-submodule of M such that $M/N = (a+N)R$ and $\mathrm{Ann}(M/N) \subseteq \mathrm{Ann}(N)$, then $M = aR \oplus N$.*

Proof. Clearly $M = aR + N$. Suppose $ar \in N$. Since N is an RD-submodule of M, $ar = nr$ for some $n \in N$. Then $(a-n)r = 0$ and $r \in \mathrm{Ann}(a-n) = \mathrm{Ann}((a-n)R)$ since R is right duo. Once we have shown $r \in \mathrm{Ann}(M/N)$, then $Nr = 0$ and therefore $ar = 0$. If $m + N \in M/N$, then $m + N = as + N$ for some $s \in R$. Since $r \in \mathrm{Ann}((a-n)R)$, we obtain $[(a-n)s]r = 0$, and $asr \in N$. Thus, $(m+N)r = (as+N)r = asr + N = N$, and $r \in \mathrm{Ann}(M/N)$. □

Corollary 3.6. *Let R be a right duo chain ring and M a finitely generated right R-module. Suppose that $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ is an RD-composition series of M with non-decreasing annihilator sequence $A_1 \subseteq \cdots \subseteq A_n$. If $M_k/M_{k-1} = (x_k + M_{k-1})R$ and $A_1 = \cdots = A_k$ for some $k \le n$, then $M_k = \bigoplus_{i=1}^{k} x_i R$. In particular, if all terms of the annihilator sequence of $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ are equal to $\mathrm{Ann}(M)$, then M is the direct sum of cyclic submodules.*

Proof. The result is trivial for $k = 1$ as $M_1 = x_1 R$ is cyclic. We show that M_k is a direct sum of cyclic modules using Proposition 3.5. By assumption, $\mathrm{Ann}(M) = \mathrm{Ann}(x_1 R) = \mathrm{Ann}(M_1) = A_1 = A_2 = \cdots = A_k$. Since M_{k-1} is an RD-submodule of M, $M_k/M_{k-1} = (x_k + M_{k-1})R$ is cyclic, and $\mathrm{Ann}(M_k/M_{k-1}) = A_k = A_1 = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(M_{k-1})$, we obtain $M_k = M_{k-1} \oplus x_2 R$. Since M_{k-1} is a direct sum of cyclic modules by induction hypotheses, the result follows. □

Lemma 3.7. *Let R be a right duo chain ring. Every RD-composition series of a finitely generated right R-module M is isomorphic to one whose annihilator sequence is non-decreasing.*

Proof. Let $0 = M_0 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$ be an RD-composition series of M with annihilator sequence $A_1, \ldots, A_n$, and write $M_i/M_{i-1} = (x_i + M_{i-1})R$.

Suppose there exists an $i \in \{1, 2, \ldots n-1\}$ such that $A_i \not\subseteq A_{i+1}$. Since R is a right chain ring, A_{i+1} is properly contained in A_i. We replace M_i by an RD-submodule M_i' such that $M_i'/M_{i-1} \cong M_{i+1}/M_i$ and $M_{i+1}/M_i' \cong M_i/M_{i-1}$. Once this has been done, we have $\text{Ann}(M_i'/M_{i-1}) = A_{i+1} \subseteq A_i = \text{Ann}(M_{i+1}/M_i')$ as desired.

To obtain M_i', consider the RD-submodule $N = M_i/M_{i-1}$ of the module $K = M_{i+1}/M_{i-1}$. Observe that $K/N \cong M_{i+1}/M_i$, so K/N is cyclic, and $K = (x_{i+1} + M_{i-1})R + N$. Moreover, $\text{Ann}(K/N) = A_{i+1} \subseteq A_i = \text{Ann}(N)$ so that $K = N \oplus (x_{i+1} + M_{i-1})R$ by Proposition 3.5. Let $M_i' = x_{i+1}R + M_{i-1} \subseteq M_{i+1}$. Clearly, $M_i'/M_{i-1} \cong K/N \cong M_{i+1}/M_i$ is cyclic, and the same holds for $M_{i+1}/M_i' \cong (M_{i+1}/M_{i-1})/(M_i'/M_{i-1}) = K/(M_i'/M_{i-1}) \cong N = M_i/M_{i-1}$. Since M_{i-1} is an RD-submodule of M and M_i/M_{i-1} is a relatively divisible submodule of M/M_{i-1}, we obtain that M_i' is an RD-submodule of M. □

The next result establishes the isomorphism of any two RD-composition series of a finitely generated right R module over a duo chain ring.

THEOREM 3.8. *If M is a finitely generated right R-module over a right duo chain ring R, then any two RD-composition series of M are isomorphic.*

PROOF. Any RD-composition series of M has length equal to $\text{gen}(M)$. We may also assume, by Lemma 3.7, that the annihilator sequences A_i and B_i are non-decreasing and have the same length. By Theorem 3.4, we are done. □

4. Essential pure submodules

The goal of this section is to show that every finitely generated right R-module M contains an essential pure submodule of M that is the direct sum of cyclic modules. We begin with some technical results based on the discussion of the commutative case in [12]. However, several modifications are necessary to extend them to the noncommutative setting.

PROPOSITION 4.1. *Let R be a right duo chain ring, and M be a finitely generated right R-module. If $0 \neq x \in M$, then there exists $r \in R$ such that $x \in Mr \backslash MJr$.*

PROOF. Let $0 \neq x \in M$, and suppose $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is an RD-composition series of M with non-zero factors. Choose $i > 0$ such that $x \in M_i \setminus M_{i-1}$. If $i = 1$, then $x \in M_1 = x_1R$, and there exists an $r \in R$ such that $x = x_1r \in Mr$. If $x \in MJr$, then $x = mjr$ for some $j \in J$ and $x \in M_1 \cap Mjr = M_1jr$ because M_1 is a pure submodule of M. But, $M_1 = x_1R$

yields $x = x_1 sjr$ for some $s \in R$. Because J is an ideal, $sj \in J$ and $1 - sj$ is a unit of R with $x_1(1 - sj)r = 0$. Since $x_1 R$ is an RD-submodule of M, Proposition 3.1 yields $\mathrm{Ann}_R(y) \subseteq \mathrm{Ann}_R(x_1)$ for every $y \in x_1 + MJ$. Because $x_1(1 - sj) \in x_1 + MJ$ and $r \in \mathrm{Ann}_R(x_1(1 - sj))$, we have $r \in \mathrm{Ann}_R(x_1)$. But then $x = x_1 r = 0$ yields a contradiction.

Suppose now that $i > 1$. Set $\bar{x} = x + M_{i-1}$ and $\bar{M} = M/M_{i-1}$. Then $\bar{x}$ is a non-zero element of M_i/M_{i-1}. Since M_i/M_{i-1} is cyclic with generator $x_i + M_{i-1}$, we have $\bar{x} = \bar{x_i} r$ for some $r \in R$. Hence $\bar{x} \in \bar{M} r$. We claim that $x \notin \bar{M} J r$. If $x \in \bar{M} J r$, then $\bar{x} = \bar{z} j r$ for some $z \in m$ and $j \in J$, and we can write $\bar{x} = \bar{x_i} r = \bar{x_i} j r$. Arguing as above, noting that the purity of M_i/M_{i-1} in M/M_{i-1} implies that $\bar{x_i} R$ is pure in $\bar{M}$, we obtain $r \in \mathrm{Ann}(\overline{x_i})$. Therefore $x = x_i r \in M_{i-1}$, a contradiction. Hence, $x \in \bar{M} r \setminus \bar{M} J r$.

If $x \in Mr$, then the proof is complete since $x \in MJr$ implies $\bar{x} \in \bar{M} J r$, contradicting the above. If $x \notin Mr$, then $x = mr + y$ for some $m \in M$ and $y \in M_{i-1}$. By the induction hypothesis, there exists an $s \in R$ such that $y \in Ms \setminus MJs$. Clearly $y \notin Mr$ because $x \notin Mr$. Since R is a chain ring, $Rr \subseteq Rs$ or $Rs \subseteq Rr$. Because $Rs \subseteq Rr$ implies $y \in Mr$, we have $Rr \subseteq Rs$ and $r = ts$ for some $t \in R$. If $t \notin J$, then $s = t^{-1} r$ and $x \in Mr$, a contradiction. Hence $x = mr + y = m(ts) + y = (mt)s + y \in Ms \setminus MJs$ since $mts \in MJs$, and the proof is complete. □

When considering $MJ(R)r$, the question arises whether or not $aJ = Ja$ for every $a \in R$. This is not always the case, as the following example shows. Let R be the 2×2 lower triangular matrix ring with entries from a field k. By [11, Corollary 4.9], R is a right and left Artinian hereditary ring. Since R is Artinian, the Jacobson radical is equal to the nilradical N of R. Let e_{ij} be the standard matrix units. Then $N = re_{21}$, where $r \in k$. If $a = e_{11}$, then a calculation shows that $aJ = 0$. On the other hand, again by an easy calculation, $Ja = J$. Hence, $aJ \neq Ja$.

For the rings under consideration, we have

Proposition 4.2. *If R is a chain domain, then $aJ = Ja$ for every $a \in R$ if and only if R is a duo ring.*

Proof. If R is a duo ring, then $aR = Ra$ for every non-zero $a \in R$. By symmetry, it is enough to show $Ja \subseteq aJ$. Observe that Ja and aJ are right ideals of R since R is a duo ring. If Ja is not a subset of aJ, then aJ is a proper subset of Ja since R is a right chain ring. In particular, Ja/aJ is a non-zero submodule of aR/aJ in view of $Ja \subseteq Ra = aR$. Consider the epimorphism

$\phi : R/J \to aR/aJ$ defined by $\phi(r + J) = ar + aJ$. If $\phi(r + J) = 0$, then $ar \in aJ$ which implies $ar = aj$ for some $j \in J$. Thus, $a(r - j) = 0$. If $r \notin J$, then $rs = 1$ for some $s \in R$, and $0 = a(r - j) = a(rs - js) = a(1 - js)$, which is a contradiction as $1 - js$ is a unit of R. Hence, ϕ is an isomorphism. Since R is a right chain ring, J is a maximal ideal and R/J is a simple R-module. Hence, aR/aJ is simple which implies $Ja/aJ = aR/aJ$. We conclude $Ja = aR = Ra$. Thus, there exists $y \in J$ such that $ya = a$, which implies $(y - 1)a = 0$. But then $a = 0$ as $y - 1$ is a unit. Thus, $Ja \subseteq aJ$.

Conversely, suppose $aJ = Ja$ for every $a \in R$. Consider $0 \neq r \in R$. By symmetry, it suffices to show $Rr \subseteq rR$. Suppose that $x \in R$ such that $xr \notin rR$. If $x \in J$, then $xr \in Jr = rJ \subseteq rR$. Thus, x is a unit of R, and $xrR \subseteq rR$ or $rR \subseteq xrR$. Since $xr \notin rR$, we have $rR \subsetneq xrR$. Write $r = xrt$ for some $t \in R$. If $t \notin J$, then $xr = rt^{-1} \in rR$, a contradiction. Thus, we can find $t' \in J$ such that $rt = tr$ from which we get $r = xrt = xt'r$. Thus, $(1 - xt')r = 0$. Since $1 - xt'$ is a unit of R, this implies $r = 0$, a contradiction. □

The following technical result is crucial in establishing the main result. Although the proof is based on ideas found in [12], modifications are necessary for working in a non-commutative setting.

THEOREM 4.3. *Let R be a duo chain ring, and consider a finitely generated right R-module M. If $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ is an RD-composition series of M with non-decreasing annihilator sequence $A_1, \ldots, A_n$ such that M_{n-1} is not essential in M, then M has a non-zero cyclic summand.*

PROOF. Let $M = x_1R + \cdots + x_nR$. Choose $0 \neq y \in M$ with $yR \cap M_{n-1} = 0$, and consider the equation

$$y = \sum_{i=1}^{n} x_i a_i \quad (a_i \in R). \tag{1}$$

If $y \in MJ$, then there exist $r \in R$ and $x \in M \setminus MJ$ with $y = xr$ by Proposition 4.1. Since $xR \cong R/I$ for some right ideal I of R, we obtain that xR is uniserial as R is a chain ring. Therefore, yR is an essential submodule of xR, and $xR \cap M_{n-1} = 0$. Replacing y by x allows us to assume $y \in M \setminus MJ$. Thus, $a_i \notin J$ for at least one i, and choose j to be the largest index such that $a_j \notin J$. Then a_j is a unit so that we may assume without loss of generality that $a_j = 1$. If $j = n$, then $M = yR \oplus M_{n-1}$, and the proof is complete. Assume $j < n$ and set $N = \sum_{i \neq j} x_iR$. Since $M = N + yR$, we need only show that

$N \cap yR = 0$. If $N \cap yR \neq 0$, then we have a relation

(2) $$0 \neq yr = \sum_{i \neq j} x_i b_i \quad (b_i \in R).$$

We claim that if $j < h \leq n$ then there exist a relation

(3) $$x_h r_h = \sum_{i=1}^{h-1} x_i a_{h,i}$$

with

(4) $$r_h \in Jr, \quad a_{h,j} \in Rr \setminus Jr$$

The proof is by induction on $n-h$. We show first that the claim holds for $h = n$. From (1) and (2) we obtain $x_n(b_n - a_n r) = \sum_{i=1}^{n-1} x_i(a_i r - b_i) \in M_{n-1}$, where $b_j = 0$. If $b_n \in A_n = \mathrm{Ann}(M/M_{n-1})$, then $0 \neq yr = \sum_{i \neq j} x_i b_i \in yR \cap M_{n-1}$, a contradiction. Similarly, $a_n r \notin A_n$. Since R is a chain ring, either $rR \subseteq b_n R$ or $b_n R \subseteq rR$. If the first case holds, then $r = b_n t$ for some $t \in R$. Recall that $a_n \in J$ by our assumption on j. Hence, $a_n b_n \in Jb_n = b_n J$ by Proposition 4.2. Thus, $a_n b_n = b_n a_n'$ for some $a_n' \in J$. Then $b_n - a_n r = b_n - a_n b_n t = b_n - b_n a_n' t = b_n(1 - a_n' t)$. Since $a_n' \in J$, $(1 - a_n' t)$ is a unit of R. Hence, $x_n b_n (1 - a_n' t) \in M_{n-1}$, which implies $x_n b_n \in M_{n-1}$. We conclude that $b_n \in A_n$, a contradiction. Thus, $b_n R \subsetneq rR$, and $b_n = rs$ for some non-unit $s \in R$. So $b_n \in rJ = Jr$, and thus $a_n \in J$ implies that $b_n - a_n r \in Jr$.

Set $r_n = b_n - a_n r$ and note that (3) and the first part of (4) hold for $h = n$. To establish the final claim in (4), recall that $b_j = 0$ and that $a_j \notin J$. So we have that $a_j r \in Rr$. If $r = a_j r \in Jr$, then $r = jr$ for some $j \in J$ which is not possible since $1 - j$ is a unit. Setting $a_{n,j} = a_j r$, we see that the final claim in (4) holds and the result is established for $h = n$.

By induction, we have

(5) $$x_h r_h = \sum_{i=1}^{h-1} x_i a_{h,i}$$

such that $r_h \in Jr$ and $a_{h,j} \in Rr \setminus Jr$.

Since $r_h \in Jr$, we have $x_h r_h \in MJr \cap M_{h-1} = M_{h-1}Jr$ by the purity of M_{h-1}. Therefore we can write $x_h r_h = \sum_{i=1}^{h-1} x_i c_i q r$, where $q \in J, c_i \in R$. Subtracting, we see $x_{h-1}(a_{h,h-1} - c_{h-1} q r) = \sum_{i=1}^{h-2} x_i(c_i q r - a_{hi})$. By the same argument as above, using the fact that $r \notin A_{h-1}$, we show $a_{h,h-1} - c_{h-1} q r \in Jr$.

Since $c_{h-1} q r \in Jr$ as $q \in J$, the claim is reduced to showing $a_{h,h-1} \in Jr$. If $Ra_{h,h-1} \subsetneq Rr$, then $a_{h,h-1} = jr$ for some $j \in J$ and the claim follows.

Suppose $Rr \subseteq Ra_{h,h-1}$. Then $r = sa_{h,h-1}$ for some $s \in R$. Utilizing Proposition 4.2 in an identical fashion as above, we obtain $x_h a_{h,h-1} \in M_{h-1}$. Hence, $a_{h,h-1} \in A_{h-1}$. Thus one gets $r = sa_{h,h-1} \in A_{h-1}$, a contradiction. Therefore, $a_{h,h-1} - c_{h-1}qr \in Jr$. Moreover, $a_{h,j} \in Rr \setminus Jr$ implies $c_j qr - a_{h,j} \in Rr \setminus Jr$. Set $r_{h-1} = a_{h,h-1} - c_{h-1}qr$ and $a_{h-1,j} = c_j qr - a_{h,j}$ for $i \leq h-2$.

Now set $h = j + 1$. By the result above, we have a relation $x_{j+1}r_{j+1} = \sum_{i=1}^{j} x_i a_{j+1,j}$, where $r_{j+1} \in Jr$ and $a_{j+1,j} \in Rr \setminus Jr$. Repeating the argument above, using the fact that $r_{j+1} \in Jr$ and the relative divisibility of M_j, we can write $x_j(a_{j+1,j} - d_j qr) \in M_{j-1}$, where $d_j \in R$ and $q \in J$. Then $r \notin A_{j-1}$ implies that $a_{j+1,j} - d_j qr \in Jr$. But, $q \in J$ implies that $d_j qr \in Jr$ so that $a_{j+1,j} \in Jr$, a contradiction. Hence, $N \cap yR = 0$. □

We are now able to prove the non-commutative version of the following results which were shown by Salce and Zanardo ([12]) in the commutative setting. Here, the symbol $\dim(M)$ denotes the *Goldie dimension* of the R-module M.

THEOREM 4.4. *Over a duo chain ring R, every finitely generated right R-module M contains an essential pure submodule which is the direct sum of cyclic modules such that* $\dim(B) = \dim(M)$.

PROOF. We induct on $n = \operatorname{gen}(M)$. If $n = 1$, then the result is trivial, so assume $n > 1$. By induction, M_{n-1} has an essential pure submodule B' that is the direct sum of $\dim(M_{n-1})$ non-zero cyclic submodules. If M_{n-1} is essential in M, then B' is essential in M. Set $B = B'$ and note that B has finite Goldie dimension. Then $B \leq_e M$ implies that $\dim(B) = \dim(M)$, and the result follows. If M_{n-1} is not essential in M, then by Theorem 4.3, there exists $0 \neq y \in M$ and a submodule N of M such that $M = yR \oplus N$. Note $\operatorname{gen}(N) = n - 1$, so by the induction hypothesis, N contains an essential pure submodule B'', which is the direct sum of $\dim(N)$ non-zero cyclic submodules. Set $B = yR \oplus B''$. Then B is an essential pure submodule of M that is the direct sun of non-zero cyclic submodules, and the proof is complete. □

As a result of Theorem 4.4 and the observations above, we can obtain an upper estimate on the Goldie dimension of a finitely generated right R-module.

COROLLARY 4.5. *If M is a finitely generated right R-module over a duo chain ring R, then* $\dim(M) \leq \operatorname{gen}(M)$.

PROOF. Let B be an essential pure submodule which is the direct sum of non-zero cyclic submodules. Since B is pure, we have that $BJ = B \cap MJ$. Then

$(B + MJ)/MJ$ is a submodule of M/MJ, and

$$B/BJ = B/B \cap MJ \cong (B + MJ)/MJ.$$

Since M/MJ is a finite dimensional vector space over the division ring R/J, we obtain $\dim_{R/J} B/BJ \leq \dim_{R/J} M/MJ$. By the observations above, we have

$$\dim(M) = \dim(B) = \dim_{R/J} B/BJ \leq \dim_{R/J} M/MJ = \text{gen}(M). \qquad \square$$

The final result of this section is a criteria for a finitely generated right R-module over a right duo, chain ring to be a direct sum of cyclic modules.

Corollary 4.6. *A finitely generated right R-module over a duo chain ring R is the direct sum of cyclic modules if and only if* $\text{gen}(M) = \dim(M)$.

Proof. If M is the direct sum of cyclic modules, the clearly we get $\text{gen}(M) = \dim(M)$. Conversely, assume $\text{gen}(M) = \dim(M)$. By Theorem 4.4, M contains an essential pure submodule B that is the sum of non-zero cyclic submodules and $\dim(B) = \dim(M) = \text{gen}(M)$. Suppose B is a proper submodule of M and $\dim(B) = \text{gen}(M)$. Then $B/BJ \cong B/(B \cap MJ) = (B + MJ)/MJ \subsetneq M/MJ$. Consequently, $B + MJ \subsetneq M$, a contradiction. Hence, $\text{gen}(M) > \text{gen}(B) = \dim(B) = \text{gen}(M)$, an obvious contradiction. Thus, $M = B$, and the result follows. $\square$

References

[1] U. Albrecht, *Chain conditions in endomorphism rings,* Rocky Mountain J. Math. **15** (1985), no. 1, pp. 91–106.

[2] U. Albrecht – G. Törner, *Valuations and group algebras,* Rend. Sem. Mat. Univ. Padova **100** (1998), pp. 67–79.

[3] C. Bessenrodt – H. H. Brungs – G. Törner, *Right chain rings,* Pt 1, Schriftenreihe des Fachbereiches Mathematik, Universität Duisburg 74, 1985.

[4] C. Bessenrodt – H. H. Brungs – G. Törner, *Right chain rings,* Pt 2a, Schriftenreihe des Fachbereiches Mathematik, Universität Duisburg 74, 1986.

[5] C. Bessenrodt – H. H. Brungs – G. Törner, *Right chain rings,* Pt 2b, Schriftenreihe des Fachbereiches Mathematik, Universität Duisburg 74, 1986.

[6] H. H. Brungs – G. Törner, *Chain rings and prime ideals,* Arch. Math. (Basel) **27** (1976), no. 3, 253–260.

[7] N. I. Dubrovnin, *Chain domains,* Moscow Univ. Math. Bull **35** (1980), pp. 51–54.

[8] L. Fuchs – L. Salce, *Uniserial modules over valuation rings,* J. Algebra **85** (1983), no. 1, pp. 14–31

[9] L. Fuchs – L. Salce, *Modules over non-Noetherian domains,* Mathematical Surveys and Monographs 84, American Mathematical Society, Providence, R.I., 2001.

[10] P. Gabriel, *Des categories abeliennes,* Bull. Soc. Math. France **90** (1962), pp. 323–448.

[11] K. R. Goodearl, *Ring theory*, Pure and Applied Mathematics 33, Marcel Dekker, New York and Basel, 1976.

[12] L. Salce – P. Zanardo, Finitely generated modules over valuation rings. Comm. Algebra **12** (1984), no. 15-16, pp. 1795–1812.

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