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Proper base change for separated locally proper maps

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Abstract – We introduce and study the notion of a locally proper map between topological spaces. We show that fundamental constructions of sheaf theory, more precisely proper base change, projection formula, and Verdier duality, can be extended from continuous maps between locally compact Hausdorff spaces to separated locally proper maps between arbitrary topological spaces.

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Contents

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1. Introduction

The proper direct image functor $f_!$ and its right derived functor $\mathrm{R} f_!$ are defined for any continuous map $f: Y \to X$ of locally compact Hausdorff spaces, see [6, 10, 11]. Instead of $f_!$ and $\mathrm{R} f_!$ we will use the notation $f_{(!)}$ and $f_!$ for these functors. They are embedded into a whole collection of formulas known as the six-functor-formalism of Grothendieck. Under the assumption that the proper direct image functor $f_{(!)}$ has finite cohomological dimension, Verdier proved that its derived functor $f_!$ admits a right adjoint $f^!$.

In this article we introduce in 2.3 the notion of a locally proper map between topological spaces and show that the above results even hold for arbitrary topological spaces if all maps whose proper direct image functors are involved are locally proper and separated. Every continuous map between locally compact Hausdorff spaces is separated and locally proper by 2.11, and in this case our results specialize to the classical theory.

The basic properties of locally proper maps are established in Section 2. In particular a map from a topological space to a one point space is separated and locally proper if and only if our space is locally compact Hausdorff. Since the properties of being separated and locally proper are moreover stable under base change by 9.11 and 2.7, a separated locally proper map can be interpreted as a continuous family of locally compact Hausdorff spaces.

Our main sheaf theoretic results concerning separated locally proper maps are the proper base change 4.4, the projection formula 6.2, the derived proper base change 5.10, the derived projection formula 6.3, and Verdier duality 7.7. Theorem 8.3 explains how all the previous results work in the setting of unbounded derived categories.

In the following we list three applications of our extension of the classical theory described above.

Locally closed embeddings. An embedding $i: Z \hookrightarrow X$ of topological spaces is always separated, and by 2.5 it is locally proper if and only if $i(Z)$ is a locally closed subset of X. In this case the functor $i_{(!)} = i_!$ is the extension by zero and the functor $\mathcal{H}^p i^!$ maps a sheaf to its p-th local cohomology sheaf.

Homotopy invariance of sheaf cohomology. Homotopy invariance of sheaf cohomology can be deduced, as explained in [6, 2.7], from the fact that given a space X and a complex of sheaves $\mathcal{F}$ on X, for the projection $\pi: X \times [0, 1] \to X$

the unit of the adjunction of derived direct and inverse image is an isomorphism $\mathcal{F} \xrightarrow{\sim} \pi_* \pi^* \mathcal{F}$. Now for an arbitrary topological space X the map π is always proper and separated, so proper base change in the generality of this article allows to reduce the claim to the case when X is a one point space, which is treated in the literature, say in [6].

Equivariant derived categories. Fix a topological group G and let X be a G-space, i. e. a topological space with a continuous G-action $G \times X \to X$. Let $\mathrm{E}\,G$ be a contractible space with a topologically free G-action, e. g. the Milnor construction. Then by homotopy invariance of sheaf cohomology the map $\mathrm{E}\,G \to \{\mathrm{pt}\}$ is acyclic, and so is $p\colon \mathrm{E}\,G \times X \to X$. Let $q\colon \mathrm{E}\,G \times X \to \mathrm{E}\,G \times_G X$ be the quotient map. Then the bounded below equivariant derived category can by [2, 2.9.4] be described as the full triangulated subcategory of $D^+(\mathrm{E}\,G \times_G X)$ given by

$$\begin{aligned} D^+_G(X) := \{\mathcal{F} \in D^+(\mathrm{E}\,G \times_G X) \mid {} & \text{there exists } \mathcal{G} \in D^+(X) \\ & \text{such that } q^*\mathcal{F} \cong p^*\mathcal{G}\}. \end{aligned}$$

Now let $f\colon X \to Y$ be a G-equivariant continuous map of locally compact Hausdorff G-spaces or even a G-equivariant separated locally proper map of arbitrary G-spaces. We obtain a diagram

$$\begin{array}{ccccc} \mathrm{E}\,G \times_G X & \longleftarrow & \mathrm{E}\,G \times X & \longrightarrow & X \\ \big\downarrow \bar{f} & & \big\downarrow \hat{f} & & \big\downarrow f \\ \mathrm{E}\,G \times_G Y & \longleftarrow & \mathrm{E}\,G \times Y & \longrightarrow & Y \end{array}$$

whose vertical maps are induced by f. Both squares are cartesian so that $\hat{f}$ and $\bar{f}$ are separated locally proper by 2.7, 2.9, 9.11. Note however that, given a locally compact Hausdorff space Z, the spaces $\mathrm{E}\,G \times Z$ and $\mathrm{E}\,G \times_G Z$ are, in general, not locally compact Hausdorff. Derived proper base change 5.10 in the generality of this article nevertheless shows that $\bar{f}_!$ induces a functor $f_!\colon D^+_G(X) \to D^+_G(Y)$ which generalizes the proper direct image functor of [2].

In the hope that the reader is now sufficiently motivated, let us just add that from our point of view the main new ingredients are the definition of a locally proper map and the proof of the underived form of proper base change. Once this is done, we just have to document that the standard arguments work in this generality as well.

2. Locally proper maps

2.1. A topological space is called *compact* if every open covering has a finite subcovering. A topological space is called *locally compact* if every neighborhood of any point contains a compact neighborhood of this point. We do not require the Hausdorff property in either case.

2.2. The definitions of proper and separated maps and their basic properties are recalled in Section 9.

2.3 Definition. A map $f: Y \to X$ of topological spaces is called *locally proper* if it is continuous and if given any point $y \in Y$ and any neighborhood V of y there are neighborhoods $A \subset V$ of y and U of $f(y)$ such that $f(A) \subset U$ and the induced map $f: A \to U$ is proper.

2.4. The constant map from a topological space Y to a space consisting of a single point is locally proper if and only if Y is locally compact.

2.5 Lemma. *An embedding $i: Y \hookrightarrow X$ of topological spaces is locally proper if and only if $i(Y)$ is a locally closed subset of X.*

Proof. We can assume that i is the inclusion of a subset Y of X with its induced topology. Recall from 9.3 that an embedding is proper if and only if it is closed.

Assume that i is locally proper. Given $y \in Y$ there are a neighborhood A of y in Y and a neighborhood U of y in X with $A \subset U$ such that the inclusion $A \subset U$ is closed. By replacing U with a smaller open neighborhood of y and A by its intersection with this neighborhood we can assume that U is open in X. Since A is a neighborhood of y there is an open subset $W \subset X$ such that $y \in W \cap Y \subset A$. Then the inclusion $W \cap A \subset W \cap U$ is closed and $W \cap A = W \cap Y = (W \cap U) \cap Y$. This shows that Y is a locally closed subset of X.

The converse implication is obvious. □

2.6. Every composition of locally proper maps is locally proper. This follows easily using that a composition of proper maps is proper by 9.3 and that any base change of a proper map is proper by 9.4.

2.7 Lemma (Local properness and base change). *Local properness is stable under base change. More precisely, let*

$$\begin{array}{ccc} W & \xrightarrow{q} & Y \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{p} & X \end{array}$$

be a cartesian diagram of topological spaces and continuous maps. If f is locally proper so is g.

Proof. Let $w \in W$ together with a neighborhood $S \subset W$ be given. Let us identify $W = Z \times_X Y$. Then $w = (g(w), q(w))$. We find neighborhoods $V \subset Y$ of $q(w)$ and $U \subset Z$ of $g(w)$ such that $U \times_X V \subset S$. By replacing S by $U \times_X V$ we can assume that the diagram

$$\begin{array}{ccc} S & \xrightarrow{q} & V \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ U & \xrightarrow{p} & X \end{array}$$

is cartesian. We find neighborhoods $A \subset V$ of $q(w)$ and $N \subset X$ of $f(q(w))$ such that $f : A \to N$ is proper. The same is true for the map $g : q^{-1}(A) \to p^{-1}(N)$ obtained by base change, by 9.4. □

2.8. The property of being locally proper is local on the source. Namely, let $f : Y \to X$ be a map and let $\mathcal{V}$ be an open covering of Y. Then f is locally proper if and only if its restrictions $f|_V : V \to Y$ are locally proper for all $V \in \mathcal{V}$, as follows easily using 2.5.

2.9. The property of being locally proper is local on the target. Namely, let $f : Y \to X$ be a map and let $\mathcal{U}$ be an open covering of X. Then f is locally proper if and only if the induced maps $f^{-1}(U) \to U$ are locally proper for all $U \in \mathcal{U}$, as follows easily using 2.7.

2.10 Lemma. *Let $g : Z \to Y$ and $f : Y \to X$ be continuous maps of topological spaces. Assume that $f \circ g$ is locally proper and that f is separated. Then g is locally proper.*

Proof. Let $V \subset Z$ be a given neighborhood of a point $z \in Z$. Since $f \circ g$ is locally proper, there are neighborhoods $A \subset V$ of z and U of $f(g(z))$ such that $f \circ g$ induces a proper map $A \to U$. This map factors as

$$A \xrightarrow{g'} f^{-1}(U) \xrightarrow{f'} U$$

where g' and f' are induced by g and f respectively. Since f' is separated by 9.11, Lemma 9.12 shows that $g' \colon A \to f^{-1}(U)$ is proper. This shows that g is locally proper. □

2.11 Corollary. *Every continuous map $g \colon Z \to Y$ from a locally compact Hausdorff space Z to a Hausdorff space Y is separated and locally proper.*

Proof. Since Z is locally compact and Y is Hausdorff, g is locally proper by 2.10. Since Z is Hausdorff, g is separated. □

2.12 Proposition. *Every proper and separated map is locally proper.*

2.13. This generalizes the fact that every compact Hausdorff space is locally compact.

Proof. Let $f \colon Y \to X$ be a proper and separated map. Let $y \in Y$ together with an open neighborhood $V \subset Y$ be given. Put $x := f(y)$ and consider $Z = Y \setminus V$. For every point $z \in f^{-1}(x) \cap Z$ there is an open neighborhood $W_z \subset Y$ of z and an open neighborhood $B_z \subset V$ of y with $W_z \cap B_z = \emptyset$. Finitely many of these W_z cover the compact set $f^{-1}(x) \cap Z$. Let W be the union of these sets and B the intersection of the corresponding sets B_z. Then $y \in B$, B is open in V, $(f^{-1}(x) \cap Z) \subset W$, W is open in Y, and $B \cap W = \emptyset$. Now we find an open neighborhood $U \subset X$ of x with $(f^{-1}(U) \cap Z) \subset W$, we may for example take the complement of the closed subset $f(Z \cap (Y \setminus W))$ of X. Then the closure A of $B \cap f^{-1}(U)$ in $f^{-1}(U)$ has empty intersection with $f^{-1}(U) \cap Z$ and hence is a neighborhood of y contained in V such that the induced map $f \colon A \to U$ is proper. □

3. Proper direct image

3.1. By a sheaf we mean a sheaf of abelian groups. The constant sheaf with stalk $\mathbb{Z}$ on a topological space X is denoted $\mathbb{Z}_X$. Given a sheaf $\mathcal{F}$ on X we abbreviate as usual $\Gamma\mathcal{F} = \Gamma(\mathcal{F}) = \Gamma(X;\mathcal{F}) = \mathcal{F}(X)$ its set of global sections and write $\mathcal{F}(Z) = \Gamma(Z;\mathcal{F}) = \Gamma(Z;\mathcal{F}|_Z)$ if $Z \subset X$ is an arbitrary subset of X.

We write $\mathrm{Sh}(X)$ for the abelian category of sheaves on X. Given a complex $\mathcal{F}$ in $\mathrm{Sh}(X)$ we denote its p-th cohomology sheaf by $\mathcal{H}^p(\mathcal{F})$. We write $C(X)$ for the abelian category of complexes in $\mathrm{Sh}(X)$, denote the corresponding homotopy category by $K(X)$ and the corresponding derived category by $D(X)$. Let $D^+(X) \subset D(X)$ be the full triangulated subcategory consisting of complexes with bounded below cohomology sheaves.

Given a continuous map f of topological spaces we use the notation $f^{(*)}$ and $f_{(*)}$ for the inverse and direct image functors of sheaves, in contrast to the usual notation f^{-1} and f_* in the literature, and denote their derived functors by f^* and f_*, in contrast to the usual notation $\mathrm{L}f^*$ and $\mathrm{R}f_*$.

Given a category $\mathcal{C}$ and objects $A, B \in \mathcal{C}$ we denote the set of morphisms by $\mathcal{C}(A, B)$ or $\mathrm{Hom}_{\mathcal{C}}(A, B)$. We write $\mathrm{Sh}_X(A, B)$ instead of $(\mathrm{Sh}(X))(A, B)$.

3.2 Definition. Let $f : Y \to X$ be a continuous map of topological spaces. The *proper direct image* $f_{(!)} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is defined by

$$(f_{(!)}\mathcal{F})(U) := \{s \in \mathcal{F}(f^{-1}(U)) \mid f : (\operatorname{supp} s) \longrightarrow U \text{ is proper}\}.$$

Since by 9.5 properness is local on the target, $f_{(!)}\mathcal{F} \subset f_{(*)}\mathcal{F}$ is a subsheaf of sets and by 9.7 even a subsheaf of abelian groups. Obviously, $f_{(!)}$ is a left exact functor. For proper f we have $f_{(!)} = f_{(*)}$ by 9.3. Given a sheaf $\mathcal{F}$ on Y we denote by

$$\Gamma_!\mathcal{F} = \Gamma_!(\mathcal{F}) = \Gamma_!(Y; \mathcal{F}) = (c_{(!)}\mathcal{F})(X)$$

the abelian group of sections with compact support where c is the unique map from Y to a space consisting of a single point. If $Z \subset Y$ is any subset we abbreviate $\Gamma_!(Z; \mathcal{F}) = \Gamma_!(Z; \mathcal{F}|_Z)$.

3.3. Usually the functor $f_{(!)}$ is studied for continuous maps of locally compact Hausdorff spaces. Our aim is to show that $f_{(!)}$ behaves well in a more general setting, namely for separated locally proper maps f.

3.4. Let $i : Y \hookrightarrow X$ be an embedding of a locally closed subset. Then $i_{(!)}\mathcal{F}$ is the extension of $\mathcal{F} \in \mathrm{Sh}(Y)$ by zero which is denoted by $i_!\mathcal{F}$ in [5, Exp. I] and by $\mathcal{F}^X$ in [4, Theorem 2.9.2]. The functor $i_{(!)} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is exact and has a right adjoint functor $i^{(!)}$ which is denoted $i^!$ in [5, Exp. I]. If Y is closed in X we have $i_{(!)} = i_{(*)}$. If Y is open in X we have $i^{(!)} = i^{(*)}$.

3.5. Let $f : Y \to X$ be a continuous map and $\mathcal{F}$ a sheaf on Y. A global section $s \in (f_{(*)}\mathcal{F})(X)$ is the same thing as a global section $s' \in \mathcal{F}(Y)$. The supports of these sections are related by $\overline{f(\operatorname{supp} s')} = \operatorname{supp} s$.

3.6 Theorem (Iterated proper images). *Given continuous maps* $g\colon Z \to Y$ *and* $f\colon Y \to X$ *with* f *separated, the usual identity* $f_{(*)} \circ g_{(*)} = (f \circ g)_{(*)}$ *of functors* $\mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ *restricts to an identity*

$$f_{(!)} \circ g_{(!)} = (f \circ g)_{(!)}.$$

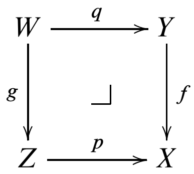
Proof. Let $\mathcal{F}$ be a sheaf on Z and $U \subset X$ an open subset and let $s \in (f_{(*)}(g_{(*)}\mathcal{F}))(U)$ be a section. We can interpret s also as a section $s' \in (g_{(*)}\mathcal{F})(f^{-1}(U))$ or as a section $s'' \in \mathcal{F}(g^{-1}(f^{-1}(U)))$, and also as a section $t \in ((f \circ g)_{(*)}\mathcal{F})(U)$.

Assume that s is in $(f_{(!)}(g_{(!)}\mathcal{F}))(U)$. Equivalently, this means that s' is in $(g_{(!)}\mathcal{F})(f^{-1}(U))$ with $f\colon(\operatorname{supp} s') \to U$ proper or, equivalently, that s'' is in $\mathcal{F}(g^{-1}(f^{-1}(U)))$ with $g\colon(\operatorname{supp} s'') \to f^{-1}(U)$ proper and $f\colon g(\operatorname{supp} s'') \to U$ proper. Here we use 3.5 and the fact that $g(\operatorname{supp} s'')$ is closed in $f^{-1}(U)$. Then the induced map $g\colon(\operatorname{supp} s'') \to g(\operatorname{supp} s'')$ is proper by 9.12 and so is the composition $f \circ g\colon(\operatorname{supp} s'') \to U$ by 9.3. This means that $t \in ((f \circ g)_{(!)}\mathcal{F})(U)$. We have not yet used that f is separated.

Conversely, assume that $t \in ((f \circ g)_{(!)}\mathcal{F})(U)$. This means that the composition $f \circ g\colon(\operatorname{supp} s'') \to U$ is proper. Since f is separated, $g\colon(\operatorname{supp} s'') \to f^{-1}(U)$ is proper by 9.12, so s' is in $(g_{(!)}\mathcal{F})(f^{-1}(U))$ and $(\operatorname{supp} s') = g(\operatorname{supp} s'')$. This and 9.6 imply that $f\colon(\operatorname{supp} s') \to U$ is proper. Hence $s \in (f_{(!)}(g_{(!)}\mathcal{F}))(U)$. □

4. Proper base change

4.1. In the following we will often work with a cartesian diagram



of topological spaces and continuous maps. Let us refer to it as "given a cartesian diagram $f \circ q = p \circ g$" without mentioning the spaces involved.

4.2 Lemma. *Given a cartesian diagram* $f \circ q = p \circ g$ *of topological spaces as in* 4.1 *the identity* $f_{(*)} \circ q_{(*)} = p_{(*)} \circ g_{(*)}$ *induces a morphism*

$$f_{(!)} \circ q_{(*)} \longrightarrow p_{(*)} \circ g_{(!)}$$

of functors $\mathrm{Sh}(W) \to \mathrm{Sh}(X)$.

Proof. Let $\mathcal{G}$ be a sheaf on W. It is enough to prove the claim on global sections, i. e. that the identity $\Gamma f_{(*)} q_{(*)} \mathcal{G} = \Gamma p_{(*)} g_{(*)} \mathcal{G}$ induces a morphism $\Gamma f_{(!)} q_{(*)} \mathcal{G} \to \Gamma p_{(*)} g_{(!)} \mathcal{G}$. Similarly as before, we interpret $\Gamma f_{(!)} q_{(*)} \mathcal{G}$ as the group of all $s \in \Gamma \mathcal{G}$ with $f : \overline{q(\operatorname{supp} s)} \to X$ proper. The cartesian diagram

$$\begin{array}{ccc} q^{-1}(\overline{q(\operatorname{supp} s)}) & \longrightarrow & \overline{q(\operatorname{supp} s)} \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & X \end{array}$$

and 9.3, 9.4 show that $g : (\operatorname{supp} s) \to Z$ is proper. Hence $s \in \Gamma p_{(*)} g_{(!)} \mathcal{G}$. □

4.3. Given a cartesian diagram $f \circ q = p \circ g$ of topological spaces as in 4.1 there is a canonical morphism

$$p^{(*)} f_{(!)} \longrightarrow g_{(!)} q^{(*)}$$

of functors obtained as follows: the adjunction $(q^{(*)}, q_{(*)})$ and 4.2 provide morphisms $f_{(!)} \to f_{(!)} q_{(*)} q^{(*)} \to p_{(*)} g_{(!)} q^{(*)}$ so that we can use the adjunction $(p^{(*)}, p_{(*)})$.

4.4 Theorem (**Proper base change**). *Let a cartesian diagram $f \circ q = p \circ g$ of topological spaces as in 4.1 be given. Assume that the vertical maps f and g are separated locally proper. Then the morphism constructed in 4.3 is an isomorphism*

$$p^{(*)} \circ f_{(!)} \xrightarrow{\sim} g_{(!)} \circ q^{(*)}$$

of functors $\operatorname{Sh}(Y) \to \operatorname{Sh}(Z)$.

4.5. If f and g are even separated and proper, we obtain by 2.12 an isomorphism $p^{(*)} f_{(*)} \xrightarrow{\sim} g_{(*)} q^{(*)}$ of functors.

Proof. We need to show that our morphism of functors evaluated at $\mathcal{F} \in \operatorname{Sh}(Y)$ is an isomorphism

$$p^{(*)} f_{(!)} \mathcal{F} \xrightarrow{\sim} g_{(!)} q^{(*)} \mathcal{F}.$$

A morphism of sheaves is an isomorphism if and only if it induces isomorphisms on all stalks. Hence we can assume without loss of generality that Z consists of a single point $x \in X$. In this case the theorem claims that the obvious map is an isomorphism

$$(f_{(!)} \mathcal{F})_x \xrightarrow{\sim} \Gamma_!(f^{-1}(x); \mathcal{F}).$$

Injectivity: Let $U \subset X$ be an open neighborhood of x and $s \in \mathcal{F}(f^{-1}(U))$ a section with $(\operatorname{supp} s) \to U$ proper but $s|_{f^{-1}(x)} = 0$. Then x is not in the closed subset $f(\operatorname{supp} s) \subset U$. If $V \subset U$ is the open complement of this subset, the restriction of our section $s \in (f_{(!)}\mathcal{F})(U)$ to V alias $f^{-1}(V)$ is zero.

Surjectivity: Let $s \in \Gamma_!(f^{-1}(x); \mathcal{F})$. Denote by $\overline{\mathcal{F}}$ the étale space over Y associated to $\mathcal{F}$. We view s as a continuous section $s\colon f^{-1}(x) \to \overline{\mathcal{F}}$ with compact support $K := \operatorname{supp} s \subset f^{-1}(x)$. Since f is separated, 9.10 and 10.1 allow us to extend $s|_K$ to a continuous section $\hat{s}\colon C \to \overline{\mathcal{F}}$ where $C \subset Y$ is an open neighborhood of K. By shrinking C we can assume in addition that $\hat{s}$ and s coincide on $C \cap f^{-1}(x)$ (the subset of $C \cap f^{-1}(x)$ where the restrictions of s and $\hat{s}$ coincide is open in $C \cap f^{-1}(x)$ and contains K, hence is of the form $C' \cap C \cap f^{-1}(x)$ for some open subset $K \subset C' \subset Y$; replace C by C'). Hence s and $\hat{s}$ glue to a continuous section $C \cup f^{-1}(x) \to \overline{\mathcal{F}}$.

We claim that there is a subset $B \subset Y$ and an open subset $U \subset X$ with $K \subset B^\circ \subset B \subset C$ and $f(B) \subset U$ such that $f\colon B \to U$ is proper.

Indeed, since f is locally proper, any $y \in K$ has a neighborhood A_y in Y contained in C such that there is an open neighborhood $U_y \subset X$ of $f(y) = x$ with $f(A_y) \subset U_y$ and $f\colon A_y \to U_y$ proper. Since f is separated, $A_y \subset f^{-1}(U_y)$ is a closed subset by 9.3, 9.11, and 9.12. Finitely many A_y° cover K, and we define U as the intersection of the corresponding sets U_y and B as the union of the corresponding sets A_y intersected with $f^{-1}(U)$. Then $f\colon B \to U$ is proper by 9.7, proving the claim.

Certainly $B^\circ \subset f^{-1}(U)$ is an open subset, and $B \subset f^{-1}(U)$ is closed by 9.12. Let $\partial B = B \setminus B^\circ$ be the boundary of B with respect to $f^{-1}(U)$. Since $(\operatorname{supp} \hat{s})$ is closed in C, the intersection $(\operatorname{supp} \hat{s}) \cap \partial B$ is closed in B and hence the map $f\colon (\operatorname{supp} \hat{s}) \cap \partial B \to U$ is proper. Its image does not contain x because $f^{-1}(x) \cap (\operatorname{supp} \hat{s}) \cap \partial B = K \cap \partial B = \emptyset$. By replacing U with its open subset $U \setminus f((\operatorname{supp} \hat{s}) \cap \partial B)$ we can assume that $(\operatorname{supp} \hat{s}) \cap \partial B = \emptyset$. Hence the continuous map $\hat{s}|_B\colon B \to \overline{\mathcal{F}}$ and the zero section $f^{-1}(U) \setminus B^\circ \to \overline{\mathcal{F}}$ coincide on ∂B and glue to a continuous map $f^{-1}(U) \to \overline{\mathcal{F}}$ alias an element of $\mathcal{F}(f^{-1}(U))$ whose support is equal to $(\operatorname{supp} \hat{s}) \cap B$ and hence proper over U. We deduce surjectivity. □

4.6 Lemma. *If $f\colon Y \to X$ is a separated locally proper map, the proper direct image functor $f_{(!)}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ commutes with filtered colimits.*

Proof. Let $\mathcal{F}\colon I \to \mathrm{Sh}(Y)$ be a filtered diagram. We claim that the obvious morphism is an isomorphism

$$\varinjlim (f_{(!)}\mathcal{F}_i) \xrightarrow{\sim} f_{(!)}(\varinjlim \mathcal{F}_i).$$

Since the inverse image functor of sheaves commutes with arbitrary colimits, proper base change 4.4 allows to reduce to the case that X consists of a single point. Then Y is a locally compact Hausdorff space and the claim is well known and recalled in 10.2. □

5. Derived proper direct image

5.1 Definition. Given a continuous map $f: Y \to X$, a sheaf on Y is called *f-c-soft* if its restriction to every fiber of f is c-soft in the sense of 10.3.

5.2. This definition seems to be useful just for separated locally proper maps.

5.3. If our map $f: Y \to X$ is separated, every flabby sheaf and in particular every injective sheaf on Y is f-c-soft by 10.1.

5.4. Given a separated locally proper map $f: Y \to X$, arbitrary direct sums alias coproducts, and even filtered colimits of f-c-soft sheaves are f-c-soft again. In fact, the inverse image functor of sheaves commutes with arbitrary colimits, and every filtered colimit of c-soft sheaves on a locally compact Hausdorff space is c-soft by 10.4.

5.5 Lemma. *Let $f: Y \to X$ be a separated locally proper map. Then any f-c-soft sheaf on Y is $f_{(!)}$-acyclic.*

Proof. Let $\mathcal{F} \in \mathrm{Sh}(Y)$ be f-c-soft and $\mathcal{F} \hookrightarrow \mathcal{I}^*$ an injective resolution. We need to show that $f_{(!)}\mathcal{F} \hookrightarrow f_{(!)}\mathcal{I}^*$ is exact. We test this on the stalks at an arbitrary $x \in X$. Let j be the inclusion $f^{-1}(x) \hookrightarrow Y$. By proper base change 4.4 it is enough to show exactness of $\Gamma_!(j^{(*)}\mathcal{F}) \hookrightarrow \Gamma_!(j^{(*)}\mathcal{I}^*)$. But this follows from 5.3 and [6, Proposition 2.5.8, Corollary 2.5.9]. □

5.6. Let $g: Z \to Y$ be a continuous map of locally compact Hausdorff spaces. If $\mathcal{F} \in \mathrm{Sh}(Z)$ is c-soft so is $g_{(!)}\mathcal{F} \in \mathrm{Sh}(Y)$ by [6, Proposition 2.5.7].

5.7 Lemma. *Let $g: Z \to Y$ and $f: Y \to X$ be separated locally proper maps. If $\mathcal{F} \in \mathrm{Sh}(Z)$ is an $(f \circ g)$-c-soft sheaf on Z, its proper direct image $g_{(!)}\mathcal{F}$ is an f-c-soft sheaf on Y.*

Proof. Given a point $x \in X$ consider the following diagram with two cartesian squares.

$$\begin{array}{ccc} g^{-1}(f^{-1}(x)) & \overset{k}{\hookrightarrow} & Z \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle g} \\ f^{-1}(x) & \overset{j}{\hookrightarrow} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} \\ \{x\} & \overset{i}{\hookrightarrow} & X \end{array}$$

Proper base change 4.4 shows $j^{(*)}g_{(!)}\mathcal{F} \cong v_{(!)}k^{(*)}\mathcal{F}$. By assumption, the sheaf $k^{(*)}\mathcal{F}$ is c-soft and both $g^{-1}(f^{-1}(x))$ and $f^{-1}(x)$ are locally compact Hausdorff spaces. Hence 5.6 shows that $v_{(!)}k^{(*)}\mathcal{F} \cong j^{(*)}g_{(!)}\mathcal{F}$ is c-soft. This proves the claim. □

5.8 Definition. If $f: Y \to X$ is a continuous map we denote the right derived functor $\mathrm{R}f_{(!)}: D^+(Y) \to D^+(X)$ of the left exact proper direct image functor $f_{(!)}: \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ by $f_! := \mathrm{R}f_{(!)}$.

5.9 Theorem (Iterated derived proper images). *Let both $g: Z \to Y$ and $f: Y \to X$ be separated locally proper maps. Then the identity $f_{(!)} \circ g_{(!)} = (f \circ g)_{(!)}$ from 3.6 gives rise to an isomorphism*

$$(f \circ g)_! \xrightarrow{\sim} f_! \circ g_!$$

of triangulated functors $D^+(Z) \to D^+(X)$.

Proof. It is enough to show that the functor $g_{(!)}$ maps injective sheaves to $f_{(!)}$-acyclic sheaves. Every injective sheaf $\mathcal{I} \in \mathrm{Sh}(Z)$ is $(f \circ g)$-c-soft by 5.3. Then $g_{(!)}\mathcal{I} \in \mathrm{Sh}(Y)$ is f-c-soft by 5.7 and hence $f_{(!)}$-acyclic by 5.5. □

5.10 Theorem (Derived proper base change). *Let a cartesian diagram $f \circ q = p \circ g$ of topological spaces as in 4.1 be given. Assume that the vertical maps f and g are separated and locally proper. Then, in the space of triangulated functors $D^+(Y) \to D^+(Z)$, the obvious morphisms are isomorphisms*

$$\mathrm{R}(p^{(*)} \circ f_{(!)}) \xrightarrow{\sim} p^* \circ f_! \quad \textit{and} \quad \mathrm{R}(g_{(!)} \circ q^{(*)}) \xrightarrow{\sim} g_! \circ q^*$$

so that proper base change 4.4 yields an isomorphism

$$p^* \circ f_! \xrightarrow{\sim} g_! \circ q^*.$$

Proof. If $\mathcal{I}$ is an injective sheaf on Y, it is f-c-soft by 5.3, its inverse image $q^{(*)}\mathcal{I}$ on W is g-c-soft and hence $g_{(!)}$-acyclic by 5.5. This shows that

$$\mathrm{R}(g_{(!)} \circ q^{(*)}) \longrightarrow g_! \circ q^*$$

is an isomorphism. The rest of the proof is obvious. □

6. Projection formula

6.1. Let $f: Y \to X$ be a continuous map and $\mathcal{F} \in \mathrm{Sh}(Y)$ and $\mathcal{G} \in \mathrm{Sh}(X)$. Then we obtain a canonical morphism

$$(f_{(*)}\mathcal{F}) \otimes \mathcal{G} \longrightarrow f_{(*)}(\mathcal{F} \otimes f^{(*)}\mathcal{G})$$

of sheaves by adjunction from the composition

$$f^{(*)}(f_{(*)}\mathcal{F} \otimes \mathcal{G}) \xrightarrow{\sim} f^{(*)} f_{(*)}\mathcal{F} \otimes f^{(*)}\mathcal{G} \longrightarrow \mathcal{F} \otimes f^{(*)}\mathcal{G},$$

where the first morphism is the usual isomorphism encoding the compatibility of tensor product and inverse image and the second morphism comes from the adjunction. This morphism obviously induces a morphism

$$(f_{(!)}\mathcal{F}) \otimes \mathcal{G} \longrightarrow f_{(!)}(\mathcal{F} \otimes f^{(*)}\mathcal{G})$$

on the level of proper direct images.

6.2 Theorem (Projection formula). *Let $f: Y \to X$ be a separated locally proper map and $\mathcal{F} \in \mathrm{Sh}(Y)$ and $\mathcal{G} \in \mathrm{Sh}(X)$. If $\mathcal{G}$ is flat the morphism from 6.1 is an isomorphism*

$$(f_{(!)}\mathcal{F}) \otimes \mathcal{G} \xrightarrow{\sim} f_{(!)}(\mathcal{F} \otimes f^{(*)}\mathcal{G}).$$

If $\mathcal{F}$ is in addition assumed to be f-c-soft, the same is true for $\mathcal{F} \otimes f^{(*)}\mathcal{G}$.

Proof. Proper base change 4.4 reduces the first claim to the case that X consists of a single point which is known, see [6, Proposition 2.5.13]. The last claim follows from [6, Proposition 2.5.12]. □

6.3 Theorem (Derived projection formula). *Let $f: Y \to X$ be separated locally proper. Then there is a canonical isomorphism*

$$(f_!\mathcal{F}) \otimes^{\mathrm{L}} \mathcal{G} \xrightarrow{\sim} f_!(\mathcal{F} \otimes^{\mathrm{L}} f^*\mathcal{G})$$

which is natural in $\mathcal{F} \in D^+(Y)$ and $\mathcal{G} \in D^+(X)$.

Proof. We can assume that $\mathcal{F}$ is a bounded below complex of injective sheaves and that $\mathcal{G}$ is a bounded below complex of flat sheaves. Then the claim follows from the underived projection formula 6.2 and 5.5. □

7. Verdier duality

7.1. If Y is a topological space, $\mathcal{G}$ a sheaf on Y and $j: V \subset Y$ the embedding of an open subset we define $\mathcal{G}_{V \subset Y} := j_{(!)} j^{(*)} \mathcal{G} = j_{(!)} j^{(!)} \mathcal{G} \in \mathrm{Sh}(Y)$, cf. 3.4. Since both $j_{(!)}$ and $j^{(*)} = j^{(!)}$ are exact we have $\mathcal{G}_{V \subset Y} := j_! j^* \mathcal{G}$ in the derived category. We write $\mathbb{Z}_{V \subset Y}$ instead of $(\mathbb{Z}_Y)_{V \subset Y} = j_{(!)} \mathbb{Z}_V$.

7.2 Lemma. *Let $f: Y \to X$ be separated and locally proper. Then a sheaf $\mathcal{G} \in \mathrm{Sh}(Y)$ is f-c-soft if and only if the sheaf $\mathcal{G}_{V \subset Y}$ is $f_{(!)}$-acyclic for all open subsets $V \subset Y$. In particular, if $\mathcal{G}$ is f-c-soft then so is $\mathcal{G}_{V \subset Y}$.*

Proof. Denote by $i = i_x: \{x\} \hookrightarrow X$ the embedding of a point $x \in X$. A sheaf $\mathcal{F} \in \mathrm{Sh}(Y)$ is $f_{(!)}$-acyclic if and only if $\mathcal{H}^\nu i_x^* f_! \mathcal{F} \neq 0$ implies $\nu = 0$ for all $\nu \in \mathbb{Z}$ and $x \in X$.

Let $j: V \subset Y$ be the inclusion of an open subset and consider for arbitrary $x \in X$ the following diagram with two cartesian squares.

$$\begin{array}{ccc}
V \cap f^{-1}(x) & \xrightarrow{\quad l \quad} & V \\
\downarrow u & & \downarrow j \\
f^{-1}(x) & \xhookrightarrow{\quad k \quad} & Y \\
\downarrow g & & \downarrow f \\
\{x\} & \xhookrightarrow{\quad i \quad} & X
\end{array}$$

Several proper base changes 5.10 show

$$i^* f_! \mathcal{G}_{V \subset Y} \cong g_! k^* \mathcal{G}_{V \subset Y} = g_! k^* j_! j^* \mathcal{G} \cong g_! u_! l^* j^* \mathcal{G} \cong g_! u_! u^* k^* \mathcal{G}.$$

Using the above criterion, we deduce that $\mathcal{G}_{V \subset Y}$ is $f_{(!)}$-acyclic if and only if the sheaf $(k^* \mathcal{G})_{(V \cap f^{-1}(x)) \subset f^{-1}(x)}$ is $\Gamma_!$-acyclic for every $x \in X$. Now we use [6, Exercise II.6]. For the last claim use that proper base change 4.4 implies that $(\mathcal{G}_{U \subset X})_{V \subset X} \cong \mathcal{G}_{U \cap V \subset X}$ for open subsets U and V of X. $\square$

7.3 Lemma. *Let $f: Y \to X$ be a separated locally proper map with $f_{(!)}$ of finite cohomological dimension $\leq d$. If*

$$\mathcal{G}^0 \longrightarrow \cdots \longrightarrow \mathcal{G}^{d-1} \longrightarrow \mathcal{G}^d \longrightarrow 0$$

is an exact sequence in $\mathrm{Sh}(Y)$ and $\mathcal{G}^0, \mathcal{G}^1, \ldots, \mathcal{G}^{d-1}$ are f-c-soft then so is $\mathcal{G}^d$.

Proof. We use 7.2. Let $V \subset Y$ be an open subset. We have to show that $(\mathcal{G}^d)_{V\subset Y}$ is $f_{(!)}$-acyclic. The sequence

$$(\mathcal{G}^0)_{V\subset Y} \longrightarrow \cdots \longrightarrow (\mathcal{G}^{d-1})_{V\subset Y} \longrightarrow (\mathcal{G}^d)_{V\subset Y} \longrightarrow 0$$

is exact in $\mathrm{Sh}(Y)$ and all sheaves $(\mathcal{G}^0)_{V\subset Y}, \ldots, (\mathcal{G}^{d-1})_{V\subset Y}$ are $f_{(!)}$-acyclic. Let $\mathcal{K}^i$ be the kernel of $\mathcal{G}^i \to \mathcal{G}^{i+1}$ for $i = 0, \ldots, d-1$. For $p > 0$ we obtain isomorphisms

$$\mathrm{R}^p f_{(!)}((\mathcal{G}^d)_{V\subset Y}) \xrightarrow{\sim} \mathrm{R}^{p+1} f_{(!)}(\mathcal{K}^{d-1}_{V\subset Y}) \xrightarrow{\sim} \ldots \xrightarrow{\sim} \mathrm{R}^{p+d} f_{(!)}(\mathcal{K}^0_{V\subset Y}) = 0.$$

This shows that $(\mathcal{G}^d)_{V\subset Y}$ is $f_{(!)}$-acyclic. □

7.4 Lemma. *Let $f\colon Y \to X$ be a separated locally proper map with $f_{(!)}$ of finite cohomological dimension. Let $\mathcal{K} \in \mathrm{Sh}(Y)$ be a flat and f-c-soft sheaf. Then $\mathcal{G} \otimes \mathcal{K}$ is f-c-soft for any sheaf $\mathcal{G} \in \mathrm{Sh}(Y)$.*

Proof. Any sheaf $\mathcal{G}$ on Y has a resolution

$$\ldots \longrightarrow \mathcal{G}^{-1} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow 0$$

where each $\mathcal{G}^j$ is a direct sum of sheaves of the form $\mathbb{Z}_{V\subset Y}$ with $V \subset Y$ open. We obtain an exact sequence

$$\ldots \longrightarrow \mathcal{G}^{-1} \otimes \mathcal{K} \longrightarrow \mathcal{G}^0 \otimes \mathcal{K} \longrightarrow \mathcal{G} \otimes \mathcal{K} \longrightarrow 0.$$

By 7.3 it is sufficient to show that each $\mathcal{G}^j \otimes \mathcal{K}$ is f-c-soft. But $\mathcal{G}^j \otimes \mathcal{K}$ is isomorphic to a direct sum of sheaves of the form $\mathbb{Z}_{V\subset Y} \otimes \mathcal{K} \cong \mathcal{K}_{V\subset Y}$ which are f-c-soft by 7.2, so that we can use 5.4. □

7.5 Proposition. *Let $f\colon Y \to X$ be a separated locally proper map with $f_{(!)}$ of finite cohomological dimension, and let $\mathcal{K} \in \mathrm{Sh}(Y)$ be a flat and f-c-soft sheaf. Then the functor*

$$f^{\mathcal{K}}_{(!)} := f_{(!)}(- \otimes \mathcal{K})\colon \mathrm{Sh}(Y) \longrightarrow \mathrm{Sh}(X)$$

preserves colimits and therefore by 11.3 admits a right adjoint $f^{(!)}_{\mathcal{K}}$. Furthermore $f^{\mathcal{K}}_{(!)}$ is exact and therefore its right adjoint $f^{(!)}_{\mathcal{K}}$ maps injective sheaves to injective sheaves.

7.6. Any morphism of functors induces a morphism in the opposite direction between the adjoint functors, if they exist. In particular, any morphism $\mathcal{K} \to \mathcal{L}$ of flat f-c-soft sheaves will lead to a morphism $f^{(!)}_{\mathcal{L}} \to f^{(!)}_{\mathcal{K}}$.

Proof. The functor $(- \otimes \mathcal{K})$ preserves colimits, is exact, and maps every sheaf to an f-c-soft sheaf by 7.4. Then $f_{(!)}^{\mathcal{K}}$ is exact by 5.5 and preserves colimits, because it preserves filtered colimits by 4.6, in particular direct sums, and is right exact. Therefore, we can apply 11.3. The remaining claim is obvious. □

7.7 Theorem (Verdier duality). *Let $f: Y \to X$ be a separated locally proper map with $f_{(!)}: \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ of finite cohomological dimension. Then the derived proper direct image functor $f_!: D^+(Y) \to D^+(X)$ has a right adjoint functor*

$$f^!: D^+(X) \longrightarrow D^+(Y).$$

Proof. Let d be the cohomological dimension of $f_{(!)}$. Let $\mathbb{Z}_Y \hookrightarrow \mathcal{K}$ alias

$$0 \longrightarrow \mathbb{Z}_Y \hookrightarrow \mathcal{K}^0 \longrightarrow \mathcal{K}^1 \longrightarrow \ldots \longrightarrow \mathcal{K}^d \longrightarrow 0$$

be the Godement resolution truncated in degree d: by this we mean that each $\mathcal{K}^i$ for $i = 0, \ldots, d-1$ is the sheaf of not necessarily continuous sections of the cokernel of the previous map and that $\mathcal{K}^d$ is the cokernel of the previous map. Then $\mathcal{K}$ consists of flat sheaves, by 10.5, and even of f-c-soft sheaves, by 5.3 and 7.3.

Given $\mathcal{F} \in C^+(X)$ we construct a double complex with entries $f_{\mathcal{K}^{-p}}^{(!)}(\mathcal{F}^q)$ by applying our functors from 7.5. We denote its total complex by $f_{\mathcal{K}}^{(!)}(\mathcal{F})$. In this way we obtain a functor

$$f_{\mathcal{K}}^{(!)}: C^+(X) \longrightarrow C^+(Y)$$

which is right adjoint to the functor

$$f_{(!)}^{\mathcal{K}} := f_{(!)}(- \otimes \mathcal{K}): C^+(Y) \longrightarrow C^+(X)$$

and transforms complexes of injective sheaves to complexes of injective sheaves by 7.5.

Let $\mathcal{G} \in C^+(Y)$ be arbitrary. Since $\mathcal{K}$ is a bounded complex of flat sheaves we can assume that $\mathcal{G} \otimes^{\mathrm{L}} \mathcal{K} = \mathcal{G} \otimes \mathcal{K}$ in $D^+(Y)$. Since $\mathcal{K}$ is a bounded complex of flat and f-c-soft sheaves, 7.4 and 5.5 show that $\mathcal{G} \otimes \mathcal{K}$ consists of f-c-soft and hence $f_{(!)}$-acyclic sheaves. The quasi-isomorphism $\mathbb{Z}_Y \to \mathcal{K}$ then shows that the obvious morphisms

$$f_{(!)}^{\mathcal{K}}(\mathcal{G}) = f_{(!)}(\mathcal{G} \otimes \mathcal{K}) \longrightarrow f_!(\mathcal{G} \otimes \mathcal{K}) = f_!(\mathcal{G} \otimes^{\mathrm{L}} \mathcal{K}) \leftarrow f_!(\mathcal{G})$$

are isomorphisms in $D^+(X)$. Let

$$f_{\mathcal{K}}^!: D^+(X) \longrightarrow D^+(Y)$$

be the right derived functor of $f_{\mathcal{K}}^{(!)}: K^+(X) \to D^+(Y)$. As usual it may be computed using injective resolutions.

Now let $\mathcal{G} \in C^+(Y)$ be arbitrary and $\mathcal{F} \in C^+(X)$ a complex of injective sheaves. Then the facts stated above show that all maps in the following diagram are isomorphisms.

$$
\begin{array}{ccc}
\mathrm{Hom}_{K(X)}(f_{(!)}(\mathcal{G} \otimes \mathcal{K}), \mathcal{F}) & \xleftarrow{\sim} & \mathrm{Hom}_{K(Y)}(\mathcal{G}, f^{(!)}_{\mathcal{K}}(\mathcal{F})) \\
\downarrow \wr & & \\
\mathrm{Hom}_{D(X)}(f_{(!)}(\mathcal{G} \otimes \mathcal{K}), \mathcal{F}) & & \downarrow \wr \\
\uparrow \wr & & \\
\mathrm{Hom}_{D(X)}(f_!(\mathcal{G} \otimes^{\mathrm{L}} \mathcal{K}), \mathcal{F}) & & \\
\downarrow \wr & & \\
\mathrm{Hom}_{D(X)}(f_!(\mathcal{G}), \mathcal{F}) & & \mathrm{Hom}_{D(Y)}(\mathcal{G}, f^!_{\mathcal{K}}(\mathcal{F}))
\end{array}
$$

All maps are natural in complexes $\mathcal{G} \in C^+(Y)$ and complexes of injective sheaves $\mathcal{F} \in C^+(X)$. This yields the desired adjunction by setting $f^! = f^!_{\mathcal{K}}$. □

8. The case of unbounded derived categories

8.1. We finally explain how our results generalize to unbounded derived categories as soon as the relevant involved maps f are separated locally proper with $f_{(!)}$ of finite cohomological dimension.

8.2. The derived functors f^*, f_*, $\mathrm{R}\mathcal{H}om$, $\otimes^{\mathrm{L}}$ all exist on the level of unbounded derived categories (see [7, Ch. 18]) and so does $f_!$ by 12.1 and 12.4.

8.3 Theorem (cf. [10, Theorem B]). *Let $f: Y \to X$ be a separated locally proper map such that $f_{(!)}$ has finite cohomological dimension.*

(a) *The derived proper direct image functor $f_!: D(Y) \to D(X)$ has a right adjoint functor $f^!: D(X) \to D(Y)$.*

(b) *If $g: Z \to Y$ is another separated locally proper map such that $g_{(!)}$ has finite cohomological dimension then $f \circ g$ is separated locally proper with $(f \circ g)_{(!)}$ of finite cohomological dimension and there are isomorphisms $(f \circ g)_! \xrightarrow{\sim} f_! \circ g_!$ and $(f \circ g)^! \xleftarrow{\sim} g^! \circ f^!$ of triangulated functors.*

(c) *Let a cartesian diagram $f \circ q = p \circ g$ of topological spaces as in 4.1 be given. Then g is separated locally proper with $g_{(!)}$ of finite cohomological dimension and there are isomorphisms*

$$p^* \circ f_! \xrightarrow{\sim} g_! \circ q^* \quad \textit{resp.} \quad f^! \circ p_* \xleftarrow{\sim} q_* \circ g^!$$

in the space of triangulated functors from $D(Y)$ to $D(Z)$ resp. from $D(Z)$ to $D(Y)$.

(d) *For all $\mathcal{A} \in D(Y)$ and $\mathcal{B}, \mathcal{C} \in D(X)$ there are natural isomorphisms*

$$(f_!\mathcal{A}) \otimes^{\mathrm{L}} \mathcal{B} \xrightarrow{\sim} f_!(\mathcal{A} \otimes^{\mathrm{L}} f^*\mathcal{B}),$$

$$\mathrm{R}\mathcal{H}om\,(f_!\mathcal{A}, \mathcal{B}) \xleftarrow{\sim} f_*\,\mathrm{R}\mathcal{H}om\,(\mathcal{A}, f^!\mathcal{B}),$$

$$f^!\,\mathrm{R}\mathcal{H}om\,(\mathcal{B}, \mathcal{C}) \xleftarrow{\sim} \mathrm{R}\mathcal{H}om\,(f^*\mathcal{B}, f^!\mathcal{C}).$$

8.4. Verdier duality for unbounded derived categories has been proved in [10] for continuous maps f between locally compact Hausdorff spaces that satisfy a condition that is slightly weaker than our condition on the cohomological dimension of $f_{(!)}$. It is also possible to deduce it from Brown representability for well generated triangulated categories [9, Theorem 1.17], [8, Theorem 0.2].

Proof. (a). The proof of Verdier duality in the bounded case given in 7.7 generalizes easily: When we omit all upper indices + there, 12.4.(b) and the quasi-isomorphism $\mathbb{Z}_Y \to \mathcal{K}$ still show that the obvious morphisms

$$f^{\mathcal{K}}_{(!)}(\mathcal{G}) = f_{(!)}(\mathcal{G} \otimes \mathcal{K}) \longrightarrow f_!(\mathcal{G} \otimes \mathcal{K}) = f_!(\mathcal{G} \otimes^{\mathrm{L}} \mathcal{K}) \longleftarrow f_!(\mathcal{G})$$

are isomorphisms in $D(X)$. Thus $f^{\mathcal{K}}_{(!)}$ maps acyclic complexes to acyclic complexes, and this implies that the functor

$$f^{(!)}_{\mathcal{K}}: C(X) \longrightarrow C(Y)$$

maps h-injective complexes to h-injective complexes. Apart from these additions, the argument is the same.

It is enough to prove the first of the isomorphisms stated in each of the claims (b), (c), and (d) because the remaining isomorphisms are formal consequences using the Yoneda lemma and various adjunctions, most prominently the Verdier duality adjunction $(f_!, f^!)$.

(b). The composition $f \circ g$ is separated locally proper by 2.6 and 9.11; moreover, $(f \circ g)_{(!)}$ has finite cohomological dimension by 5.9 and 8.5. Let $\mathcal{F} \in C(Z)$ be a fibrant object. Then all components of $\mathcal{F}$ are injective sheaves by 12.3. As in the proof of 5.9 we see that all components of $g_{(!)}\mathcal{F}$ are $f_{(!)}$-acyclic. Lemma 12.4.(b) shows that the obvious morphism

$$f_{(!)}(g_{(!)}\mathcal{F}) \longrightarrow f_!(g_{(!)}\mathcal{F})$$

is an isomorphism in $D(X)$. Since $\mathcal{F}$ is h-injective we have $g_{(!)}\mathcal{F} \xrightarrow{\sim} g_!\mathcal{F}$ in $D(Y)$ and $f_{(!)}(g_{(!)}\mathcal{F}) = (f \circ g)_{(!)}(\mathcal{F}) \xrightarrow{\sim} (f \circ g)_!(\mathcal{F})$ in $D(X)$ where 3.6 is used for the equality.

(c). Clearly, g is separated locally proper by 9.11 and 2.7. Derived proper base change 5.10 reduces the question whether $g_{(!)}$ has finite cohomological dimension to the case that $Z = \{x\}$ for some $x \in X$ and $W = f^{-1}(x)$. Then $q\colon W \to Y$ is an embedding and any sheaf $\mathcal{E} \in \mathrm{Sh}(W)$ satisfies $\mathcal{E} \xrightarrow{\sim} q^{(*)}q_{(*)}\mathcal{E} = q^*q_{(*)}\mathcal{E}$. This and derived proper base change again shows $g_!\mathcal{E} \xrightarrow{\sim} g_!q^*q_{(*)}\mathcal{E} \xleftarrow{\sim} p^*f_!q_{(*)}\mathcal{E}$. This shows that $g_{(!)}$ has finite cohomological dimension.

Let Z again be arbitrary. Let $\mathcal{F} \in C(Y)$ be a fibrant object. As in the proof of 5.10 we see that $q^{(*)}\mathcal{F}$ consists of $g_{(!)}$-acyclic sheaves. Then

$$g_{(!)}(q^{(*)}\mathcal{F}) \xrightarrow{\sim} g_!(q^{(*)}\mathcal{F}) \xrightarrow{\sim} g_!(q^*\mathcal{F})$$

in $D(Z)$ where the first isomorphism comes from 12.4.(b) and the second one from the isomorphism $q^{(*)}\mathcal{F} \xrightarrow{\sim} q^*\mathcal{F}$ in $D(W)$. Now use proper base change 4.4.

(d). Following [10, Proposition 6.18] we prove the derived projection formula. From 6.1 we obtain a natural morphism

$$(\star) \qquad (f_{(!)}\mathcal{A}) \otimes \mathcal{B} \longrightarrow f_{(!)}(\mathcal{A} \otimes f^{(*)}\mathcal{B})$$

in $C(X)$. Assume that $\mathcal{A}$ is a complex of f-c-soft sheaves on Y and that $\mathcal{B}$ is an object of $\overrightarrow{\mathfrak{P}}(X)$ in the notation of [10, Sect. 5]. In particular, $\mathcal{B}$ is h-flat, and $\mathcal{A}$ consists of $f_{(!)}$-acyclic sheaves by 5.5. Hence the left hand side of $(\star)$ computes $(f_!\mathcal{A}) \otimes^{\mathrm{L}} \mathcal{B}$ by 12.4.(b). On the other hand, $f^{(*)}\mathcal{B}$ is certainly h-flat so that $\mathcal{A} \otimes^{\mathrm{L}} f^{(*)}\mathcal{B} = \mathcal{A} \otimes f^{(*)}\mathcal{B}$. We claim that $\mathcal{A} \otimes f^{(*)}\mathcal{B}$ has f-c-soft components. If $\mathcal{B}'$ is a bounded above complex of sheaves on X whose components are direct sums of sheaves of the form $\mathbb{Z}_{U \subset X}$ with $U \subset X$ open, i. e. $\mathcal{B}' \in \mathfrak{P}(X)$ in the notation of [10], then all components of $\mathcal{A} \otimes f^{(*)}\mathcal{B}'$ are f-c-soft as direct sums of f-c-soft sheaves $\mathcal{A}^p \otimes f^{(*)}\mathcal{B}'^q$, by 5.4 and 6.2. Recall that $\overrightarrow{\mathfrak{P}}(X)$ is the closure of $\mathfrak{P}(X)$ under certain filtered colimits (see [10, 2.9]) in $C(X)$. Since $(\mathcal{A} \otimes -)$ commutes with filtered colimits and $f^{(*)}$ commutes with all colimits, this implies our claim that $\mathcal{A} \otimes f^{(*)}\mathcal{B}$ has f-c-soft components (using 5.4 again). Then 12.4.(b) again shows that the right hand side of ($\star$) computes $f_!(\mathcal{A} \otimes f^*\mathcal{B})$.

Hence it is certainly enough to show that ($\star$) is an isomorphism. Since all the functors $f_{(!)}$, $\otimes$ and $f^{(*)}$ commute with filtered colimits, we can assume without loss of generality that $\mathcal{B} \in \mathfrak{P}(X)$ and even that $\mathcal{B} = \mathbb{Z}_{U \subset X}$ for some open subset $U \subset X$. But in this case ($\star$) is an isomorphism by 6.2. This establishes the derived projection formula. □

8.5 Corollary. *In the setting of 8.3 let d be the cohomological dimension of $f_{(!)}$. Then*

$$f_!(D^{\geq 0}(Y)) \subset D^{\geq 0}(X),$$
$$f_!(D^{\leq 0}(Y)) \subset D^{\leq +d}(X),$$
$$f^!(D^{\geq 0}(X)) \subset D^{\geq -d}(Y).$$

Proof. The first claim holds (for any continuous map f) because any object of $D^{\geq 0}(X)$ is isomorphic to a complex of injective sheaves whose components in degrees < 0 are zero. We have seen in the above proof that $f_!(\mathcal{G}) \cong f_{(!)}^{\mathcal{K}}(\mathcal{G})$ in $D(X)$. Hence $f_!(D^{\leq 0}(X)) \subset D^{\leq +d}(Y)$. Then $f^!(D^{\geq 0}(X)) \subset D^{\geq -d}(Y)$ follows from the adjunction $(f_!, f^!)$ or from the explicit construction of the functor $f^!$. □

9. Reminders from topology

9.1. A map $f : Y \to X$ of topological spaces is called *proper* if it is continuous and the map $f \times \mathrm{id} : Y \times Z \to X \times Z$ is closed for each topological space Z. All properties of proper maps we mention in the following can be found in [3, I.§ 10] or [13, Section 005M].

9.2. A topological space X is compact if and only if the constant map from X to a space consisting of a single point is proper.

9.3. An embedding is proper if and only if it is closed. Every composition of proper maps is proper.

9.4. The property of being proper is stable under base change: given a proper map $Y \to X$ and an arbitrary continuous map $Z \to X$ the map $Z \times_X Y \to Z$ is proper. In particular, a map is proper if and only if it is universally closed. There are two other important characterizations of proper maps. A continuous map is proper if and only if it is closed and the inverse image of every compact subset is compact. A continuous map is proper if and only if it is closed and all its fibers are compact.

9.5. Properness is local on the target: let $f: Y \to X$ be a map and $\mathcal{U}$ an open covering of X. Then f is proper if and only if the induced maps $f^{-1}(U) \to U$ are proper for all $U \in \mathcal{U}$. Note however that properness is not local on the source (cf. 2.8 for our usage of this notion).

9.6. If a composition $f \circ g$ of continuous maps is proper and g is surjective, then f is proper.

9.7. If $Y \to X$ and $Y' \to X$ are proper maps, so is $(Y \sqcup Y') \to X$. In particular, given a continuous map $Z \to X$ and subspaces $Y, Y' \subset Z$ with $Y \to X$ and $Y' \to X$ proper, the map $Y \cup Y' \to X$ is proper by 9.6.

9.8 Definition. A map $f: Y \to X$ of topological spaces is called *separated* if it is continuous and the diagonal map $Y \to Y \times_X Y$ is a closed embedding. The second condition is satisfied if and only if the diagonal is a closed subset of $Y \times_X Y$.

9.9 Definition. A subset A of a topological space Y is called *relatively Hausdorff* if any two distinct points of A have disjoint neighborhoods in Y.

9.10. A continuous map is separated if and only if all its fibers are relatively Hausdorff.

9.11. The constant map from a topological space X to a space consisting of a single point is separated if and only if X is a Hausdorff space. Every embedding is separated. Every composition of separated maps is separated. The property of being separated is stable under base change and local on the target, but not local on the source. If a composition $f \circ g$ of continuous maps is separated, g is separated. In particular, any continuous map whose source is a Hausdorff space is separated.

9.12 Lemma. *If a composition $f \circ g$ of continuous maps is proper and f is separated, then g is proper.*

9.13. This generalizes [3, Proposition I.10.1.5.(c)].

9.14. If the target of f consists of a single point, this lemma specializes to the fact that a continuous map from a compact space to a Hausdorff space is closed and even proper.

Proof. Let $g\colon Z \to Y$ and $f\colon Y \to X$. Consider the cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & Y \\ {\scriptstyle (\mathrm{id},g)}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{id},\mathrm{id})} \\ Z \times_X Y & \xrightarrow{g\times \mathrm{id}} & Y \times_X Y \end{array}$$

of topological spaces. The diagonal map $Y \hookrightarrow Y \times_X Y$ is a closed embedding and hence so is the map $(\mathrm{id}, g)\colon Z \to Z \times_X Y$ obtained by base change using 9.11. The morphism g is the composition of this latter map with the proper map $Z \times_X Y \to Y$, using 9.3 and 9.4. □

10. Reminders from sheaf theory

10.1 Proposition (Extension of sections on compacta). *Let $\mathcal{F}$ be a sheaf of sets on a topological space X. If $K \subset X$ is a compact relatively Hausdorff subset of X, any section $s \in \mathcal{F}(K)$ is the restriction of a section of $\mathcal{F}$ on an open neighborhood of K in X.*

Proof. Let $s \in \mathcal{F}(K)$. For any $x \in K$ there is an open neighborhood U_x of x in X and a section $s_x \in \mathcal{F}(U_x)$ such that $s_x|_{K\cap U_x} = s|_{K\cap U_x}$. Since K is locally compact as a compact Hausdorff space, there is a compact neighborhood K_x of x in K that is contained in $K \cap U_x$. Since K is compact, finitely many $K_{x_1}, \dots, K_{x_n}$ cover K. Put $K_i := K_{x_i}$, $U_i := U_{x_i}$ and $s_i := s_{x_i}$. We have $s_i|_{K_i} = s|_{K_i}$.

If $n = 1$ we are done. Otherwise compactness of $K_1 \cap K_2$ yields an open neighborhood W of $K_1 \cap K_2$ in $U_1 \cap U_2$ with $s_1|_W = s_2|_W$. Moreover, there are disjoint open neighborhoods $U_i' \subset U_i$ of $K_i \setminus W$ for $i = 1, 2$. Here we use that K is relatively Hausdorff and that $K_i \setminus W$ is compact. Then the three sections $s_1|_{U_1'}$, $s_2|_{U_2'}$ and $s_1|_W = s_2|_W$ glue to a section on $U_1' \cup U_2' \cup W$ which extends $s|_{K_1\cup K_2}$. An easy induction finishes the proof. □

10.2 Lemma. *On a locally compact Hausdorff space X, taking global sections with compact support commutes with filtered colimits, i. e.*

$$\varinjlim(\Gamma_!\mathcal{F}_i) \xrightarrow{\sim} \Gamma_!(\varinjlim \mathcal{F}_i)$$

for any filtered diagram $\mathcal{F}\colon I \to \mathrm{Sh}(X)$.

Proof. We use in the proof that the colimit of a system of sheaves is the sheafification of the colimit in the category of presheaves and that the canonical morphism from a presheaf to its sheafification induces isomorphisms on all stalks. We can assume without loss of generality that I is a partially ordered set.

Injectivity: Let $s \in \Gamma_!(X;\mathcal{F}_j)$ for some $j \in I$. Assume that s goes to zero in $\Gamma_!(X;\varinjlim \mathcal{F}_i)$. For every point $x \in X$ there are an open neighborhood $U(x)$ and an index $i(x)$ with $s \mapsto 0 \in \mathcal{F}_{i(x)}(U(x))$. Finitely many $U(x)$ cover the compact set $(\operatorname{supp} s)$. Let i be bigger than the finitely many $i(x)$ involved. Then $s \mapsto 0 \in \mathcal{F}_i(X)$ showing injectivity.

Surjectivity for X compact and Hausdorff: Let $s \in \Gamma(X;\varinjlim \mathcal{F}_i)$ be given. For every point $x \in X$ there are a neighborhood $U(x)$ in X, an index $i(x)$ and a section $s(x) \in \mathcal{F}_{i(x)}(U(x))$ with $s(x) \mapsto s|_{U(x)}$. Since X is locally compact we may assume that the neighborhoods $U(x)$ are compact. Since X is compact there is a finite subset $E \subset X$ such that the $U(x)$ with $x \in E$ cover X. Injectivity proved above shows that for any $x, y \in E$ there is some $j \in I$ such that the images of $s(x)$ and $s(y)$ in $\mathcal{F}_j(U(x) \cap U(y))$ coincide. Since E is finite we can assume that the index j works for all $x, y \in E$. But then the images of the $s(x)$ in $\mathcal{F}_j(U(x))$ for $x \in E$ glue to a global section of $\mathcal{F}_j$ which represents the inverse image of s we are looking for.

Surjectivity for X locally compact and Hausdorff: Let $s \in \Gamma_!(X;\varinjlim \mathcal{F}_i)$ be given. Since $(\operatorname{supp} s)$ is compact and X is locally compact and Hausdorff we find $U \subset X$ open with compact closure $\overline{U}$ and $(\operatorname{supp} s) \subset U$. We have already proved surjectivity of $\varinjlim \big(\Gamma(\overline{U};\mathcal{F}_i\big) \to \Gamma\big(\overline{U};\varinjlim \mathcal{F}_i\big)$. Hence there is an index j and a section $\tilde{s} \in \Gamma(\overline{U};\mathcal{F}_j)$ with $\tilde{s} \mapsto s|\overline{U}$. Then $\tilde{s}|\partial\overline{U} \mapsto 0 \in \Gamma(\partial\overline{U};\varinjlim \mathcal{F}_i)$ and injectivity proved above shows that $\tilde{s}|\partial\overline{U} \mapsto 0 \in \Gamma(\partial\overline{U};\mathcal{F}_l)$ for some $l \geq j$. Hence the image of $\tilde{s}$ in $\Gamma(\overline{U};\mathcal{F}_l)$ and the zero section $0 \in \Gamma(X \setminus U;\mathcal{F}_l)$ glue to a global section $\hat{s} \in \Gamma_!(X;\mathcal{F}_l)$ with compact support and $\hat{s} \mapsto s$. This proves the lemma. □

10.3 Definition. A sheaf is called *c-soft* (for compact-soft) if any section over a compact set comes from a global section.

10.4 Lemma (filtered colimits of c-soft sheaves). *On a locally compact Hausdorff space X, any filtered colimit of c-soft sheaves is c-soft.*

Proof. Let $\mathcal{F}\colon I \to \mathrm{Sh}(X)$ be a filtered diagram of c-soft sheaves. The inverse image functor of sheaves commutes with arbitrary colimits because it has a right adjoint functor. Let $u\colon K \hookrightarrow X$ be the embedding of a compact subset. Hence the obvious morphism is an isomorphism $\varinjlim u^{(*)}\mathcal{F}_i \xrightarrow{\sim} u^{(*)} \varinjlim \mathcal{F}_i$. This and 10.2

provide isomorphisms

$$\varinjlim \Gamma(K; u^{(*)}\mathcal{F}_i) \xrightarrow{\sim} \Gamma(K; \varinjlim u^{(*)}\mathcal{F}_i) \xrightarrow{\sim} \Gamma(K; u^{(*)} \varinjlim \mathcal{F}_i)$$

or concisely an isomorphism

$$\varinjlim \Gamma(K; \mathcal{F}_i) \xrightarrow{\sim} \Gamma(K; \varinjlim \mathcal{F}_i).$$

It follows that every section t of the group on the right is the image of a section $s \in \Gamma(K; \mathcal{F}_i)$ for some $i \in I$. Since $\mathcal{F}_i$ is c-soft there is a global section $\tilde{s} \in \Gamma(X; \mathcal{F}_i)$ with $\tilde{s}|_K = s$. The image of $\tilde{s}$ in $\Gamma(X; \varinjlim \mathcal{F}_i)$ then restricts to t. □

10.5 Lemma. *Let $\mathcal{F}$ be a sheaf on a topological space X and $\mathcal{G}\mathcal{F}$ the sheaf of not necessarily continuous sections of $\mathcal{F}$ from* [4, II.4.3]. *If $\mathcal{F}$ is flat, so are $\mathcal{G}\mathcal{F}$ and the cokernel* $\operatorname{cok}(\mathcal{F} \hookrightarrow \mathcal{G}\mathcal{F})$ *of the canonical monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}\mathcal{F}$.*

Proof. An abelian group is flat if and only if it is torsion-free, and these properties are preserved under products. For $x \in X$ we have

$$(\mathcal{G}\mathcal{F})_x = \varinjlim_{x \in U} (\mathcal{G}\mathcal{F})(U) = \varinjlim_{x \in U} \prod_{u \in U} \mathcal{F}_u$$

where U ranges over the open subsets of X containing x. Assume that $\mathcal{F}$ is flat. Then all $\mathcal{F}_u$ are flat abelian groups and it is easy to see that $(\mathcal{G}\mathcal{F})_x$ is torsion-free and hence flat. Hence $\mathcal{G}\mathcal{F}$ is flat. Since $\mathcal{F} \hookrightarrow \mathcal{G}\mathcal{F}$ induces split injections $\mathcal{F}_x \hookrightarrow (\mathcal{G}\mathcal{F})_x$ on each stalk the cokernel has flat stalks. □

11. Representability and adjoints

11.1 Lemma (Representability Lemma). *Let X be a topological space. A functor μ:* $\operatorname{Sh}(X) \to \mathrm{Ab}^{\mathrm{op}}$ *is representable if and only if it preserves all colimits.*

11.2. We learnt the above lemma from [12, Exp. 4, 1.0]. Our proof completes the proof given there.

Proof. It is clear that any representable functor commutes with colimits. Assume that μ is representable, i. e. there is a sheaf $\mathcal{C} \in \operatorname{Sh}(X)$ together with natural isomorphisms

$$\mu(\mathcal{F}) \xrightarrow{\sim} \operatorname{Sh}_X(\mathcal{F}, \mathcal{C})$$

for $\mathcal{F} \in \operatorname{Sh}(X)$. Plugging in $\mathcal{F} = \mathbb{Z}_{U \subset X}$ for $U \subset X$ open we obtain an isomorphism $\mu(\mathbb{Z}_{U\subset X}) \xrightarrow{\sim} \mathcal{C}(U)$. If $V \subset U$ is an open subset, we obtain in this way the horizontal isomorphisms in the commutative diagram

$$\begin{array}{ccc} \mu(\mathbb{Z}_{U\subset X}) & \xrightarrow{\sim} & \mathcal{C}(U) \\ \downarrow & & \downarrow \mathrm{res}_U^V \\ \mu(\mathbb{Z}_{V\subset X}) & \xrightarrow{\sim} & \mathcal{C}(V) \end{array}$$

whose left vertical morphism is the image under μ of the obvious morphism $\mathbb{Z}_{V\subset X} \to \mathbb{Z}_{U\subset X}$.

Now assume that μ preserves all colimits. Define a presheaf $\mathcal{C}_\mu$ by

$$\mathcal{C}_\mu(U) := \mu(\mathbb{Z}_{U\subset X})$$

with restriction maps coming from the morphisms $\mathbb{Z}_{V\subset X} \to \mathbb{Z}_{U\subset X}$. We first claim that $\mathcal{C}_\mu$ is a sheaf. Let $\mathcal{U}$ be an open covering of an open subset $V \subset X$. Consider the obvious coequalizer diagram

$$\bigoplus_{(U,U')\in\mathcal{U}^2} \mathbb{Z}_{(U\cap U')\subset X} \rightrightarrows \bigoplus_{U\in\mathcal{U}} \mathbb{Z}_{U\subset X} \to \mathbb{Z}_{V\subset X}.$$

It presents $\mathbb{Z}_{V\subset X}$ as a colimit. Since μ commutes with colimits we see that $\mathcal{C}_\mu$ is a sheaf.

Now let us show that $\mathcal{C}_\mu$ represents μ. Let Op_X be the category of open subsets of X with inclusions as morphisms and let $J\colon \mathrm{Op}_X \to \mathrm{Sh}(X)$ be the functor $U \mapsto \mathbb{Z}_{U\subset X}$. For any $\mathcal{F} \in \mathrm{Sh}(X)$ consider the morphisms

$$\begin{array}{ccc} \mu(\mathcal{F}) & \xrightarrow{\sim} & \mathrm{Fun}_{(\mathrm{Sh}(X),\mathrm{Ab}^{\mathrm{op}})}(\mu, \mathrm{Sh}_X(-,\mathcal{F})) \\ & & \downarrow \circ J \\ \mathrm{Sh}_X(\mathcal{F},\mathcal{C}_\mu) & \xleftarrow{\sim} & \mathrm{Fun}_{(\mathrm{Op}_X,\mathrm{Ab}^{\mathrm{op}})}(\mu\circ J, \mathrm{Sh}_X(J(-),\mathcal{F})) \end{array}$$

The upper horizontal morphism is the isomorphism from the Yoneda lemma, the vertical morphism is obvious, and the lower horizontal morphism comes from $\mu \circ J = \mathcal{C}_\mu$ and the isomorphism $\mathrm{Sh}_X(J(-),\mathcal{F}) \xrightarrow{\sim} \mathcal{F}(-)$.

We obtain a morphism $\gamma\colon \mu \to \mathrm{Sh}_X(-,\mathcal{C}_\mu)$ of functors $\mathrm{Sh}(X) \to \mathrm{Ab}^{\mathrm{op}}$. Since γ is an isomorphism at each object of the form $\mathbb{Z}_{U\subset X}$ and since both functors μ and $\mathrm{Sh}_X(-,\mathcal{C}_\mu)$ commute with colimits, γ is an isomorphism and $\mathcal{C}_\mu$ represents μ. □

11.3 Corollary ([12, Exp. 4, 1.1]). *Let X be a topological space and $\mathcal{B}$ an abelian category. A functor $\Lambda\colon \mathrm{Sh}(X) \to \mathcal{B}$ has a right adjoint if and only if it preserves colimits.*

11.4. Again, if we know a right adjoint R exists, it is not difficult to describe it: We just need to apply the adjunction isomorphism $\mathcal{B}(\Lambda\mathcal{F}, B) \xrightarrow{\sim} \mathrm{Sh}_X(\mathcal{F}, RB)$ to the sheaf $\mathcal{F} = \mathbb{Z}_{U\subset X}$ to find isomorphisms $\mathcal{B}(\Lambda\mathbb{Z}_{U\subset X}, B) \xrightarrow{\sim} (RB)(U)$ which describe the functor R quite explicitly.

Proof. Any left adjoint functor certainly preserves colimits. If Λ preserves colimits, apply Lemma 11.1 to the functor $\mathcal{B}(\Lambda(-), B)$ where $B \in \mathcal{B}$ is fixed. □

12. Remarks on derived functors

12.1. We will apply the results of this subsection mainly to the category $\mathrm{Sh}(X)$ of sheaves on a topological space X. This category is a Grothendieck (abelian) category by [7, 18.1.6.(v)]. We do not need the full strength of Proposition 12.2 below; the results of [7] are sufficient for our purposes, cf. 12.3.

12.2 Proposition ([1, Proposition 3.13]). *Let $\mathcal{A}$ be a Grothendieck category. Then there is a cofibrantly generated model structure on $C(\mathcal{A})$ such that the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms. We call it the injective model structure on $C(\mathcal{A})$.*

12.3. The fibrant objects of this model structure on $C(\mathcal{A})$ are precisely the objects that are h-injective and componentwise injective ([7, Proposition 14.1.6 in the setting of Section 14.3] where the class of trivial cofibrations is called QM and the fibrant objects are called QM-injective). Any bounded below complex of injective objects is h-injective ([7, 13.2.4]) and hence fibrant. The axioms of a model structure imply that any object of $C(\mathcal{A})$ admits a trivial cofibration to a fibrant object. This is also proved in [7, Theorem 14.1.7].

12.4 Lemma. *Let $\mathcal{A}$ be a Grothendieck category and $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor to an abelian category $\mathcal{B}$. We also denote by $F\colon K(\mathcal{A}) \to D(\mathcal{B})$ the composition of the induced functor $K(\mathcal{A}) \to K(\mathcal{B})$ on homotopy categories with the canonical functor $K(\mathcal{B}) \to D(\mathcal{B})$.*

(a) *$F\colon K(\mathcal{A}) \to D(\mathcal{B})$ admits a right derived functor $\mathrm{R}F\colon D(\mathcal{A}) \to D(\mathcal{B})$.*

(b) *If $F\colon \mathcal{A} \to \mathcal{B}$ is of finite cohomological dimension, then for all componentwise F-acyclic complexes $S \in C(\mathcal{A})$ the obvious morphism $F(S) \to \mathrm{R}F(S)$ in $D(\mathcal{B})$ is an isomorphism.*

PROOF. We equip $C(\mathcal{A})$ with the injective model structure from Proposition 12.2. As mentioned in 12.3, its fibrant objects are h-injective so that we get (a).

(b). Let $\mathcal{S} \subset C(\mathcal{A})$ be the full subcategory of componentwise F-acyclic complexes. Let $T \in \mathcal{S}$ be acyclic. We claim that $F(T) = 0$ in $D(\mathcal{B})$. Let K^i be the kernel of $T^i \to T^{i+1}$. The short exact sequence $K^{i-1} \hookrightarrow T^{i-1} \twoheadrightarrow K^i$ provides isomorphisms $H^p(\mathrm{R}F(K^i)) \xrightarrow{\sim} H^{p+1}(\mathrm{R}F(K^{i-1}))$ for all $p > 0$. By induction we obtain $H^p(\mathrm{R}F(K^i)) \xrightarrow{\sim} H^{p+N}(\mathrm{R}F(K^{i-N}))$ for all $p > 0$ and all $N \in \mathbb{N}$. Since F has finite cohomological dimension this shows that K^i is F-acyclic. Hence we obtain short exact sequences $F(K^{i-1}) \hookrightarrow F(T^{i-1}) \twoheadrightarrow F(K^i)$ and see that $F(T)$ is acyclic.

Now let $S \to I$ be a fibrant resolution of some object $S \in \mathcal{S}$. By 12.3 any fibrant object is componentwise injective thus we have $I \in \mathcal{S}$. Hence the cone of $S \to I$ is an acyclic object of $\mathcal{S}$ and mapped to zero in $D(\mathcal{B})$ by the above claim, so that $F(S) \to F(I)$ is a quasi-isomorphism. □

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