

The thermistor problem with Robin boundary condition

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ABSTRACT – We study the thermistor problem with Robin boundary condition for the temperature. A theorem of existence is proved using the compensated compactness method. For the one-dimensional case a result of non-existence and non-uniqueness is also given.

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1. Introduction

The name “thermistor” is used to indicate a three-dimensional body made up of a mixture of semiconducting materials for which the electrical conductivity depends sharply on the temperature [6]. We represent the body of the thermistor by Ω , an open and bounded subset of \mathbf{R}^3 . The regular boundary Γ of Ω consists of two disjoint surfaces Γ_1, Γ_2 , the electrodes of the device to which a fixed difference V of electric potential is applied.

If \mathbf{J} denotes the density of the electric current, $\mathbf{E} = -\nabla\varphi$ the electric field and φ the electric potential, we have by Ohm’s law

$$(1.1) \quad \mathbf{J} = -\sigma(u)\nabla\varphi,$$

where u is the temperature. Moreover, if \mathbf{q} is the heat flow density, the law of Fourier gives

$$(1.2) \quad \mathbf{q} = -\kappa\nabla u,$$

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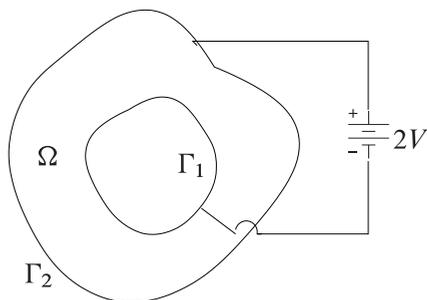


Figure 1

where κ is the thermal conductivity assumed here to be constant. Conservation of electricity and energy give

$$(1.3) \quad \nabla \cdot \mathbf{J} = 0,$$

$$(1.4) \quad \nabla \cdot \mathbf{q} = \mathbf{E} \cdot \mathbf{J},$$

where $\mathbf{E} \cdot \mathbf{J}$ represents the Joule heating. To determine u and φ in Ω we need to add to (1.3) and (1.4) boundary conditions for u and φ on Γ . If the thermistor is connected to a voltage generator, as we assume, the boundary condition for φ is

$$\varphi = \begin{cases} -\frac{V}{2} & \text{on } \Gamma_1, \\ \frac{V}{2} & \text{on } \Gamma_2. \end{cases}$$

In most of the papers on the thermistor problem (see e.g. [2], [5], [1], [4] and [8]) the boundary condition for the temperature is supposed to be of the Dirichlet type. Here we assume, as in [3] and [12], the physically more realistic Robin boundary condition

$$\kappa \frac{du}{dn} = h(g - u) \quad \text{on } \Gamma,$$

where g is a function assigned on Γ , which is related to the temperature of the medium surrounding the thermistor, and h is a positive constant. Thus, for the determination of $u(x)$ and $\varphi(x)$, $x = (x_1, x_2, x_3) \in \Omega$ we have, under stationary condition, the following problem (P)

$$(1.5) \quad \nabla \cdot (\sigma(u) \nabla \varphi) = 0 \quad \text{in } \Omega,$$

$$\varphi = \begin{cases} -\frac{V}{2} & \text{on } \Gamma_1, \\ \frac{V}{2} & \text{on } \Gamma_2, \end{cases}$$

$$(1.6) \quad -\nabla \cdot (\kappa \nabla u) = \sigma(u) |\nabla \varphi|^2 \quad \text{in } \Omega,$$

$$\kappa \frac{du}{dn} = h(g - u) \quad \text{on } \Gamma,$$

where (1.5) is obtained inserting (1.1) into (1.3), and (1.6) follows by (1.2) and (1.4). We assume hereafter that, in a suitable system of units, $\kappa = 1$ and $h = 1$ and suppose $g \in C^{0,\alpha}(\Gamma)$, $\sigma(t) \in C^1([m, \infty))$ where $m = \inf_{\Gamma} g > 0$. Moreover, we assume

$$\begin{aligned} \sigma_1 &\geq \sigma(t) > 0 && \text{for all } t \geq m, \\ t\sigma(t) &\geq \sigma_0 > 0 && \text{for all } t \geq m. \end{aligned}$$

The main difficulty in problem (P) lies in the quadratic growth of the gradient in the right hand side of equation (1.6) and in the degenerate character of equation (1.5) in view of (1.7).

In Section 2 we recall the a priori estimates, based on the maximum principle, which hold for the stationary solution of the heat equation with Robin's boundary conditions.

Section 3 deals with a sequence of regularising higher order approximating problem (P_ϵ) . We prove that each (P_ϵ) has at least one solution. The limit for $\epsilon \rightarrow 0$ is obtained in Section 4.

In Section 5 we examine a one-dimensional counter-example to problem (P) . We prove that if $\sigma(u) = e^u$ and therefore the condition (1.7) is not satisfied, there exists a critical value of the voltage V^* such that the problem (P) has two solutions if $0 < V < V^*$ and no solution if $V > V^*$.

Let φ_0 be the solution of the Dirichlet problem

$$\Delta \varphi_0 = \begin{cases} 0 & \text{in } \Omega, \\ -\frac{V}{2} & \text{on } \Gamma_1, \\ \frac{V}{2} & \text{on } \Gamma_2, \end{cases}$$

and define

$$\psi = \varphi - \varphi_0.$$

We take as weak formulation (PW) of problem (P) the following

$$(1.7a) \quad \psi \in H_0^1(\Omega),$$

$$(1.7b) \quad \int_{\Omega} \sigma(u) \nabla \psi \cdot \nabla \chi dx = - \int_{\Omega} \sigma(u) \nabla \varphi_0 \cdot \nabla \chi dx, \quad \text{for all } \chi \in H_0^1(\Omega),$$

$$(1.8a) \quad u \in H^1(\Omega),$$

$$(1.8b) \quad \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Gamma} uv d\Gamma \\ = \int_{\Omega} \sigma(u) |\nabla(\psi + \varphi_0)|^2 v dx + \int_{\Gamma} gv d\Gamma, \\ \text{for all } v \in H^1(\Omega) \cap L^\infty(\Omega).$$

It is easily seen that if $(\psi(x), u(x))$ is a regular solution of problem (PW) it solves problem (P). We assume hereafter that Γ_1 and Γ_2 are so regular that the following inequality holds.

$$(1.9) \quad \int_{\Omega} u^2 dx \leq C \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} u^2 d\Gamma \right), \quad \text{for all } u \in H^1(\Omega).$$

In particular, if $\Gamma_1, \Gamma_2 \in C^1$ then (1.9) is true (see e.g. [9]).

2. A priori bound based on the maximum principle

In the next two Lemmas we give a minimum principle for the stationary solution of the heat conduction problem with Robin boundary condition, referring to the book [10] for more details. The physical meaning is simple: if the body is heated by a positive internal source (like the electrical Joule heating) the temperature distribution cannot drop below the minimum value of the temperature of the surrounding medium evaluated on the surface of the body. The first Lemma supposes a regular solution, the second one gives an analogous result for a H^1 solution.

LEMMA 2.1. *Let $u(x) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be non-constant in $\bar{\Omega}$. Suppose $f(x) \in C^0(\Omega)$, $g(x) \in C^0(\Gamma)$ and*

$$(2.1) \quad -\Delta u = f \geq 0 \quad \text{in } \Omega,$$

$$(2.2) \quad \frac{du}{dn} = g - u \quad \text{on } \Gamma.$$

Then

$$(2.3) \quad u(x) \geq m \quad \text{for all } x \in \bar{\Omega}, \quad \text{where } m = \min_{x \in \Gamma} g(x).$$

PROOF. By (2.1) we have, for all $x \in \bar{\Omega}$,

$$(2.4) \quad u(x) \geq \min_{x \in \Gamma} u(x)$$

and let

$$\min_{x \in \Gamma} u(x) = u(x_0), \quad x_0 \in \Gamma.$$

Suppose, by contradiction, that there exists $x^* \in \bar{\Omega}$ such that

$$(2.5) \quad u(x^*) < m.$$

By (2.4) we have $u(x^*) \geq u(x_0)$. Hence $m - u(x_0) > 0$, thus, by (2.2),

$$(2.6) \quad \frac{du}{dn}(x_0) = g(x_0) - u(x_0) \geq m - u(x_0) > 0.$$

But x_0 is a minimum point of $u(x)$ in $\bar{\Omega}$ and therefore, by the maximum principle in Hopf form, we have $\frac{du}{dn}(x_0) < 0$ and (2.3) follows. \square

Next we give a “ H^1 -version” of the previous lemma. We follow the ideas on weak maximum principle of the book [7] (page 35), to which we refer in particular for the definition of inequality in the sense of H^1 . The weak formulation of problem (2.1), (2.2) is the following

$$(2.7a) \quad u \in H^1(\Omega),$$

$$(2.7b) \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Gamma} uv \, d\Gamma = \int_{\Omega} fv \, dx + \int_{\Gamma} gv \, d\Gamma, \quad \text{for all } v \in H^1(\Omega).$$

We note that the equation (2.7a) is invariant under the transformation $U = u + m$, $G = g - m$. Thus we can assume, without loss of generality,

$$(2.8) \quad m = \inf_{\Gamma} g = 0.$$

We have

LEMMA 2.2. *Let*

$$(2.9) \quad f \in L^2(\Omega), \quad f \geq 0, \quad g \in L^2(\Gamma).$$

Suppose u to satisfies (2.7a) and (2.8), then

$$(2.10) \quad u \geq 0 \quad \text{in } \Omega.$$

PROOF. We apply the truncation method. Let $\zeta(x) = \inf_{\Omega}(u(x), 0)$ and

$$\Omega_- = \{x \in \Omega, u(x) \leq 0, \text{ in the sense of } H^1(\Omega)\}.$$

We have $\zeta(x) \in H^1(\Omega)$ and $\zeta(x) \leq 0$. If we prove that $\zeta(x) = 0$ in Ω then (2.10) follows. To this end we choose $v = \zeta$ into (2.7a). We obtain, by (2.8) and (2.9),

$$\begin{aligned} (2.11) \quad 0 &\leq \int_{\Omega_-} |\nabla \zeta|^2 dx + \int_{\Gamma_-} \zeta^2 d\Gamma \\ &= \int_{\Omega_-} f \zeta dx + \int_{\Gamma_-} g \zeta d\Gamma \\ &\leq 0, \end{aligned}$$

where Γ_- is the boundary of Ω_- . By (2.11) we have

$$\int_{\Gamma_-} \zeta^2 d\Gamma = 0, \quad \int_{\Omega_-} |\nabla \zeta|^2 dx = 0.$$

This implies $\zeta(x) \in H_0^1(\Omega_-)$ and $\nabla \zeta = 0$ a.e. in Ω . Therefore $\zeta(x) = 0$. \square

In the next Lemma we give a fourth order version of Lemma 2.1. This will be useful in the sequel.

LEMMA 2.3. *Let $u(x) \in C^4(\Omega) \cap C^3(\bar{\Omega})$ be a solution of the problem*

$$\begin{aligned} \Delta \Delta u - \Delta u &= f \quad \text{in } \Omega \\ \Delta u &= 0 \quad \text{on } \Gamma \\ -\frac{d\Delta u}{dn} + \frac{du}{dn} &= g - u \quad \text{on } \Gamma \end{aligned}$$

with $f(x) \in C^0(\Omega)$, $f(x) \geq 0$ and $g \in C^0(\Gamma)$. Then

$$(2.12) \quad u(x) \geq m = \inf_{\Gamma} g.$$

PROOF. Define $\zeta = \Delta u$. We have

$$(2.13) \quad \Delta \zeta - \zeta = f \quad \text{in } \Omega,$$

$$(2.14) \quad \zeta = 0 \quad \text{on } \Gamma$$

$$(2.15) \quad \Delta u = \zeta \quad \text{in } \Omega,$$

$$(2.16) \quad -\frac{du}{dn} = u - g - \frac{d\zeta}{dn} \quad \text{on } \Gamma.$$

By (2.13) and (2.14) we have

$$(2.17) \quad \zeta < 0 \quad \text{in } \Omega.$$

Moreover, by the maximum principle in Hopf form and since all the points of Γ are points of absolute maximum for $\zeta(x)$, we have

$$(2.18) \quad \frac{d\zeta}{dn} > 0 \quad \text{on } \Gamma.$$

By (2.15) and (2.17) we get

$$u(x) \geq \min_{\Gamma} u(x).$$

Choose $x_0 \in \Gamma$ such that $\min_{\Gamma} u(x) = u(x_0)$. Again by Hopf maximum principle we have

$$(2.19) \quad \frac{du}{dn}(x_0) < 0.$$

Assume, by contradiction, that there exists $x^* \in \bar{\Omega}$ such that $u(x^*) < m$. We have $u(x_0) \leq u(x^*) < m$. Hence $u(x_0) - m < 0$. By (2.16), (2.18), and (2.19) we obtain

$$0 < \frac{d\zeta}{dn}(x_0) - \frac{du}{dn}(x_0) = u(x_0) - g(x_0) \leq m - g(x_0) \leq 0.$$

Thus (2.12) holds. □

3. Existence of a weak solution for the approximating problem

The existence of a solution to problem (PW) is proved considering the following family (P_ϵ) of regularising problems

$$\begin{aligned} \nabla \cdot (\sigma(u_\epsilon) \nabla \varphi_\epsilon) &= 0 \quad \text{in } \Omega \\ \varphi_\epsilon &= \begin{cases} -\frac{V}{2} & \text{on } \Gamma_1, \\ \frac{V}{2} & \text{on } \Gamma_2, \end{cases} \\ \epsilon \Delta \Delta u_\epsilon - \Delta u_\epsilon &= \sigma(u_\epsilon) |\nabla \varphi_\epsilon|^2 \quad \text{in } \Omega, \\ \Delta u_\epsilon &= 0 \quad \text{on } \Gamma, \\ -\epsilon \frac{d\Delta u_\epsilon}{dn} + \frac{du_\epsilon}{dn} + u_\epsilon &= g \quad \text{on } \Gamma. \end{aligned}$$

To give a weak formulation to problem (P_ϵ) we consider in $C^\infty(\bar{\Omega})$ the scalar product

$$(3.1) \quad ((u, v)) = \int_{\Omega} \Delta u \Delta v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Gamma} uv \, d\Gamma$$

and define \mathcal{H} as the completion of $C^\infty(\bar{\Omega})$ with respect to (3.1). In weak form problem (P_ϵ) can be restated as the problem (PW_ϵ) , where $\varphi_\epsilon = \psi_\epsilon + \varphi_0$, given by

$$(3.2a) \quad \psi_\epsilon \in H_0^1,$$

$$(3.2b) \quad \int_{\Omega} \sigma(u_\epsilon) \nabla \psi_\epsilon \cdot \nabla \chi \, dx = - \int_{\Omega} \sigma(u_\epsilon) \nabla \varphi_0 \cdot \nabla \chi \, dx, \quad \text{for all } \chi \in H_0^1,$$

$$(3.3a) \quad u_\epsilon \in \mathcal{H},$$

$$(3.3b) \quad \begin{aligned} & \in \int_{\Omega} \Delta u_\epsilon \Delta v \, dx + \int_{\Omega} \nabla u_\epsilon \cdot \nabla v \, dx + \int_{\Gamma} u_\epsilon v \, d\Gamma \\ & = \int_{\Omega} \sigma(u_\epsilon) |\nabla \varphi_\epsilon|^2 v \, dx + \int_{\Gamma} g v \, d\Gamma, \quad \text{for all } v \in \mathcal{H}. \end{aligned}$$

LEMMA 3.1. *If $(\psi_\epsilon, u_\epsilon)$ is a solution of the problem (PW_ϵ) we have*

$$(3.4) \quad \|\psi_\epsilon\|_{L^\infty(\Omega)} \leq C_0,$$

$$(3.5) \quad \int_{\Omega} |\nabla u_\epsilon|^2 \, dx \leq C_1,$$

$$(3.6) \quad \int_{\Omega} |\Delta u_\epsilon|^2 \, dx \leq C_2,$$

$$(3.7) \quad \int_{\Omega} |\nabla \psi_\epsilon|^2 \, dx \leq C_3,$$

$$(3.8) \quad \int_{\Gamma} |u_\epsilon|^2 \, dx \leq C_4,$$

where C_j are different constants depending only on the data.

PROOF. Equation (3.2) can be restated as

$$\varphi_\epsilon - \varphi_0 \in H_0^1(\Omega), \quad \int_{\Omega} \sigma(u_\epsilon) \nabla \varphi_\epsilon \cdot \nabla \chi \, dx = 0, \quad \text{for all } \chi \in H_0^1(\Omega).$$

On the other hand, $u_\epsilon \in H^2(\Omega)$ and, as we are in dimension 3, we have $u_\epsilon \in C^{0,\alpha}(\bar{\Omega})$, thus $\inf_\Omega \sigma(u_\epsilon(x)) > 0$ by (1.7). By the Stampacchia maximum principle (see [7]) we have $\|\varphi_\epsilon\| \leq \sup_\Gamma |\varphi_0|$. Moreover, by the classical maximum principle

$$\|\varphi_0\|_{L^\infty(\Omega)} \leq \frac{V}{2}.$$

Thus (3.4) follows. With the choice $\chi = \psi_\epsilon$ in (3.2) we obtain

$$\int_\Omega \sigma(u_\epsilon) |\nabla \psi_\epsilon|^2 dx \leq A.$$

Moreover, setting $\chi = \psi_\epsilon u_\epsilon$ in (3.2) we obtain, by (1.7),

$$(3.9) \quad \int_\Omega |\nabla \psi_\epsilon|^2 dx \leq A \int_\Omega u_\epsilon \sigma(u_\epsilon) |\nabla \psi_\epsilon|^2 dx \leq B \left[1 + \left(\int_\Omega |\nabla u_\epsilon|^2 dx \right)^{1/2} \right].$$

With $u_\epsilon = v$ in (3.3) we have, by (1.9) and (3.9),

$$\begin{aligned} & \epsilon \int_\Omega |\Delta u_\epsilon|^2 dx + \int_\Omega |\nabla u_\epsilon|^2 dx + \int_\Gamma u_\epsilon^2 d\Gamma \\ &= \int_\Omega \sigma(u_\epsilon) |\nabla \psi_\epsilon + \nabla \varphi_0|^2 u dx \\ &\leq C \left[1 + \left(\int_\Omega |\nabla u_\epsilon|^2 dx \right)^{1/2} + \left(\int_\Omega u_\epsilon^2 dx \right)^{1/2} \right] \\ &\leq C^* \left[1 + \left(\int_\Omega |\nabla u_\epsilon|^2 dx \right)^{1/2} + \left(\int_\Gamma u_\epsilon^2 d\Gamma \right)^{1/2} \right]. \end{aligned}$$

Hence (3.6), (3.7), and (3.8) follow. □

We use the following slightly modified version of the Schaefer fixed point principle [2] and the ‘‘a priori’’ estimates of Lemma 3.1 to prove that problem (P_ϵ) has at least one solution.

THEOREM 3.2. *Let \mathcal{B} be a Banach space and $T(w, \lambda)$ a continuous and compact map from $\mathcal{B} \times [0, 1]$ in \mathcal{B} such that*

$$\bar{u} \in \mathcal{B}, \quad T(w, 0) = \bar{u}, \quad \text{for all } w \in \mathcal{B}.$$

If there exists a constant M such that

$$\|u\|_{\mathcal{B}} \leq M$$

for all $(u, \lambda) \in \mathcal{B} \times [0, 1]$ such that $u = T(u, \lambda)$, then the equation $u = T(u, 1)$ has at least one solution.

Let $\tilde{\sigma}(t) \in C^1(\mathbf{R}^1)$ be an extension of $\sigma(t)$ such that

$$(3.10) \quad \tilde{\sigma}(t) = \sigma(t), \quad \text{for all } t \geq m > 0,$$

$$(3.11) \quad \sigma_1 \geq \tilde{\sigma}(t) > 0, \quad \text{for all } t \in \mathbf{R}^1,$$

$$t\tilde{\sigma}(t) \geq \sigma_0 > 0, \quad \text{for all } t \in \mathbf{R}^1.$$

LEMMA 3.3. *For every $\epsilon > 0$ there exists at least one solution of the problem*

$$(3.12a) \quad \psi_\epsilon \in H_0^1,$$

$$(3.12b) \quad \int_{\Omega} \tilde{\sigma}(u_\epsilon) \nabla \psi_\epsilon \cdot \nabla \chi \, dx = - \int_{\Omega} \tilde{\sigma}(u_\epsilon) \nabla \varphi_0 \cdot \nabla \chi \, dx, \quad \text{for all } \chi \in H_0^1,$$

$$(3.13a) \quad u_\epsilon \in \mathcal{H},$$

$$(3.13b) \quad \begin{aligned} & \epsilon \int_{\Omega} \Delta u_\epsilon \Delta v \, dx + \int_{\Omega} \nabla u_\epsilon \cdot \nabla v \, dx + \int_{\Gamma} u_\epsilon v \, d\Gamma \\ & = \int_{\Omega} \tilde{\sigma}(u_\epsilon) |\nabla(\psi_\epsilon + \varphi_0)|^2 v \, dx + \int_{\Gamma} g v \, d\Gamma, \quad \text{for all } v \in \mathcal{H}. \end{aligned}$$

PROOF. Let $\mathcal{B} = H^1(\Omega) \cap L^\infty(\Omega)$ and define the map $u = T(w, \lambda)$ from $\mathcal{B} \times [0, 1]$ in \mathcal{B} via the linear problem

$$(3.14a) \quad \psi_\epsilon \in H_0^1,$$

$$(3.14b) \quad \int_{\Omega} \tilde{\sigma}(\lambda w) \nabla \psi_\epsilon \cdot \nabla \chi \, dx = - \int_{\Omega} \tilde{\sigma}(\lambda w) \nabla \varphi_0 \cdot \nabla \chi \, dx, \quad \text{for all } \chi \in H_0^1,$$

$$(3.15a) \quad u_\epsilon \in \mathcal{H},$$

$$(3.15b) \quad \epsilon \int_{\Omega} \Delta u_\epsilon \Delta v \, dx + \int_{\Omega} \nabla u_\epsilon \cdot \nabla v \, dx + \int_{\Gamma} u_\epsilon v \, d\Gamma$$

$$(3.15c) \quad = \int_{\Omega} \tilde{\sigma}(\lambda w) |\nabla(\psi_\epsilon + \varphi_0)|^2 v \, dx + \int_{\Gamma} g v \, d\Gamma, \quad \text{for all } v \in \mathcal{H}.$$

Since $w \in L^\infty(\Omega)$, we have by (3.11)

$$(3.16) \quad \tilde{\sigma}(\lambda w) \geq \sigma_w > 0$$

with σ_w a constant. Therefore, for every $w \in \mathcal{B}$ problem (3.14) has one and only one solution by the Lax–Milgram lemma. On the other hand, the left hand side of (3.15) defines a bilinear form which is continuous and coercive in \mathcal{H} , whereas the right hand side of the same equation is a linear continuous functional in \mathcal{H} . Therefore (3.15) is also uniquely solvable and $T(w, \lambda)$ is well-defined in $\mathcal{B} \times [0, 1]$.

Let $\lambda = 0$ and define \bar{u} as solution of problem (3.14), (3.15) when $\lambda = 0$. We have $T(w, 0) = \bar{u}$ for all $w \in \mathcal{B}$. Moreover, $T(w, \lambda)$ is continuous. For, let $\epsilon > 0$ be fixed, and $(w_n, \lambda_n) \rightarrow (w^*, \lambda^*)$ in $\mathcal{B} \times [0, 1]$ as $n \rightarrow \infty$. We claim that $u_n \rightarrow u^*$ in $H^1 \cap L^\infty$, where $u_n = T(w_n, \lambda_n)$ and (ψ^*, u^*) is the solution of the problem

$$\begin{aligned} \psi^* &\in H_0^1, \\ \int_{\Omega} \tilde{\sigma}(\lambda^* w^*) \nabla \psi^* \cdot \nabla \chi \, dx &= - \int_{\Omega} \tilde{\sigma}(\lambda^* w^*) \nabla \varphi_0 \cdot \nabla \chi \, dx, \quad \text{for all } \chi \in H_0^1, \\ u^* &\in \mathcal{H}, \\ \epsilon \int_{\Omega} \Delta u^* \Delta v \, dx + \int_{\Omega} \nabla u^* \cdot \nabla v \, dx + \int_{\Gamma} u^* v \, d\Gamma \\ &= \int_{\Omega} \tilde{\sigma}(\lambda^* w^*) |\nabla(\psi^* + \varphi_0)|^2 v \, dx + \int_{\Gamma} g v \, d\Gamma, \quad \text{for all } v \in \mathcal{H}. \end{aligned}$$

Since $\tilde{\sigma}(\lambda_n w_n)$ converges to $\tilde{\sigma}(\lambda^* w^*)$ in $L^\infty(\Omega)$ we easily verify that $\psi_n \rightarrow \psi^*$ in $H_0^1(\Omega)$. Hence we have

$$\tilde{\sigma}(\lambda_n w_n) |\nabla(\psi_n + \varphi_0)|^2 \longrightarrow \tilde{\sigma}(\lambda^* w^*) |\nabla(\psi^* + \varphi_0)|^2 \text{ in } L^1(\Omega).$$

Therefore, $u_n \rightarrow u^*$ in \mathcal{H} and, as a consequence, in $H^1(\Omega) \cap L^\infty(\Omega)$. This proves the continuity of $T(\lambda, w)$, which is also compact by the estimates (3.5), (3.6), and (3.7) since bounded set of \mathcal{H} are compact in $H^1(\Omega) \cap L^\infty(\Omega)$, in dimension 3, by the Rellich–Kondrachov theorem.

Finally, repeating with minor changes the proof of Lemma 3.1 and recalling (3.5), (3.6), (3.7), and (3.8), we can prove that all solutions of the equation $u = T(\lambda, u)$ are bounded in the \mathcal{B} -norm by a constant not depending on λ . We conclude that problem (3.12), (3.13) has at least one solution (ψ, u) for every $\epsilon > 0$ by Theorem 3.2. □

The solution of the problem (3.12), (3.13) can be regularised. Since $\tilde{\sigma}(u) \in L^\infty(\Omega)$ we have $\tilde{\sigma}(u)|\nabla\varphi|^2 \in L^1(\Omega)$. Hence $u \in H^{4,1}(\Omega)$ and $u \in C^{0,\alpha}(\bar{\Omega})$. This, in turn, implies $\nabla\varphi \in C^{1,\alpha}(\bar{\Omega})$ by the Schauder estimates. Therefore $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$. Recalling the Lemma 2.3 we conclude that $u_\epsilon(x) \geq m$ and therefore $\tilde{\sigma}(u_\epsilon) = \sigma(u_\epsilon)$. Hence problem (PW_ϵ) has at least one solution for every $\epsilon > 0$.

4. Existence of a solution for problem (PW)

THEOREM 4.1. *If $g \in C^{0,\alpha}(\Gamma)$ and $\sigma(t)$ satisfies*

$$\sigma_1 \geq \sigma(t) > 0 \quad \text{for all } t \geq m = \inf_{\Gamma} g > 0,$$

$$t\sigma(t) \geq \sigma_0 > 0 \quad \text{for all } t \geq m = \inf_{\Gamma} g > 0,$$

then there exists at least one solution to problem (PW) .

PROOF. Let $(\psi_\epsilon, u_\epsilon)$ be the solution of the problem (PW_ϵ) . By (3.5), (3.6), (3.7) and (3.8) we can extract from $(\psi_\epsilon, u_\epsilon)$ a subsequence, not relabelled, such that as $\epsilon \rightarrow 0$,

$$(4.1) \quad u_\epsilon \rightharpoonup u^* \text{ weakly in } H^1(\Omega), \quad u_\epsilon \rightarrow u^* \text{ strongly in } L^2(\Omega) \text{ and a.e.,}$$

$$(4.2) \quad \begin{aligned} u_\epsilon|_{\Gamma} &\rightharpoonup u^*|_{\Gamma} \text{ weakly in } L^2(\Gamma), \quad \sigma(u_\epsilon) \rightarrow \sigma(u^*) \\ &\text{strongly in } L^p(\Omega), \quad 2 \leq p < \infty, \end{aligned}$$

$$(4.3) \quad \psi_\epsilon \rightharpoonup \psi^* \text{ weakly in } H_0^1(\Omega).$$

By (4.2) and (4.3), we have (1.7) by (3.2), i.e.

$$(4.4) \quad \int_{\Omega} \sigma(u^*) \nabla \psi^* \cdot \nabla \chi \, dx = - \int_{\Omega} \sigma(u^*) \nabla \varphi_0 \cdot \nabla \chi \, dx, \quad \text{for all } \chi \in H_0^1(\Omega).$$

By (3.6) we have

$$\epsilon \int_{\Omega} \Delta u_\epsilon \Delta v \, dx \leq \epsilon \|\Delta u_\epsilon\|_{L^2} \|\Delta v\|_{L^2} \leq \epsilon^{1/2} C_2 \|\Delta v\|_{L^2} \rightarrow 0.$$

Moreover, by (4.1)

$$\int_{\Omega} \nabla u_\epsilon \cdot \nabla v \, dx \rightarrow \int_{\Omega} \nabla u^* \cdot \nabla v \, dx, \quad \text{for all } v \in \mathcal{H}$$

and, by (4.2)

$$\int_{\Gamma} u_\epsilon v \, d\Gamma \rightarrow \int_{\Gamma} u^* v \, d\Gamma, \quad \text{for all } v \in \mathcal{H}.$$

It remains to pass to the limit in the first term in the R.H.S. of equation (3.3). This requires more attention. Setting $\chi = \psi^*$ in equation (4.4) we have

$$(4.5) \quad \int_{\Omega} \sigma(u^*) |\nabla \psi^*|^2 dx = - \int_{\Omega} \sigma(u^*) \nabla \varphi_0 \cdot \nabla \psi^* dx.$$

With $\chi = \psi^*$ in (3.2) we have, by (4.5),

$$(4.6) \quad \begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \sigma(u_{\epsilon}) |\nabla \psi_{\epsilon}|^2 dx &= - \lim_{\epsilon \rightarrow 0} \int_{\Omega} \sigma(u_{\epsilon}) \nabla \varphi_0 \cdot \nabla \psi_{\epsilon} dx \\ &= - \int_{\Omega} \sigma(u^*) \nabla \psi_0 \cdot \nabla \psi^* dx \\ &= \int_{\Omega} \sigma(u^*) |\nabla \psi^*|^2 dx. \end{aligned}$$

Recalling that $\psi_{\epsilon} = \varphi_{\epsilon} - \varphi_0$ and $\psi^* = \varphi^* - \varphi_0$ we have, by (4.6),

$$(4.7) \quad \lim_{\epsilon \rightarrow 0} \int_{\Omega} \sigma(u_{\epsilon}) |\nabla \varphi_{\epsilon}|^2 dx = \int_{\Omega} \sigma(u^*) |\nabla \varphi^*|^2 dx.$$

Define the vector fields

$$\begin{aligned} \mathbf{f}^* &= (f_1^*, f_2^*, f_3^*), \quad \mathbf{f}_{\epsilon} = (f_{1\epsilon}, f_{2\epsilon}, f_{3\epsilon}), \\ f_j^* &= \sqrt{\sigma(u^*)} \partial \varphi^* / \partial x_j, \quad f_{j\epsilon} = \sqrt{\sigma(u_{\epsilon})} \partial \varphi_{\epsilon} / \partial x_j, \quad j = 1, 2, 3. \end{aligned}$$

We have

$$\|\mathbf{f}_{\epsilon}\|_{(L^2(\Omega))^3}^2 = \sum_{j=1}^3 \int_{\Omega} f_{j\epsilon}^2 dx = \int_{\Omega} \sigma(u_{\epsilon}) |\nabla \varphi_{\epsilon}|^2 dx$$

and, by (4.7)

$$(4.8) \quad \lim_{\epsilon \rightarrow 0} \|\mathbf{f}_{\epsilon}\|_{(L^2(\Omega))^3}^2 = \|\mathbf{f}^*\|_{(L^2(\Omega))^3}^2.$$

On the other hand,

$$(4.9) \quad \mathbf{f}_{\epsilon} \rightarrow \mathbf{f} \quad \text{weakly in } (L^2(\Omega))^3.$$

For, let $\mathbf{g} \in (L^2(\Omega))^3$. By (4.2) and (4.3) we have

$$(4.10) \quad \begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\Omega} \mathbf{f}_{\epsilon} \cdot \mathbf{g} dx &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} \sqrt{\sigma(u_{\epsilon})} \nabla \varphi_{\epsilon} \cdot \mathbf{g} dx \\ &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} \sqrt{\sigma(u^*)} \nabla \varphi_{\epsilon} \cdot \mathbf{g} dx \\ &\quad + \lim_{\epsilon \rightarrow 0} \int_{\Omega} [\sqrt{\sigma(u_{\epsilon})} - \sqrt{\sigma(u^*)}] \nabla \varphi_{\epsilon} \cdot \mathbf{g} dx \\ &= \int_{\Omega} \mathbf{f}^* \cdot \mathbf{g} dx. \end{aligned}$$

Hence, by (4.10) and (4.8), we have

$$(4.11) \quad \lim_{\epsilon \rightarrow 0} \|\mathbf{f}_\epsilon - \mathbf{f}^*\|_{(L^2(\Omega))^3} = 0.$$

Moreover,

$$(4.12) \quad \begin{aligned} \int_{\Omega} |\sigma(u_\epsilon)| \nabla \varphi_\epsilon|^2 - \sigma(u^*) |\nabla \varphi^*|^2| \, dx &= \left\| \sum_{i=1}^3 (f_{\epsilon i}^2 - f_i^{*2}) \right\|_{L^1(\Omega)} \\ &= \int_{\Omega} \left| \sum_{i=1}^3 (f_{\epsilon i} - f_i^*)(f_{\epsilon i} + f_i^*) \right| \, dx \\ &\leq \|\mathbf{f}_\epsilon - \mathbf{f}^*\|_{(L^2(\Omega))^3} \|\mathbf{f}_\epsilon + \mathbf{f}^*\|_{(L^2(\Omega))^3}. \end{aligned}$$

If $v \in \mathcal{H}$ we have, by (4.12) and (4.11),

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \left| \int_{\Omega} [\sigma(u_\epsilon) |\nabla \varphi_\epsilon|^2 - \sigma(u^*) |\nabla \varphi^*|^2] v \, dx \right| \\ &\leq \sup_{\Omega} |v| \lim_{\epsilon \rightarrow 0} \int_{\Omega} |\sigma(u_\epsilon) |\nabla \varphi_\epsilon|^2 - \sigma(u^*) |\nabla \varphi^*|^2| \, dx \\ &= 0. \end{aligned}$$

Hence (1.8) follows by (3.3) as $\epsilon \rightarrow 0$. □

REMARK 4.2. In the proof of Theorem 4.1 we note that $f_{j\epsilon}^2$ does not converges in $L^1(\Omega)$ to f_j^{2*} . Only the sum $\sum_{j=1}^3 f_{j\epsilon}^2$ converges toward $\sum_{j=1}^3 f_j^{2*}$. This situation is typical of the compensated compactness see [11].

5. A one-dimensional example of non-existence and non-uniqueness

To show that the assumption (1.7) is essential to the existence of a solution we examine in this section the one-dimensional counterpart of problem (P) with $\Omega = (-1, 1)$, $g = 0$ and $\sigma(u) = e^u$. We refer for this case also to the paper [8]. We obtain the two-point problem (ODE)

$$(5.1) \quad (e^u \varphi')' = 0 \quad \text{in } (-1, 1)$$

$$(5.2) \quad \varphi(-1) = -\frac{V}{2}, \quad \varphi(1) = \frac{V}{2}$$

$$(5.3) \quad u'' + e^u \varphi'^2 = 0 \quad \text{in } (-1, 1)$$

$$(5.4) \quad u'(-1) = u(-1)$$

$$(5.5) \quad -u'(1) = u(1).$$

Only the parameter V has been retained. It will be a bifurcation parameter.

LEMMA 5.1. *If $(\varphi(x), u(x))$ is a solution of problem (ODE) then*

$$(5.6) \quad u(x) = u(-x)$$

and

$$(5.7) \quad \varphi(x) = -\varphi(-x).$$

PROOF. By (5.1) we have

$$(5.8) \quad e^u \varphi' = \mu,$$

where the constant μ (certainly positive in view of (5.2)) represents physically the total electric current crossing the device. Using (5.3) and (5.8) we obtain

$$(5.9) \quad u'' + \mu^2 e^{-u} = 0$$

$$(5.10) \quad u'(-1) = (-1)$$

$$(5.11) \quad -u'(1) = u(1).$$

Given μ , the solution of the problem (5.9), (5.10), (5.11) cannot have more than one solution. For, let $u_1(x)$, $u_2(x)$ be two solutions and $w(x) = u_1(x) - u_2(x)$. By the mean value theorem we have $e^{-u_1} - e^{-u_2} = e^{-\xi} w$. Thus w is solution of the problem

$$w'' + \mu^2 e^{-\xi} w = 0, \quad w'(-1) = w(-1), \quad -w'(1) = w(1).$$

Multiplying this equation by w , integrating by parts over $[-1, 1]$ and taking into account the boundary conditions, we obtain

$$\int_{-1}^1 w'^2 dx + w(1)^2 + w(-1)^2 + \int_{-1}^1 e^{-\xi} w^2 dx = 0.$$

Therefore we have $w'(x) = 0$, $w(-1) = 0$. Thus $w(x) = 0$ in $[-1, 1]$. If we define $v(x) = u(-x)$ it is easy to verify that v is a solution of the problem (5.9), (5.10), (5.11). Hence, by the proven uniqueness, (5.6) follows. Moreover, (5.7) is a consequence of (5.1) and (5.2). \square

As a consequence of the Lemma 5.1 we can take the condition

$$(5.12) \quad u'(0) = 0$$

instead of (5.4). Multiplying the equation (5.9) by u' we obtain that $u'^2 - 2\mu^2 e^{-u}$ is constant in $[-1, 1]$ and in view of (5.12) we may only have

$$(5.13) \quad u'^2 - 2\mu^2 e^{-u} = -C^2, \quad C > 0.$$

Thus by (5.12) and (5.13) we have

$$(5.14) \quad u(0) = \log \frac{2\mu^2}{C^2}.$$

Moreover, for the concavity of $u(x)$ we have for $x \in (0, 1)$,

$$(5.15) \quad u'(x) = -\sqrt{2\mu^2 e^{-u} - C^2}.$$

Integrating (5.15) and taking into account (5.14) we arrive at

$$(5.16) \quad x = \frac{2}{C} \arctan \frac{\sqrt{2\mu^2 e^{-u} - C^2}}{C}, \quad 0 < x < 1.$$

Solving (5.16) with respect to u we find

$$(5.17) \quad u(x, \mu, C) = \ln 2\mu^2 - \ln \left[C^2 \left(\tan^2 \frac{Cx}{2} + 1 \right) \right], \quad u'(x, \mu, C) = -C \tan \left(\frac{Cx}{2} \right).$$

By (5.5) we get

$$(5.18) \quad C \tan \frac{C}{2} = \ln 2\mu^2 - \ln \left[C^2 \left(\tan^2 \frac{C}{2} + 1 \right) \right].$$

Solving (5.18) with respect to μ we have

$$(5.19) \quad \mu = \frac{C}{\sqrt{2}} \sqrt{e^{C \tan \frac{C}{2}} \left(\tan^2 \frac{C}{2} + 1 \right)}$$

and substituting in (5.17)

$$(5.20) \quad u(x, C) = \ln \left[e^{C \tan \frac{C}{2}} \frac{\cos^2 \left(\frac{Cx}{2} \right)}{\cos^2 \left(\frac{C}{2} \right)} \right].$$

Substituting (5.19) and (5.20) in (5.8) we obtain

$$(5.21) \quad \varphi'(x, C) = \frac{C \left| \cos \frac{C}{2} \right|}{\cos^2 \frac{Cx}{2}} \frac{1}{\sqrt{2e^{C \tan \frac{C}{2}}}}.$$

Integrating (5.21) with the initial condition $\varphi(0) = 0$ ($\varphi(x)$ is odd) we obtain

$$\varphi(x, C) = \sqrt{2e^{-C \tan \frac{C}{2}}} \left| \cos \frac{C}{2} \right| \tan \frac{Cx}{2}.$$

The constant C , still undetermined, is found with condition $\varphi(1, C) = \frac{V}{2}$. This gives the equation in the unknown C

$$(5.22) \quad g(C) = \frac{V}{2}, \quad \text{where } g(C) = \sqrt{2e^{-C \tan \frac{C}{2}}} \left| \cos \frac{C}{2} \right| \tan \frac{C}{2}.$$

See Figure 2 for the graph of $g(C)$ in the interval $[0, \pi]$.

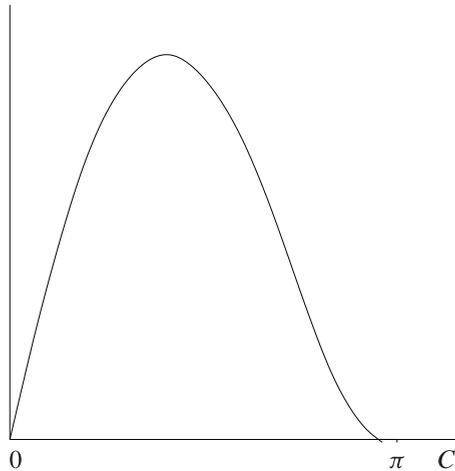


Figure 2

In fact, the only solutions of the equation (5.22) of interest to the problem (ODE) are those in the interval $[0, \pi]$ since all the others solutions give singularities in $\varphi(x)$ and are, therefore, to be excluded. Thus we have the following

LEMMA 5.2. *There exists a critical value $V^* > 0$ such that the problem (ODE) has two solutions if $0 < V < V^*$, one solution if $V = V^*$ and no solution when $V > V^*$.*

Moreover, we have, by (5.20),

$$\lim_{C \rightarrow 0^+} u(0, C) = 0, \quad \lim_{C \rightarrow \pi^-} u(0, C) = \infty.$$

This means that, in term of the parameter V , we obtain the following bifurcation diagram:

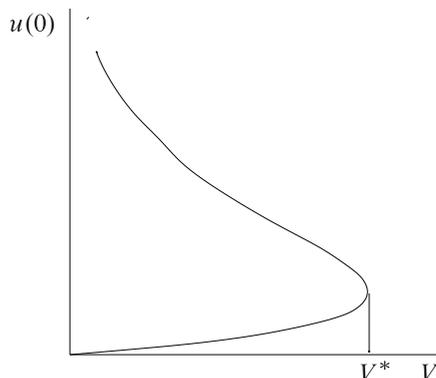


Figure 3

REMARK 5.3. Let us consider the case $0 < V < V^*$ and let $C_1(V) < C_2(V)$ be the two corresponding solutions of equation (5.22). We have

$$\lim_{V \rightarrow 0^+} C_1(V) = 0, \quad \lim_{V \rightarrow 0^+} C_2(V) = \pi.$$

On the other hand,

$$\lim_{C \rightarrow \pi^-} \mu(C) = \infty.$$

Recalling that μ represents the total current crossing the thermistor, the presence of a solution $(\varphi_2(x, V), u_2(x, V))$ of (POD) for which

$$\lim_{V \rightarrow 0^+} u_2(0, V) = \infty$$

is not surprising. It reflects the starting assumption that the device is connected to a voltage generator which can furnish an arbitrarily large amount of electric current.

REMARK 5.4. When $\sigma(u) = e^u$ it would be interesting to prove a bifurcation diagram as in Figure 2 in the three-dimensional case. Or at least to show that there exists a critical V^* such that no solution exists for $V > V^*$.

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