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On finite p-groups that are the product of a subgroup of class two and an abelian subgroup of order p^3

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Abstract – In this note it is shown that if $G = AB$ is a finite p-group that is the product of an abelian subgroup A of order p^3 and a subgroup B of nilpotency class two, then G can have derived length at most three.

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The question of the relationship between the derived length of a finite group that is the product of two nilpotent subgroups and the nilpotency classes of the factors goes back to the well-known conjecture, mentioned by Scott in [8] (p. 385) and by Kegel in [6], that the derived length of the product should be bounded by the sum of the classes of the factors. The examples of Cossey and Stonehewer [2] eventually showed that the derived length of a product can exceed the sum of the classes of the factors. In particular, Cossey and Stonehewer constructed examples of finite p-groups of derived length four that can be expressed as the product of an abelian subgroup and a subgroup of class two. In these examples the abelian factor is quite large, being of order p^{p^3} for p odd, and the question arises as to how small the abelian factor can be in such a product of derived length four. Another significant result concerning factorised finite p-groups $G = AB$, where A is abelian, is that of Morigi ([7], Theorem 2, or [1], Theorem 3.3.11), which shows that if $|B'| = p^n$ then G can have derived length at most $n + 2$, while the results of Jabara ([4], [5] or [1], Corollary 3.3.25) show that if B has rank 2 and class k then $G^{(2k)} = 1$, whereas if B has an abelian subgroup of index p^{n-1} then G has derived length at most $2n$.

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The purpose of the present note is to consider one very particular case, namely finite p-groups $G = AB$ that are the product of an abelian subgroup A of order p^3 and a subgroup B of class two. It will be shown that such groups can have derived length at most three. Taken in conjunction with Morigi's result, this shows in particular that a finite p-group of derived length four that factorises as the product of an abelian subgroup and a subgroup of class two will require an abelian factor of order at least p^4 and a factor of class two whose derived subgroup is of order at least p^2.

We first examine a rather restricted normal product of two subgroups of class at most two.

Lemma 1. *Let the finite group G be the product $G = HK$, for subgroups H and K such that*

(i) *$H \trianglelefteq G$ and $K \trianglelefteq G$;*

(ii) *$|G : H| = |G : K| = p$, where p is a prime;*

(iii) *H and K are nilpotent of class at most two.*

Then $G^{(2)} = 1$ and $|G'| \leq p|H'K'|$.

Proof. We have

$$G/H \cong G/K \cong C_p,$$

which is abelian, so $H'K' \leqslant G' \leqslant H \cap K$. Since H and K both have class at most two we have $H' \leqslant Z(H)$ and $K' \leqslant Z(K)$, whence $H'K' \leqslant Z(H \cap K)$. In addition we see that $[G, H \cap K] = [H, H \cap K][K, H \cap K] \leqslant H'K'$. Thus

$$(H \cap K)/H'K' \leqslant Z(G/H'K').$$

Now $H/(H \cap K) \cong HK/K = G/K \cong C_p$, so

$$|G : H \cap K| = |G : H||H : H \cap K| = p^2.$$

Since $G/(H \cap K)$ can be embedded in $G/H \times G/K$ ($\cong C_p \times C_p$) we have, by comparison of orders,

$$G/(H \cap K) = H/(H \cap K) \times K/(H \cap K) \cong C_p \times C_p.$$

We let $1 \neq x \in H \backslash (H \cap K)$ and $1 \neq y \in K \backslash (H \cap K)$. Then $G = \langle x, y, H \cap K \rangle$ and $G'/H'K' = \langle [x, y] \rangle H'K'/H'K' \leqslant (H \cap K)/H'K' \leqslant Z(G/H'K')$. In addition $x^p H'K' \in (H \cap K)/H'K'$. Thus, since $[x, y]H'K' \in Z(G/H'K')$, we have

$$[x, y]^p H'K' = [x^p, y]H'K' = 1_{G/H'K'}.$$

Furthermore, since $[x, y] \in H \cap K$ and $H'K' \leqslant Z(H \cap K)$, we see that $G' = \langle [x, y], H'K' \rangle$ is abelian, whence $G^{(2)} = 1$. Finally we observe that $G'/H'K' = \langle [x, y] \rangle H'K'/H'K'$ is cyclic of order at most p, so it follows that $|G'| \leq p|H'K'|$. □

Corollary 2. *Let G be a finite p-group and let H be a subgroup of G such that*

(i) $|G : H| \leq p^2$;

(ii) *H has class at most two.*

Then $G^{(3)} = 1$.

Proof. If $H \trianglelefteq G$ then G/H is abelian and the result follows. If $H \ntrianglelefteq G$ then, since G is a finite p-group and $|G : H| \leq p^2$, we must have $|G : N_G(H)| = |N_G(H) : H| = p$, so $H \trianglelefteq N_G(H) \trianglelefteq G$. In addition, for $x \in G \backslash N_G(H)$ we have, by comparison of orders, $N_G(H) = HH^x$. Since H and H^x have class at most two and are (normal) subgroups of index p in $N_G(H)$, we see by Lemma 1 that $N_G(H)$ has derived length at most two. Since $G/N_G(H) \cong C_p$ is abelian, it follows that $G^{(3)} = 1$. □

An easy, and no doubt well-known, consequence of Corollary 2 is that a finite p-group that is the product of a subgroup of class two and an abelian subgroup of order p^2 can have derived length at most three.

We note the following elementary consequence of the famous Theorem of Itô ([3], Satz 1, or [1], Theorem 3.1.7):

Lemma 3. *Let the group $G = AB$ be the product of the subgroups A and B such that*

(i) *A is abelian*;

(ii) *$(B')^G$ is abelian.*

Then $G^{(3)} = 1$.

Proof. We see that $G/(B')^G = (A(B')^G/(B')^G)(B(B')^G/(B')^G)$ is the product of two abelian subgroups. By Itô's Theorem we have $G^{(2)} \leqslant (B')^G$. Since $(B')^G$ is abelian, we conclude that $G^{(3)} = 1$. □

We will use the next lemma in the proof of our main result, Theorem 5.

Lemma 4. *Let the group $G = AB$ be the product of the subgroups A and B such that*

(i) *A is abelian*;

(ii) *$B' \leqslant Z(B)$*;

(iii) *B is subnormal of defect at most two in G.*

Then $(B')^G$ is abelian (so $G^{(3)} = 1$ by Lemma 3).

Proof. Since B has defect at most two, there exists a normal subgroup $N \trianglelefteq G$ such that $B \trianglelefteq N \trianglelefteq G$. Then $N/B = (N \cap A)B/B$ is isomorphic to a factor group of A, so $N' \leqslant B$. But $N' \trianglelefteq G$ so $(B')^G \leqslant N'$ ($\leqslant B$). Hence $B' \leqslant Z(B) \cap (B')^G \leqslant Z((B')^G) \trianglelefteq G$, so we conclude that $(B')^G$ is abelian. □

Our main result provides some information about the structure of a finite p-group that is the product of an abelian subgroup of order p^3 and a subgroup of class two.

Theorem 5. *Let $G = AB$ be a finite p-group for subgroups A and B such that*

(i) *A is abelian*;

(ii) $|A| = p^3$;

(iii) $A \cap B = 1$;

(iv) $B' \leqslant Z(B)$.

Then one of the following holds:

1. *G has a subgroup B_1 (with possibly $B_1 = B$) such that*
   (a) *$G = AB_1$ and $A \cap B_1 = 1$*;
   (b) $B_1' \leqslant Z(B_1)$;
   (c) $(B_1)' \leqslant B'$;
   (d) *$(B_1')^G$ is abelian*;

or

2. *G has a subgroup B_2 such that*
   (a) $B_2' \leqslant Z(B_2)$;
   (b) $|B_2^G : B_2| = p$;
   (c) $|G : B_2^G| = p^2$.

PROOF. We use induction on $|B'|$ and note that if $(B')^G$ is abelian (in particular if $B' = 1$) then we may let $B_1 = B$ and see that the result is trivial. We therefore assume that $(B')^G$ is non-abelian. By Lemma 4 we may also assume that B has defect three in G. Now $|A| = p^3$, so we may further assume that there exist elements $w, x, y \in A$ for which the following is satisfied:

$$\begin{aligned} A &= \langle w, x, y\rangle \\ \langle w\rangle &= N_A(B) \cong C_p; \\ \langle w, x\rangle &= N_A(\langle w\rangle B); \\ y^p &\in \langle w, x\rangle, \quad \text{but } y \notin \langle w, x\rangle; \\ x^p &\in \langle w\rangle, \qquad \text{but } x \notin \langle w\rangle. \end{aligned}$$

We may further assume that $\langle w\rangle B = N_G(B)$ and $\langle w, x\rangle B = N_G(\langle w\rangle B) \trianglelefteq G$. Thus $B' \leqslant (\langle w, x\rangle B)' \trianglelefteq G$, so $(B')^G \leqslant (\langle w, x\rangle B)'$. But $\langle w, x\rangle B/\langle w\rangle B$ is abelian, so $(\langle w, x\rangle B)' \leqslant \langle w\rangle B$. Hence, in particular, we have $(B')^G \leqslant \langle w\rangle B$ and, by conjugation, $(B')^G \leqslant (\langle w\rangle B)^y = \langle w\rangle B^y$.

If $C_A(B') \neq 1$, then B is a proper subgroup of $C_G(B')$, and thus also of $N_{C_G(B')}(B) = N_G(B) \cap C_G(B')$. By comparison of orders it follows that $N_G(B) = \langle w\rangle B \leqslant C_G(B')$. Hence $B' \leqslant Z(\langle w\rangle B)$. But $(B')^G \leqslant \langle w\rangle B$, so $B' \leqslant Z(\langle w\rangle B) \cap (B')^G \leqslant Z((B')^G)$ and $(B')^G$ is in fact abelian, in contradiction to our assumption. Hence we may assume that $C_A(B') = 1$ so, in particular, we have $B = C_G(B')$ (and, for $g \in G$, $B^g = C_G((B')^g)$).

If $(B')^G \leqslant B$ then $B' \leqslant Z(B) \cap (B')^G \leqslant Z((B')^G)$ and $(B')^G$ is again abelian (which is excluded). Thus, since $(B')^{\langle w,x\rangle B} \leqslant ((\langle w\rangle B)')^{\langle w,x\rangle B} = (\langle w\rangle B)' \leqslant B$, we may further assume that $(B')^y \not\leqslant B$. Now, if $B^y \leqslant \langle w\rangle B$, then we obtain $(B')^y \leqslant (\langle w\rangle B)' \leqslant B$, which has been ruled out. Hence it follows that $B^y \not\leqslant \langle w\rangle B$.

Since $|\langle w\rangle B : B| = |\langle w, x\rangle B : \langle w\rangle B| = p$, we then have, by comparison of orders:

$$\langle w\rangle B = BB^x = B(B')^y$$

and

$$\langle w, x\rangle B = \langle w\rangle BB^y = B(B')^y B^y = BB^y.$$

In particular it follows that

$$p^2|B| = |\langle w, x\rangle B| = \frac{|B||B^y|}{|B \cap B^y|} = \frac{|B|}{|B \cap B^y|}|B^y|,$$

and so

$$|B : B \cap B^y| = \frac{|B|}{|B \cap B^y|} = p^2.$$

Now $|(B')^y : (B')^y \cap B| = |B(B')^y : B| = |\langle w\rangle B : B| = p$. Therefore, since $B = C_G(B')$, we have

$$|(B')^y : C_{(B')^y}(B')| = |(B')^y : (B')^y \cap C_G(B')| = |(B')^y : (B')^y \cap B| = p.$$

In particular we see that B' and $(B')^y$ do not centralise each other. Thus, since $C_G((B')^y) = B^y$, we have $B' \nleqslant B^y$, so $\langle w\rangle B^y = B^y B'$. It then similarly follows that

$$|B' : B' \cap B^y| = p.$$

In addition since $(B')^G \leqslant \langle w\rangle B = N_G(B)$, we have $B' \trianglelefteq (B')^G$. By conjugation $(B')^y$ is also normal in $(B')^G$, so B' and $(B')^y$ normalise each other. We note that

$$(B')^y \cap B = (B')^y \cap B \cap B^y \leqslant (B')^y \cap B'(B \cap B^y) \leqslant (B')^y \cap B,$$

and so

$$(B')^y \cap B'(B \cap B^y) = (B')^y \cap B.$$

We let $H = \langle B', (B')^y, B \cap B^y\rangle$. Then H can be expressed as the product $H = (B')^y B'(B \cap B^y)$ and we have

$$\begin{aligned}
|H| &= \frac{|(B')^y||B'(B \cap B^y)|}{|(B')^y \cap B'(B \cap B^y)|} \\
&= \frac{|(B')^y||B'(B \cap B^y)|}{|(B')^y \cap B|} \\
&= p|B'(B \cap B^y)| \\
&= p\frac{|B'||B \cap B^y|}{|B' \cap B \cap B^y|} \\
&= p\frac{|B'|}{|B' \cap B^y|}|B \cap B^y| \\
&= p^2|B \cap B^y|.
\end{aligned}$$

But $|B : B \cap B^y| = p^2$, so it follows that

$$|H| = p^2\Big(\frac{|B|}{p^2}\Big) = |B|.$$

Now B' and $(B')^y$ normalise each other, so

$$[B', (B')^y] \leqslant B' \cap (B')^y \leqslant Z(B) \cap Z(B^y) \leqslant Z(BB^y) = Z(\langle w, x\rangle B).$$

We similarly have

$$(B \cap B^y)' \leqslant B' \cap (B')^y \leqslant Z(\langle w, x\rangle B).$$

Hence, bearing in mind that both B' and $(B')^y$ are centralised by $B \cap B^y$, we have

$$\begin{aligned} H' &= \langle B', (B')^y, B \cap B^y\rangle' \\ &= [B', (B')^y](B \cap B^y)' \\ &\leqslant Z(\langle w, x\rangle B) \cap (B')^y \cap B'(\leqslant Z(H)). \end{aligned}$$

Thus H has class at most two. In addition we see that w centralises H' and that $H' \leqslant B'$. However w does not centralise B', so we further conclude that H' is a proper subgroup of B'.

If $A \cap H = 1$ then, since $|H| = |B|$, we have $G = AH$. As $|H'| < |B'|$ and $H' \leqslant B'$, we can then use induction on $|B'|$ to show that the result holds. Thus we may assume that $A \cap H \neq 1$. Since $H = \langle B', (B')^y, B \cap B^y\rangle \leqslant B(B')^y = \langle w\rangle B$, we have $A \cap H \leqslant A \cap \langle w\rangle B = \langle w\rangle$. But $\langle w\rangle$ has order p, so

$$A \cap H = \langle w\rangle.$$

We further have $|\langle w\rangle B : H| = |\langle w\rangle B : B| = p$, so $H \trianglelefteq \langle w\rangle B$. Therefore $\langle w\rangle B = HB$ is the (normal) product of the subgroups H and B, both of class at most two and of index p in $\langle w\rangle B$. Hence, by Lemma 1, we have $|(\langle w\rangle B)'| \le p|H'B'|$. But $H' \leqslant B'$, so we conclude

$$|(\langle w\rangle B)'| \le p|B'|.$$

If $(B')^x = B'$ then $x \in N_G(B')$, so x normalises $C_G(B') = B$, which is ruled out. Thus B' is a proper subgroup of $B'(B')^x$. Now $B'(B')^x \leqslant (\langle w\rangle B)'(= (BB^x)')$ and, from above, $|(\langle w\rangle B)'| \le p|B'|$, so, by comparison of orders, we have

$$(\langle w\rangle B)' = B'(B')^x.$$

Hence $C_G((\langle w\rangle B)') = C_G(B'(B')^x) = C_G(B') \cap C_G((B')^x) = B \cap B^x$. But $(\langle w\rangle B)' \trianglelefteq \langle w, x\rangle B$, so $B \cap B^x \trianglelefteq \langle w, x\rangle B$. Since both B and B^x have index p in $\langle w\rangle B$, we have $B' \leqslant B \cap B^x$. Thus if $B \cap B^x \trianglelefteq G$, then $(B')^G \leqslant B$, which has already been have ruled out. Therefore we may assume that $B \cap B^x \ntrianglelefteq G$, so $\langle w, x\rangle B = N_G(B \cap B^x)$.

We have $B^y \leqslant \langle w, x\rangle B = N_G(B \cap B^x)$ and, since $(B')^y \nleqslant B$, we also have $(B')^y \nleqslant B \cap B^x$. Hence $B^y(B \cap B^x)/(B \cap B^x)$ is a non-abelian group. But, since $|B : B \cap B^x| = |BB^x : B^x| = |\langle w\rangle B : B^x| = p$, we have

$$|\langle w, x\rangle B : B \cap B^x| = |\langle w, x\rangle B : B||B : B \cap B^x| = p^2 p = p^3.$$

Therefore, since $B^y(B \cap B^x)/(B \cap B^x)$ is non-abelian, we have by comparison of orders:

$$B^y(B \cap B^x)/(B \cap B^x) = \langle w, x\rangle B/(B \cap B^x),$$

or, equivalently:

$$\langle w, x\rangle B = B^y(B \cap B^x).$$

Now $(\langle w\rangle B)' = B'[\langle w\rangle, B]$ and, from above, $(\langle w\rangle B)' = B'(B')^x$. Hence if $B \leqslant C_G([\langle w\rangle, B])$ then B centralises $(\langle w\rangle B)'$ and, in particular, one gets $B \leqslant C_G((B')^x) = B^x$, which is ruled out. Thus $B \nleqslant C_G([\langle w\rangle, B])$. But we have $[\langle w\rangle, B] = [\langle w\rangle, AB] = [\langle w\rangle, G] \trianglelefteq G$ and $B \cap B^x = C_G((\langle w\rangle B)') \leqslant C_G([\langle w\rangle, B])$ so, by normality, $(B \cap B^x)^G \leqslant C_G([\langle w\rangle, B])$. It then follows that $B \nleqslant (B \cap B^x)^G$.

We now let

$$T = (B \cap B^x)^G.$$

Then $T \leqslant \langle w, x\rangle B$ but, since $B \nleqslant T$, we have $T \neq \langle w\ x\rangle B$. Now,

$$\langle w, x\rangle B/(B \cap B^x) = B^y(B \cap B^x)/(B \cap B^x)$$

is a non-abelian group of order p^3, so $Z(\langle w, x\rangle B/(B \cap B^x)) \cong C_p$. In addition $|\langle w\rangle B/(B \cap B^x)| = |\langle w, x\rangle(B \cap B^x)/(B \cap B^x)| = p^2$. Hence, by comparison of orders:

$$\begin{aligned} Z(\langle w, x\rangle B/(B \cap B^x)) &= \langle w\rangle B/(B \cap B^x) \cap \langle w, x\rangle(B \cap B^x)/(B \cap B^x) \\ &= \langle w\rangle(B \cap B^x)/(B \cap B^x)(\cong C_p). \end{aligned}$$

Now $B \cap B^x \ntrianglelefteq G$, so $B \cap B^x$ is a proper subgroup of T $(= (B \cap B^x)^G)$. Thus, by normality,

$$1 \neq T/(B \cap B^x) \cap Z(\langle w, x\rangle B/(B \cap B^x)),$$

so:

$$\langle w\rangle(B \cap B^x)/(B \cap B^x) \leqslant T/(B \cap B^x).$$

Since $B \not\leqslant T$ we see, by comparison of orders, that $B \cap T = B \cap B^x$. Thus $BT/T \cong B/(B\cap T) \cong B/(B\cap B^x) \cong C_p$. We then also have $(BT/T)^{yT} = B^yT/T \cong C_p$. Hence if $T = \langle w\rangle(B \cap B^x)$, then $|\langle w, x\rangle B : T| = p^2$, so B^yT is a proper subgroup of $\langle w, x\rangle B$. But $\langle w, x\rangle B = B^y(B \cap B^x) \leqslant B^yT$ and a contradiction ensues. Therefore $\langle w\rangle(B \cap B^x)$ is a proper subgroup of T and, since $T \neq \langle w, x\rangle B$, we conclude that $|\langle w, x\rangle B : T| = p$. In particular we see that $|G : T| = p^2$.

Since $w \in T = (B \cap B^x)^G \leqslant C_G([\langle w\rangle, B]) \leqslant C_G([\langle w\rangle, B \cap B^x]$, we see that $\langle w\rangle(B \cap B^x)$ centralises $[\langle w\rangle, B \cap B^x]$. We further have $(B \cap B^x)' \leqslant B' \cap (B')^x$, which is centralised by $BB^x = \langle w\rangle B$ so, in particular, $\langle w\rangle(B \cap B^x)$ centralises $(B \cap B^x)'$. Hence

$$(\langle w\rangle(B \cap B^x))' = (B \cap B^x)'[\langle w\rangle, B \cap B^x] \leqslant Z(\langle w\rangle(B \cap B^x)).$$

Thus $\langle w\rangle(B \cap B^x)$ has class at most two. Now $\langle w\rangle(B \cap B^x)/(B \cap B^x) \cong C_p$, so $\langle w\rangle(B \cap B^x)$ is a (normal) subgroup of index p in T. Thus, letting

$$B_2 = \langle w\rangle(B \cap B^x),$$

we have $B_2' \leqslant Z(B_2)$ and, since $T = (B \cap B^x)^G = (\langle w\rangle(B \cap B^x))^G = B_2^G$, we conclude that $|B_2^G : B_2| = p$ and $|G : B_2^G| = p^2$, as desired. □

Corollary 6. *Let the finite p-group $G = AB$ be the product of an abelian subgroup A of order p^3 and a subgroup B of class two. Then $G^{(3)} = 1$.*

Proof. If $A \cap B \neq 1$ then $|G : B| \leq p^2$, and the result follows from Corollary 2. If $A \cap B = 1$ then, by Theorem 5, either $G = AB_1$ where $(B_1')^G$ is abelian and the result follows from Lemma 3, or G has a subgroup B_2, of class at most two, such that $|B_2^G : B_2| = p$ and $|G : B_2^G| = p^2$. In the latter case $B_2 \trianglelefteq B_2^G$ and, letting $y \in G\backslash N_G(B_2)$, we see that $B_2^G = B_2B_2^y$ is the normal product of two subgroups of class at most two and index p. Thus Lemma 1 applies and we have $(B_2^G)^{(2)} = 1$. But $|G : B_2^G| = p^2$ so G/B_2^G is abelian and we conclude that $G^{(3)} = 1$. □

The following elementary example shows that G will not necessarily have derived length three if the factor A of order p^3, as in Corollary 6, is non-abelian. We let H be be a p-group of class two and let K be a non-abelian p-group of order p^3. We then let $G = HwrK$ be the regular wreath product of H by K. Since the regular wreath product of two groups of derived length two has derived length four, G is thus a p-group of derived length four that is the product of the base group H^K, which has class two, and K which has order p^3 and is non-abelian.

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