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Characterizations of hypercyclically embedded subgroups of finite groups

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Abstract – A normal subgroup H of a finite group G is said to be *hypercyclically embedded in* G if every chief factor of G below H is cyclic. Our main goal here is to give new characterizations of hypercyclically embedded subgroups. In particular, we prove that a normal subgroup E of a finite group G is hypercyclically embedded in G if and only if for every different primes p and q and every p-element $a \in (G' \cap F^*(E))E'$, p'-element $b \in G$ and q-element $c \in G'$ we have $[a, b^{p-1}] = 1 = [a^{q-1}, c]$. Some known results are generalized.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p and q are always supposed to be primes and $\pi(G)$ denotes the set of all primes dividing $|G|$.

A normal subgroup A of G is said to be *hypercentrally* (respectivelly *hypercyclically*) *embedded in* G if either $A = 1$ or $A \neq 1$ and every chief factor of G below A is central (respectivelly cyclic) [18, p. 217].

The hypercentrally and hypercyclically embedded subgroups essentially influence on the structure of a group and they are useful for descriptions of some important classes of groups. For example, if all cyclic subgroups of G of prime order or order 4 are hypercentrally embedded in G, then G is nilpotent (N. Ito).

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If all these subgroups are hypercyclically embedded in G, then G is supersoluble (Huppert, Doerk). If all subgroups of G of prime order are normal in G, then G is soluble (Ito and Gaschütz [14, Chapter IV, 5.7]). A group G is quasinilpotent if and only if it has a normal hypercentrally embedded subgroup E such that G/E is semisimple [15, Chapter X, 13.6]. A group G is quasisupersoluble (i.e. for every non-cyclic chief factor H/K of G, every automorphism of H/K induced by an element of G is inner) if and only if it has a normal hypercyclically embedded subgroup E such that G/E is semisimple (Guo and Skiba [10]).

The study of hypercentrally embedded and hypercyclically embedded subgroups begins with the paper of Baer [2] and they have close relation to quasinormal subgroups. For instance, it was proved in [17] that if $A_G = 1$ and A is a quasinormal subgroup of G, then A is hypercentrally embedded in G; if $A_G = 1$ and A is a modular element (in the sense of Kurosh [18, p. 43]) of the subgroup lattice of G, then A is hypercyclically embedded in G [18, 5.2.5]). Some other results related to the hypercyclically embedded subgroups are discussed in the book [24] (see also the recent papers [20, 21, 22]).

In this paper we prove the following two results in this line research.

Theorem 1.1. *Let E be a normal subgroup of G. Then the following conditions are equivalent*:

(i) *E is hypercyclically embedded in G*;

(ii) *for every different primes p and q and every p-element $a \in (G' \cap F^*(E))E'$, p'-element $b \in G$ and q-element $c \in G'$ we have*

$$(*) \qquad [a, b^{p-1}] = 1 = [a^{q-1}, c];$$

(iii) *for every different primes p and q, equalities $(*)$ hold for every p-element $a \in (G' \cap F^*(E))E'$ of prime order or order 4 (if $p = 2$ and the Sylow 2-subgroups of E are non-abelian) and every p'-element $b \in G$ and q-element $c \in G'$.*

A chief factor H/K of G is called *Frattini* if $H/K \le \Phi(G/K)$.

Theorem 1.2. *Let E be a normal subgroup of G. Then every non-Frattini chief factor of G below E is cyclic if and only if, for every maximal subgroup M of G, either $E \le M$ or every non-Frattini chief factor of $G/E \cap M_G$ below $E/E \cap M_G$ is cyclic.*

As applications of Theorem 1.1 we get

COROLLARY 1.3. *G is supersoluble if and only if G has a normal subgroup E with supersoluble quotient G/E such that equalities* (∗) *hold for every different primes p and q and every p-element $a \in (G' \cap F^*(E))E'$, p'-element $b \in G$ and q-element $c \in G'$.*

Let $p_1 > \cdots > p_t$ be the set of all primes dividing $|G|$. Then G is called a *Sylow tower group* or *dispersive in the sense of Ore* if it has a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_t = G$ such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1}, for all $i = 1, \ldots, t$.

COROLLARY 1.4. *Let $\mathfrak{F}$ be one of the following classes*:

(1) *the class of all metanilpotent groups*;

(2) *the class of all nilpotent-by-abelian groups*;

(3) *the class of all dispersive in the sense of Ore groups*;

(4) *the class of all p-soluble groups G of p-length $l_p(G) \leq 1$.*

Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup E with $G/E \in \mathfrak{F}$ such that equalities (∗) *hold for every different primes p and q and every p-element $a \in (G' \cap F^*(E))E'$, p'-element $b \in G$ and q-element $c \in G'$.*

From Corollary 1.3 we get the following well-known Baer's result.

COROLLARY 1.5 ([24, Appendix 5.1]). *G is supersoluble if and only if for for every prime p and every p-element $a \in G'$ and p'-element $b \in G$ we have $[a, b^{p-1}] = 1$.*

From Theorem 1.1 we also get

COROLLARY 1.6 (Buckley [4]). *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

From Theorem 1.2 we get

COROLLARY 1.7. *G is supersoluble if and only if G has a soluble normal subgroup E with supersoluble quotient G/E such that, for every maximal subgroup M of G, either $F(E) \leq M$ or $M \cap F(E)$ is a maximal subgroup of $F(E)$.*

Corollary 1.8. *Let $\mathfrak{F}$ be one of the following classes*:

(1) *the class of all metanilpotent groups*;

(2) *the class of all nilpotent-by-abelian groups*;

(3) *the class of all dispersive in the sense of Ore groups*;

(4) *the class of all p-soluble groups G of p-length $l_p(G) \leq 1$.*

Then $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup E with $G/E \in \mathfrak{F}$ such that, for every maximal subgroup M of G, either $F(E) \leq M$ or $M \cap F(E)$ is a maximal subgroup of $F(E)$.

In the case when $E = G$ from Corollary 1.7 we get the following well-known Kramer's result.

Corollary 1.9 ([16] or Theorem 3.3 in [24, Chapter 1]). *Let G is soluble. Then G is supersoluble if and only if, for every maximal subgroup M of G, either $F(G) \leq M$ or $M \cap F(G)$ is a maximal subgroup of $F(G)$.*

All unexplained notation and terminology are standard. The reader is referred to [1], [6], or [11] if necessary.

2. Preliminaries

We use $G^{\mathcal{A}(p-1)}$ to denote the intersection of all normal subgroups R of G such that G/R is an abelian group of exponent dividing $p-1$.

Lemma 2.1 (Lemma 2.2 in [21]). *Let E be a normal p-subgroup of G. Then E is hypercyclically embedded in G if and only if*

$$(G/C_G(E))^{\mathcal{A}(p-1)} \leq O_p(G/C_G(E)).$$

Lemma 2.2 (Lemma 2.2 in [19]). *Let H be a non-identity normal subgroups of G. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be chief series of G below H. Then there exists a one-to-one correspondence between the chief factors of $\mathcal{H}_1$ and those of $\mathcal{H}_2$ such that corresponding factors are G-isomorphic and such that the Frattini (in G) chief factors of $\mathcal{H}_1$ correspond to the Frattini (in G) chief factors of $\mathcal{H}_2$.*

Lemma 2.3 ([9, Chapter 5, 3.11]). *Let P be a p-group and D a Thompson critical subgroup of P. Then D is of class at most 2 and $D/Z(D)$ is elementary abelian. Moreover, D is characteristic in P and every non-trivial p'-automorphism of P induces a non-trivial automorphism of D.*

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.4. *Let P be a p-group of class at most 2. Suppose that exp$(P/Z(P))$ divides p.*

(1) *If $p > 2$, then* $\exp(\Omega(P)) = p$.

(2) *If P is a non-abelian 2-group, then* $\exp(\Omega(P)) = 4$.

Proof. See p. 3 in [3]. □

Lemma 2.5 (Lemma 2.10 in [5]). *Let P be a normal p-subgroup of G. Let D be a characteristic subgroup of P such that every non-trivial p'-automorphism of P induces a non-trivial automorphism of D. If D is hypercyclically embedded in G, then P is hypercyclically embedded in G.*

Lemma 2.6 (Lemma 2.12 in [5]). *Let P be a normal p-subgroup of G, D a Thompson critical subgroup of P and $\Omega = \Omega(D)$. If Ω is hypercyclically embedded in G, then P is hypercyclically embedded in G.*

Recall that G is said to be a *minimal non-supersoluble group* if G is not supersoluble but every its proper subgroup is supersoluble. We shall need the following result by Doerk and Huppert.

Lemma 2.7. *Let G be a minimal non-supersoluble group. The following hold*:

(1) *G is soluble* [13].

(2) *$G^{\mathfrak{U}}$ is the unique normal Sylow subgroup of G, see* [13, 6];

(3) *$G^{\mathfrak{U}}$ is of exponent p or of exponent* 4.

Lemma 2.8 ([21, Theorem B]). *Let E a normal subgroup of G. If each chief factor of G below $F^*(E)$ is hypercyclically embedded in G, then E is hypercyclically embedded in G.*

Recall that a *formation* is a class $\mathfrak{F}$ of groups with the following properties: (i) every homomorphic image of any group $G \in \mathfrak{F}$ belongs to $\mathfrak{F}$; (ii) if G/M and G/N belong to $\mathfrak{F}$, then also $G/(M \cap N)$ belongs to $\mathfrak{F}$. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$.

LEMMA 2.9. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E \in \mathfrak{F}$. If every non-Frattini chief factor of G below E is cyclic, then $G \in \mathfrak{F}$.*

PROOF. Suppose that this lemma is false. Then $E \neq 1$. Let R be a minimal normal subgroup of G contained in E. The hypothesis holds for $(G/R, E/R)$. Hence $G/R \in \mathfrak{F}$ by induction. Since $\mathfrak{F}$ is saturated, $R \not\leq \Phi(G)$. Hence R is cyclic and so $G \in \mathfrak{F}$ by Lemma 2.16 in [23]. □

LEMMA 2.10 ([12, Theorem A]). *Let E be a soluble normal subgroup of G. If every non-Frattini chief factor of G below $F(E)$ is cyclic, then every non-Frattini chief factor of G below E is cyclic.*

3. Proofs of the results

We use $G^{\mathfrak{U}}$ to denote the intersection of all normal subgroups N of G such that G/N is is supersoluble.

PROOF OF THEOREM 1.1. First assume that E is hypercyclically embedded in G. Then E is supersoluble, so $F(E) = E^*(E)$ by [15, Chapter X, 13.6], and $E' \leq F(E)$ by [14, Chapter VI, 9.1]. Let a be any p-element in

$$(G' \cap F^*(E))E' = G' \cap F(E),$$

b any p'-element of G and c a q-element of G' ($q \neq p$). Then $a \in O_p(E)$. Let $C = C_G(O_p(E))$ and $S/C = (G/C)^{\mathfrak{A}(p-1)}$. Then S/C is a p-group by Lemma 2.1, and $b^{p-1}C \in S/C$, which imply that $b^{p-1} \in C$ since $(|S/C|, q) = 1$. Therefore $[a, b^{p-1}] = 1$. Finally, since $c \in G' \leq S$ and $q \neq p$, $c \in C$ and so $[a, c] = [a^{q-1}, c] = 1$. Thus (i) $\Longrightarrow$ (ii).

The implication (ii) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (i) Assume that this implication is false and let G be a counterexample with $|G| + |E|$ minimal. Let $F^* = F^*(E)$, p divide $|F^*|$ and P be a Sylow p-subgroup of F^*.

(1) CASE $E \neq P$

Assume that $E = P$. Then $G' \cap F^*(E))E' = G' \cap P$. Let $C = C_G(G' \cap P)$ and $S/C = (G/C)^{\mathfrak{A}(p-1)}$.

(a) G has a normal subgroup $R \leq P$ such that P/R is a non-cyclic chief factor of G, R is hypercyclically embedded in G and $V \leq R$ for any normal subgroup $V \neq P$ of G contained in P.

Indeed, let $V \neq P$ be a normal subgroup of G contained in P. Then

$$(G' \cap F^*(V))V' = G' \cap V \leq G' \cap P$$

and hence the hypothesis holds for (G, V). Therefore V is hypercyclically embedded in G by the choice of $(G, E) = (G, P)$. Now let P/R be a chief factor of G. Then R is hypercyclically embedded in G and so, in view of Lemma 2.2 and the choice of (G, P), P/R is non-cyclic. Now let $W \neq P$ be any normal subgroup of G contained in P. If $W \nleq R$, then in view of the G-isomorphism

$$P/R = WR/R \simeq W/W \cap R$$

we have P/R is cyclic. This contradiction shows that $W \leq R$.

(b) $G' \cap P = P$.

Assume that $G' \cap P < P$. Then $G' \cap P \leq Z$ by Claim (a). On the other hand, in view of the G-isomorphism $P/P \cap G' \simeq G'P/G'$ we have

$$P/P \cap G' \leq Z_{\infty}(G/P \cap G'),$$

so P is hypercyclically embedded in G. This contradiction shows that we have (b).

(c) P is of exponent p or exponent 4 (if $p = 2$ and P is a non-abelian 2-group).

Assume that this is false. Let L be a Thompson critical subgroup of P and $\Omega = \Omega(L)$. Then Ω is of exponent p or exponent 4 (if $p = 2$ and L is a non-abelian 2-group) by Lemmata 2.3 and 2.4. Hence $\Omega < P$, so Ω is hypercyclically embedded in G by Claim (a). Therefore P is hypercyclically embedded in G by Lemma 2.6, which contradicts the choice of (G, E). Hence $\Omega = P$, so P is of exponent p or exponent 4 (if $p = 2$ and P is a non-abelian 2-group).

(d) P is a minimal normal subgroup of G.

Assume that this is false. Since $P/R \leq (G/R)' = RG'/R$ by Claim (b), the hypothesis holds for $(G/R, P/R)$ by Claim (c) and so the choice of $(G, E) = (G, P)$ implies that P/R is cyclic, contrary to Claim (a). Hence we have (d).

(e) $O_{p'}(G) = 1$.

Assume that $D = O_{p'}(G) \neq 1$. Since the hypothesis holds for the couple $(G/D, DP/D)$, the choice of G implies that DP/D is hypercyclically embedded in G/D and so from the G-isomorphism $DP/D \simeq P$ we conclude that P is hypercyclically embedded in G, a contradiction.

(f) S/CG' is a p-group.

Since $CG'/C = (G/C)'$, it is enough to show that every p'-element bCG' of S/CG' has order dividing $p-1$. Without loss of generality we may assume that b is a p'-element of G and that $b \notin C$. Let $V = P\langle b\rangle$. It is cleat that the hypothesis holds for (V, P). So in the case when $V \neq G$, the choice of G implies that P is hypercyclically embedded in V and so $V/C_V(P)$ is an abelian group of exponent dividing $p-1$ by Lemma 2.1. Hence $b^{p-1} \in C_V(P) = C \cap V$, which implies that $|bCG'|$ divides $p-1$.

Now assume that $V = G$. Then, in view of Claim (e), $P = C_V(P)$ by the Hall–Higman lemma. Therefore, in view of Claims (b) and (d), for any element $a \in P$ we have $[a, b^{p-1}] = 1$, so $b^{p-1} \in C$ and so we again conclude that $|bCG'|$ divides $p-1$.

Final contradiction for (1). In view of Claim (f) and Lemma 2.1, $G'C/C$ is not a p-group. Let c be a q-element of G' such that $q \neq p$ and $c \notin C$. Then $[a, c^{p-1}] = 1 = [a^{q-1}, c]$ for every p-element $a \in P$ by Claims (b) and (d). If $q > p$, then $(p-1, q) = 1$ and so $\langle c^{p-1}\rangle = \langle c\rangle$. Thus $c \in C$. Similarly, if $p > q$ we get $[a^{q-1}, c] = 1 = [a, c]$. Thus again we have $c \in C$. This contradiction completes the proof of (1).

(2) Case $F^* = E = G$. Hence G is not soluble

Assume that $F^* \neq E$. Since F^* is characteristic in E, it is normal in G. Moreover, since $(F^*)' \leq E' \leq G'$, the hypothesis holds for (G, F^*) and for (E, F^*), so the choice of (G, E) implies that F^* is hypercyclically embedded in G. Hence E is hypercyclically embedded in G by Lemma 2.6, a contradiction. Thus $F^*(E) = E$. Finally, suppose that $E \neq G$. Then the choice of (G, E) implies that E is supersoluble, so E is nilpotent by [15, Chapter X, 13.6]. Since by (1) for any Sylow subgroup P of E we have $P \neq E$, the choice of (G, E) implies that P is hypercyclically embedded in G. Then E is hypercyclically embedded in G, a contradiction. Hence we have (2).

Final contradiction for the implication (iii) $\Longrightarrow$ (i). In view of Claim (2), G' is not supersoluble. Let H be a minimal non-supersoluble of G'. By Lemma 2.7, H is soluble, $Q = H^{\mathfrak{U}}$ is a Sylow q-subgroup of H for some prime q dividing

$|H|$ and P is of exponent p or exponent 4 (if Q is a non-abelian 2-group). Hence $H \neq G$ by Claim (2). Moreover, Claim (2) implies that

$$Q \leq H \leq G' = (G' \cap F^*(E))E'.$$

Hence the hypothesis holds for (H, P), so the choice of (G, E) implies that P is hypercyclically embedded in H and so H is supersoluble since H/Q is supersoluble. This contradiction completes the proof of the implication (iii) $\Longrightarrow$ (i). The theorem is proved. □

Proof of Theorem 1.2. First assume that, for every maximal subgroup M of G, either $E \leq M$ or every non-Frattini chief factor of $G/E \cap M_G$ below $E/E \cap M_G$ is cyclic. We shall show that in this case every non-Frattini chief factor of G below E is cyclic. Suppose that this is false and let G be a counterexample with $|G|+|E|$ minimal. Let N be a minimal normal subgroup of G contained in E and M/N a maximal subgroup of G/N such that $E/N \nleq M/N$. Then

$$(E/N)/(E/N) \cap (M/N)_{G/N} = (E/N)/(E/N) \cap (M_G/N)$$

and so from the G-isomorphism

$$(E/N)/((E \cap M_G)N/N) \simeq E/E \cap M_G$$

we get that every non-Frattini chief factor of $(G/N)/(E/N \cap (M/N)_{G/N})$ below $(E/N)/(E/N) \cap (M/N)_{G/N}$ is cyclic. Therefore the hypothesis holds for $(G/N, E/N)$, so every non-Frattini chief factor of G/N below E/N is cyclic by the choice of G. Therefore, Lemma 2.2 and the choice of G imply that $N \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $N \nleq M$. Then from

$$N(E \cap M_G)/(E \cap M_G) \leq E/E \cap M_G$$

and the G-isomorphism $N(E \cap M_G)/(E \cap M_G) \simeq N$ we get that N is cyclic. But then every non-Frattini chief factor of G below E is cyclic by Lemma 2.2.

Finally, suppose that every non-Frattini chief factor of G below E is cyclic. And let M be any maximal subgroup of G such that $E \nleq M$. We shall show that every non-Frattini chief factor of $G/E \cap M_G$ below $E/E \cap M_G$ is cyclic. Suppose that this is false and let G be a counterexample with $|G| + |E|$ minimal. Then $E \cap M_G \neq 1$. Moreover, if N a minimal normal subgroup of G contained in $E \cap M_G$, then the hypothesis holds for $(G/N, E/N)$ and $E/N \nleq M/N$. Therefore every non-Frattini chief factor of $(G/N)/(E/N \cap (M/N)_{G/N})$ below $(E/N)/(E/N) \cap (M/N)_{G/N}$ is cyclic by the choice of (G, E). Thus every non-Frattini chief factor of $G/E \cap M_G$ below $E/E \cap M_G$ is cyclic. The theorem is proved. □

Proof of Corollary 1.4. It is well known that the classes of all metanilpotent groups, of all nilpotent-by-abelian groups, of all dispersive in the sense of Ore groups and of all p-soluble groups of p-length ≤ 1 are saturated formations (see for example [7, Chapter IV]). Moreover, each of these classes contains all supersoluble groups. Therefore Corollary 1.4 follows from Theorem 1.1 and Lemma 2.9. □

Proof of Corollary 1.5. If G is supersoluble, then $G' \leq F(G) = F^*(G)$ and so, by Theorem 1.1, for every prime p and for every p-element

$$a \in G' = (G' \cap F^*(G))G'$$

and every p'-element element b of G we have $[a, b^{p-1}] = 1$.

Finally, if for every p-element $a \in G'$ and every p'-element element $b \in G$ we have $[a, b^{p-1}] = 1$, then for every different primes p and q and for every p-element $a \in (G' \cap F^*(G))G'$, p'-element element $b \in G$ and q-element $c \in G'$ we have $[a, b^{p-1}] = 1 = [a^{q-1}, c]$. Hence G is supersoluble by Theorem 1.1. □

Proof of Corollary 1.7. First assume that G is supersoluble. Take a maximal subgroup M of G such that $F(E) \nleq M$. Then

$$|G : M| = |F(E) : F(E) \cap M| = p$$

for some prime p, and so $M \cap F(E)$ is a maximal subgroup of $F(E)$.

Finally, assume that G has a soluble normal subgroup E with supersoluble quotient G/E such that for every maximal subgroup M of G either $F(E) \leq M$ or $M \cap F(E)$ is a maximal subgroup of $F(E)$. If $F(E) \nleq M$, then $G = F(E)M$. On the other hand,

$$F(E) \cap M < N_{F(E)}(F(E) \cap M)$$

since $F(E)$ is nilpotent. Hence $F(E) \cap M$ is normal in G and so

$$F(E)/F(E) \cap M = F(E)/F(E) \cap M_G$$

is cyclic. Applying Theorem 1.2, we get that every non-Frattini chief factor of G below $F(E)$ is cyclic. Therefore every non-Frattini chief factor of G below E is cyclic by Lemma 2.10, so G is supersoluble by Lemma 2.9 since G/E is supersoluble by hypothesis. □

Proof of Corollary 1.8. See the proof of Corollary 1.4. □

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