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On laws of the form $ab \equiv ba$ equivalent to the abelian law

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Abstract – N.D. Gupta has proved that groups which satisfy the laws $[x, y] \equiv [x,_n y]$ for $n = 2, 3$ are abelian. Every law $[x, y] \equiv [x,_n y]$ can be written in the form $ab \equiv ba$ where a, b belong to a free group F_2 of rank two, and the normal closure of $\langle a, b\rangle$ coincides with F_2. In this work we investigate laws of this form. In particular, we discuss certain classes of laws and show that the metabelian groups which satisfy them are abelian.

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1. Introduction

Many authors have investigated laws which are equivalent to the abelian law (cf. [6, 8, 13, 18, 25]). For example N.D. Gupta [8] showed that groups satisfying the law $[x, y] \equiv [x,_n y]$ for $n = 2, 3$ are abelian and asked whether the laws $[x, y] \equiv [x,_n y]$ for $n \geq 4$ are equivalent to the abelian law. Gupta's question is still open. Gupta in [8] also showed that other laws of the form $[x, y] \equiv [a, b, c]$, where $a, b, c \in \{x^{\pm 1}, y^{\pm 1}\}$ are equivalent to the abelian law. Gandhi in [6] and, independently, Kappe with Tomkinson in [13] gave the complete proof that every law of the form $[x, y] \equiv [a, b, c]$, where $a, b, c \in \{x^{\pm 1}, y^{\pm 1}\}$ is equivalent to the abelian law. Then Moravec [18] showed the same for the law of the form $[x, y] \equiv [a, b, c, d]$, where $a, b, c, d \in \{x^{\pm 1}, y^{\pm 1}\}$.

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All laws mentioned above have the form $[u, x^{\pm 1}] \equiv [x, y]$ or $[u, y^{\pm 1}] \equiv [x, y]$. If a law has the form $[u, y] \equiv [x, y]$, then we can transform it as follows:

$$\begin{aligned}
[u, y] \equiv [x, y] &\longleftrightarrow u^{-1}y^{-1}uy \equiv x^{-1}y^{-1}xy \\
&\longleftrightarrow u^{-1}y^{-1}u \equiv x^{-1}y^{-1}x \\
&\longleftrightarrow xu^{-1}y^{-1} \equiv y^{-1}xu^{-1} \\
&\longleftrightarrow (xu^{-1})y \equiv y(xu^{-1}).
\end{aligned}$$

The last law has the form

$$ab \equiv ba. \tag{1}$$

where a, b are words in the free group $F_2 = \langle x, y \rangle$ of rank 2.

Some authors studied general laws of the form $[x,_n y] \equiv [x,_m y]$ where $1 < n < m$ (see [1, 2, 3, 9]). These laws can also be written in the form $ab \equiv ba$ where $a = [x,_{m-1} y][x,_{n-1} y]^{-1}$, $b = y$ but for $n > 1$ they are not equivalent to the abelian law, since n-Engel groups satisfy the law $[x,_n y] \equiv [x,_m y]$ where $1 < n < m$. Moreover every finite group also satisfies a law of the form $[x,_n y] \equiv [x,_m y]$ where $1 \leq n < m$ (Nikolova in [21] calculated the minimal n for some alternating groups).

Psomopoulos in [25] investigated laws of the form $x^t[x^n, y] \equiv [x, y^m]x^s$ in rings. He showed that in many cases rings satisfying such laws are commutative. For example, if m and n are coprime then every ring satisfying the law $x^t[x^n, y] \equiv [x, y^m]x^s$ is commutative ([25], Theorem 2). He also noticed that the last statement for groups is not always true. For example, the symmetric group on three symbols, which is non-abelian, satisfies the law $x^6[x^7, y] \equiv [x, y]x^6$. However, he showed that in some cases, groups satisfying such laws are abelian. The following proposition describes such a case.

Proposition 1.1 (cf. [25], Theorem 3). *Every group satisfying a law $[x^n, y] \equiv [x, y^{n+1}]$ is abelian.*

Notice that the law $[x^n, y] \equiv [x, y^{n+1}]$ can be written in the form (1). Indeed

$$\begin{aligned}
[x^n, y] \equiv [x, y^{n+1}] &\longleftrightarrow x^{1-n}y^{-1}x^n \equiv y^{-n-1}xy^n \\
&\longleftrightarrow (xy^{-1})(x^n y^{-n-1}) \equiv (x^n y^{-n-1})(xy^{-1}).
\end{aligned}$$

The natural question is the following one.

Question 1.2. *For which words $a, b \in F_2$ is the law $ab \equiv ba$ equivalent to the abelian law?*

Definition 1.3. We call a pair $(a, b) \in F_2^2$ *an abelian pair* if every group satisfying the law $ab \equiv ba$ is abelian.

For example, if a, b are free generators of F_2 then every group satisfying the law $ab \equiv ba$ is abelian. But what happens when a, b generate F_2 as a normal subgroup (normally generate F_2)? Is such a pair also abelian? In Proposition 2.1 we give a necessary (but not sufficient) condition for a pair to be abelian. Every pair that normally generates F_2 fulfills this condition, but there are pairs satisfying this condition which do not normally generate F_2. All examples of abelian pairs that we know are pairs normally generating F_2, but there are examples of pairs that normally generate F_2 and are not abelian (see Proposition 2.3).

Since we do not know necessary and sufficient conditions for pairs to be abelian, we investigate special classes of pairs and study abelian pairs in the classes of nilpotent groups, finite groups and solvable groups.

We call a non-abelian variety *pseudo-abelian* [15] if every metabelian group in this variety is abelian. The problem of existence of such a variety, that is non-abelian, was raised in [20] (Problem 5). The first examples of non-abelian pseudo-abelian varieties were given by Olshanskii [22], [23]. It is shown in [20] (21.4) that the variety of groups is pseudo-abelian if and only if every finite group in this variety is abelian.

As we have shown above, every law $[x, y] \equiv [u, y]$ is equivalent to the law $(xu^{-1})y \equiv y(xu^{-1})$. The example below shows that a variety of groups satisfying a law $(xu^{-1})y \equiv y(xu^{-1})$ where $u \in F_2'$ is pseudo-abelian.

Example 1.4. Let u be a word in F_2'. Then every metabelian or finite group satisfying the law $(xu^{-1})y \equiv y(xu^{-1})$ is abelian. So a variety of groups satisfying such law is pseudo-abelian.

Proof. Let G be a metabelian group satisfying the law $[xu^{-1}, y] \equiv 1$. Then we have (in the subvariety consisting of all the metabelian groups satisfying $xu^{-1}y \equiv yxu^{-1}$):

$$1 \equiv [xu^{-1}, y] = [x, y]^{u^{-1}}[u^{-1}, y] = [x, y][u^{-1}, y].$$

We substitute $y \mapsto [y, x]$ and get $1 \equiv [x, [y, x]][u', [y, x]] = [y, x, x]^{-1}$. So G satisfies the law $[y,_2 x] \equiv 1$. From this and $(xu^{-1})y \equiv y(xu^{-1})$ it follows easily that G is abelian.

Let us assume that there exists a finite, nonabelian group satisfying the law $(xu^{-1})y \equiv y(xu^{-1})$. Let G be such a group of the smallest order. Then every subgroup of G is abelian. By the result of Miller and Moreno [17], a finite group of which all proper subgroups are abelian, is metabelian. So, as we have proved above, G is abelian. □

Below we present the main results of this paper.

Main results

- Let w be a word in x, y. Every metabelian (or finite) group satisfying a law
$$x^w y \equiv y x^w$$
is abelian (Theorem 3.6).
- Every metabelian (or finite) group satisfying a law
$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l),$$
where k, l, m, n are integers such that $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$ is abelian (Theorem 4.4 and Corollary 4.6).
- If $(a, b) = (x^k y^l c, x^m y^n d)$ is a pair such that $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$ and $c, d \in F_2'$ then every residually finite group satisfying the law $ab \equiv ba$ is abelian-by-(locally finite of finite exponent) (Corollary 2.8).
- Every locally graded group satisfying a law
$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l),$$
where k, l, m, n are integers such that $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$ is abelian (Theorem 4.10).

Notations

If g, h, k are elements of a group G then we write $g^h := h^{-1}gh$ and $g^{h+k} := g^h g^k$. By this convention we have $g^{hk} = (g^h)^k$. Multiplication and addition in the exponent are left and right distributive but not commutative. However, if G is a metabelian group, and g lies in the commutator subgroup of G then $g^{h+k} = g^{k+h}$ and $g^{hk} = g^{kh}$ for every $h, k \in G$.

The iterated commutators are $[g,_1 h] = [g, h]$ and $[g,_{c+1} h] = [[g,_c h], h]$ for an integer $c > 1$. It can be proved by induction on c that

$$[g,_{c+1} h] = [g, h]^{(-1+h)^c} = g^{(-1+h)^{c+1}}. \tag{2}$$

We denote a normal closure of $\langle a, b\rangle$ in F_2 by $\langle a, b\rangle^{F_2}$ and call the pair (a, b) *an annihilating pair* if $\langle a, b\rangle^{F_2} = F_2$.

We will use the commutator law (cf. [20], 33.34):

$$[xy, zt] = [x,t]^y[y,t][x,z]^{yt}[y,z]^t \tag{3}$$

and its particular forms (for $t = 1$ or $y = 1$):

$$[xy, z] = [x,z]^y[y,z],\ [x, zt] = [x,t][x,z]^t. \tag{4}$$

We will also use the law:

$$[x, y^{-1}] = [x,y]^{-y^{-1}}. \tag{5}$$

2. Properties of abelian pairs

Words $a, b \in F_2$ can be written in the forms $a = x^k y^l c$, $b = x^m y^n d$, where c, d are words in the commutator subgroup F_2'.

Proposition 2.1. *Let $a = x^k y^l c$, $b = x^m y^n d$ be words in F_2 where $c, d \in F_2'$. If (a, b) is an abelian pair then $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$.*

Proof. Let $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = t \neq \pm 1$. Then there exists a prime number p dividing t. Let G be the group of 3×3 (upper) unitriangular matrices $UT(3, p)$. The group G is nilpotent of class 2 and satisfies the following identities:

$$[xy, z] \equiv [x,z][y,z],\ [x, yz] \equiv [x,y][x,z],\ [x,y]^p \equiv 1, [xc, yd] \equiv [x,y],$$

for every $c, d \in F_2'$. Thus:

$$[a,b] = [x^k y^l c, x^m y^n d] \equiv [x,y]^{kn-ml} = [x,y]^t \equiv 1.$$

Thus the pair (a, b) is not abelian. □

Remark 2.2. Every annihilating pair $(a, b) = (x^k y^l c, x^m y^n d)$ where $c, d \in F_2'$ fulfills the condition $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$, but there are non-annihilating pairs satisfying this condition.

The next proposition shows that the above condition is not sufficient.

Proposition 2.3. *Let $a_n = x^{-1}[x,_n y]$ and $b_n = y^{-1}[y,_n x]$ where n is a positive integer. Then the symmetric group S_3 on three symbols satisfies laws $a_n b_n \equiv b_n a_n$ for $n \geq 1$. Hence every pair (a_n, b_n) is not abelian, although it satisfies the condition $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \left|\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right| = 1$.*

Proof. Let $x, y \in S_3$. If x and y are both even then it is clear that $a_n b_n = b_n a_n$. We can thus assume that x and y do not commute. If one of these is in $\langle(123)\rangle$ then either $a_n = 1$ or $b_n = 1$ and thus again $a_n b_n = b_n a_n$. We are thus only left with the case where both a_n and b_n are 2-cycles. Without loss of generality we can assume that $x = (12)$ and $y = (13)$. Direct calculations show that $a_n = b_n = (23)$ and thus again $a_n b_n = b_n a_n$. □

Remark 2.4. For $n = 1, 2$ the pair $(x^{-1}[x,_n y], y^{-1}[y,_n x])$ annihilates F_2, but there are n's for which such pairs are not annihilating. An example is $n = 5$. Other numbers n for which the pair $(x^{-1}[x,_n y], y^{-1}[y,_n x])$ does not annihilate F_2 can be found in [11, 28].

Now we describe abelian pairs in the class of nilpotent groups. A well known result is that X is a free generated set of a free nilpotent group G if and only if the image of X in G/G' generates G/G' (see for example [19] p. 261). In particular we have:

Lemma 2.5. *Let $G = F_2/\gamma_c(F_2)$ be the free nilpotent group, freely generated by g_1, g_2 and let $a_1 = g_1^k g_2^l c$, $a_2 = g_1^m g_2^n d$, where $c, d \in G'$. Then a_1, a_2 are generators of G if and only if $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$.*

Theorem 2.6. *Let $a = x^k y^l c$, $b = x^m y^n d$ be words in F_2 where $c, d \in F_2'$. Every nilpotent group satisfying the law $ab \equiv ba$ is abelian if and only if $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$.*

Proof. We showed in Proposition 2.1 that if $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| \neq \pm 1$ then there are non-abelian nilpotent groups satisfying the law $ab \equiv ba$.

Conversely, let $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. Let g, h be any elements of G. Then by Lemma 2.5 elements $a_1 = a(g, h), a_2 = b(g, h)$ generate $\langle g, h\rangle$. Hence $\langle g, h\rangle = \langle a_1, a_2\rangle$ is abelian, and so is G. □

Corollary 2.7. *Let $a = x^k y^l c$, $b = x^m y^n d$ be words in F_2 where $c, d \in F_2'$ and let $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. If G is a finite group satisfying the law $ab \equiv ba$ then every Sylow subgroup of G is abelian.*

Proof. Every Sylow subgroup P of a finite group is nilpotent. Thus by Theorem 2.6 P is abelian. □

A finite group is called an A-group if all its Sylow p-subgroups are abelian (cf. [27]). Corollary 2.7 shows that if a, b satisfy the hypothesis then every finite group satisfying the identity $ab \equiv ba$ is an A-group. Ol'shanskii in [24] studied varieties in which every group is residually finite. He showed that every group in a variety is residually finite if and only if the variety is generated by a finite A-group. So if a, b satisfy the hypotheses of Corollary 2.7 then the variety of groups satisfying the law $ab \equiv ba$ is not generated by a finite group. Such a variety contains all abelian groups, and some of them are not residually finite.

Burns and Medvedev in [4] introduced the class $\mathfrak{C}$, which is the minimal class consisting of all finite and solvable groups and is closed under operations L and R, where for a group theoretic class $\mathfrak{X}$, $L\mathfrak{X}$ denotes the class of all groups locally in $\mathfrak{X}$ and $R\mathfrak{X}$ the class of groups residually in $\mathfrak{X}$. In particular all residually and locally finite or solvable groups lie in $\mathfrak{C}$.

Corollary 2.8. *Let $a = x^k y^l c$, $b = x^m y^n d$ be two words in F_2 such that $c, d \in F_2'$ and $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. Then there exists a positive integer e such that every group from the class $\mathfrak{C}$ satisfying the law $ab \equiv ba$ is abelian-by-(locally finite of exponent e). In particular every finitely generated group in $\mathfrak{C}$ satisfying this law is abelian-by-finite.*

Proof. Let $\mathfrak{V}$ be a variety of groups satisfying the law $ab \equiv ba$, where $a = x^k y^l c, b = x^m y^n d$ and $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. Let $\mathfrak{A}_p\mathfrak{A}$ be the variety of groups which are (abelian of prime exponent p)-by-abelian. Since $\mathfrak{A}_p\mathfrak{A}$ contains a nonabelian nilpotent group of order p^3, $\mathfrak{V}$ does not contain any of $\mathfrak{A}_p\mathfrak{A}$, so by [7] every metabelian group in $\mathfrak{V}$ is nilpotent-by-(finite exponent). In particular every finitely generated metabelian group in $\mathfrak{V}$ is nilpotent-by-finite. Then, by [4] (Theorem A), there exists e such that every group in $\mathfrak{C} \cap \mathfrak{V}$ is nilpotent-by-(locally finite of exponent e). By Theorem 2.6 every nilpotent group is abelian, so every group in $\mathfrak{C} \cap \mathfrak{V}$ is abelian-by-(locally finite of finite exponent e) and every finitely generated group in $\mathfrak{C} \cap \mathfrak{V}$ is abelian-by-finite. □

Corollary 2.9. *If every metabelian group satisfying a law $ab \equiv ba$ is abelian then every group in the class $\mathfrak{C}$ satisfying the law $ab \equiv ba$ is abelian.*

Proof. First we show that every finite group satisfying $ab \equiv ba$ is abelian. Let us assume, on contrary, that there exists a finite nonabelian group satisfying $ab \equiv ba$. Let G be such a group of minimal order. Then every subgroup of G is abelian. By the result of Miller and Moreno [17], G has to be metabelian. Hence G is abelian contrary to the assumption.

It is enough to show that every finitely generated group is abelian. By Corollary 2.8 every finitely generated group in $\mathcal{C}$ satisfying the law $ab \equiv ba$ is abelian-by-finite, so it is metabelian and hence abelian. □

3. Pairs (x^w, y) where $w \in F_2$

In this section we investigate pairs (x^w, y), where $w = w(x, y)$ is a word in F_2. It is clear that every pair (x^w, y) normally generates F_2. We consider the law $x^w y \equiv yx^w$ and ask the question whether this law is equivalent to the abelian law. The main aim of this section is to show that every metabelian (or finite) group satisfying a law $x^w y \equiv yx^w$ is abelian. We also show that in many cases a pair (x^w, y) is abelian.

Remark 3.1. Take any word $w \in F_2$. Then $w = x^k c y^l$ for some integers k, l and some $c \in F_2'$. It is then clear that $x^w y \equiv yx^w$ is equivalent to $x^c y \equiv yx^c$. Notice also that this is equivalent to $xy^{c^{-1}} \equiv y^{c^{-1}} x$.

Proposition 3.2. *Let $c \in F_2'$ be a fixed point of an automorphism*

$$\delta : x \longmapsto y^{\pm 1} x^{\pm 1}, \quad y \longmapsto y^{\pm 1}.$$

Then the pair (x^c, y) is an abelian pair.

Proof. It suffices to show that every 2-generator group G satisfying the law $x^c y \equiv yx^c$ is abelian. Let G be generated by g and h. Since G satisfies the law $x^c y \equiv yx^c$ it also satisfies $(x^c y)^\delta \equiv (yx^c)^\delta$. Thus G satisfies $(y^{\pm 1}x^{\pm 1})^c y \equiv y(y^{\pm 1}x^{\pm 1})^c$. Hence we get the law

$$(y^{\pm 1})^c (x^{\pm 1})^c y \equiv y (y^{\pm 1})^c (x^{\pm 1})^c. \tag{6}$$

From $x^c y \equiv yx^c$ we have $(x^{\pm 1})^c y \equiv y(x^{\pm 1})^c$ so (6) can be written in the form:

$$(y^{\pm 1})^c y (x^{\pm 1})^c \equiv y (y^{\pm 1})^c (x^{\pm 1})^c,$$

and so

$$(y^{\pm 1})^c y \equiv y (y^{\pm 1})^c.$$

Hence for generators $g, h \in G$ we have $g^c h \equiv hg^c$ and $(h^{\pm 1})^c h = h(h^{\pm 1})^c$ (here $c = c(g, h)$). Thus h centralizes $\langle g^c, (h^{\pm 1})^c \rangle$. But $\langle g^c, (h^{\pm 1})^c \rangle = \langle g, h \rangle^c = \langle g, h \rangle$, so h belongs to the center of G and it commutes with g. □

Remark 3.3. We will get the same result if c is a fixed point of one of the following automorphisms:

$$\begin{aligned} x &\longmapsto x^{\pm 1}y^{\pm 1}, & y &\longmapsto y^{\pm 1}, \\ x &\longmapsto x^{\pm 1}, & y &\longmapsto x^{\pm 1}y^{\pm 1}, \\ x &\longmapsto x^{\pm 1}, & y &\longmapsto y^{\pm 1}x^{\pm 1}. \end{aligned}$$

Corollary 3.4. *Let c be the product of commutators of the form $[x, y^k]$. Then the pair (x^c, y) is abelian.*

Proof. Let δ be an automorphism acting on generators x, y as followsQ

$$x \longmapsto yx, \quad y \longmapsto y.$$

Then $[x, y^k]^\delta = [yx, y^k] = [x, y^k]$. So c is a fixed point of δ and by Proposition 3.2 the pair (x^c, y) is abelian. $\square$

Example 3.5. We show that every pair $(x^{y^n x}, y)$ is abelian. Let us consider the law

$$x^{y^n x} y \equiv y x^{y^n x}$$

First, let us notice that $y^n x = x[x, y^{-n}]y^n$. Hence by Remark 3.1 the law $x^{y^n x} y \equiv y x^{y^n x}$ is equivalent to $x^{[x,y^{-n}]} y = y x^{[x,y^{-n}]}$, and by Corollary 3.4 the latter is equivalent to the abelian law.

Theorem 3.6. *Let F_2 be a free group, freely generated by x and y and let w be a word in F_2. Then every solvable group satisfying the law $x^w y \equiv y x^w$ is abelian.*

Proof. It is sufficient to prove that every metabelian group satisfying the law $x^w y \equiv y x^w$ is abelian. By Remark 3.1 the law $x^w y \equiv y x^w$ is equivalent to a law $x^c y \equiv y x^c$ for some $c \in F_2'$. We substitute $[x, y]$ for x and get $[x, y]^{c'} y \equiv y[x, y]^{c'}$, which in the variety of metabelian groups is equivalent to $y[x, y] \equiv [x, y]y$ and thus to $[x, y, y] \equiv 1$. By Levi Theorem [16] (see also [20], 34.31) every group satisfying the law $[x, y, y] \equiv 1$ is nilpotent (of class 3). Hence by Theorem 2.6 the law $x^w y \equiv y x^w$ is equivalent to the abelian law. $\square$

The immediate application of Theorem 3.6 and Corollary 2.9 is the following corollary.

Corollary 3.7. *Let F_2 be a free group, freely generated by x and y and let $w \in F_2$. Then every group in the class $\mathfrak{C}$ satisfying the law $x^w y \equiv y x^w$ is abelian.*

4. Pairs of the form $(x^k y^l, x^m y^n)$

In this section we study pairs of the form $(x^k y^l, x^m y^n)$ where k, l, m, n are integers such that $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. The main aim of this section is to show that every metabelian or finite group satisfying a law $x^k y^l x^m y^n \equiv x^m y^n x^k y^l$ is abelian.

Proposition 4.1. *A pair $(x^k y^l, x^m y^n)$ is an annihilating pair if and only if $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$.*

Proof. If $(x^k y^l, x^m y^n)$ annihilates F_2 then its image in F_2/F_2' is the generating pair. Thus $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$.

Conversely, let $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$, and let G be a group with presentation $\langle x, y \mid x^k y^l, x^m y^n \rangle$. We have $x^{\pm 1} = x^{kn-ml} = (x^k)^n (x^{-m})^l = (y^{-l})^n (y^n)^l = 1$. So $x = 1$ and similarly $y = 1$. Thus G is trivial and $(x^k y^l, x^m y^n)$ annihilates F_2. □

Proposition 4.2. *A law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ is equivalent to the law $[x^k, y^n] \equiv [x^m, y^l]$.*

Proof. We transform the law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ as follows:

$$\begin{aligned} (x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l) &\longleftrightarrow x^{-m} y^l x^m y^{-l} \equiv x^{-k} y^n x^k y^{-n} \\ &\longleftrightarrow [x^m, y^{-l}] \equiv [x^k, y^{-n}]. \end{aligned}$$

After the substitution $y \mapsto y^{-1}$ we get $[x^k, y^n] \equiv [x^m, y^l]$. □

Proposition 4.3. *Let k, l, m, n be integers such that $\left|\begin{smallmatrix} k & l \\ m & n \end{smallmatrix}\right| = \pm 1$. Then a pair $(x^k y^l, x^m y^n)$ is abelian in the following cases*:

(1) *$k = m$ or $l = n$*;

(2) *$k = l$ or $m = n$*;

(3) *$k \cdot l \cdot m \cdot n = 0$.*

Proof. (1) If $k = m$ then $\pm 1 = kn - lm = mn - lm = m(n - l)$. Thus $k = m = 1$ or $k = m = -1$. We consider only the case $k = m = 1$ as the second one is similar. From $kn - lm = n - l = \pm 1$ we get the pair (xy^{t+1}, xy^t) where $t = n$ or $t = l$. This pair defines the law $xy^{t+1}xy^t \equiv xy^t xy^{t+1}$ which is equivalent to the abelian law.

The case $l = n$ is analogous.

(2) If $k = l$ then $\pm 1 = kn - ml = k(n - m)$. Hence $k = \pm 1$ and $n = m \pm 1$. Let us consider only the case $l = k = 1$ and $n = m + 1$ since the other cases are analogous. In this case we get the law $(xy)(x^m y^{m+1}) \equiv (x^m y^{m+1})(xy)$. By Proposition 4.2 the last law is equivalent to $[x, y^{m+1}] \equiv [x^m, y]$, and by Proposition 1.1 this law is equivalent to the abelian law.

The case $m = n$ is similar.

(3) If $k \cdot l \cdot m \cdot n = 0$ then at least one of the numbers k, l, m, n equals zero. It suffices to consider one of these cases since the other cases are analogous. Let for example $m = 0$ then from $\pm 1 = kn - ml = kn$ we get $k = \pm 1$ and $n = \pm 1$. Thus our pair takes the form $(x^{\pm 1} y^l, y^{\pm 1})$, and since this pair generates F_2 it is abelian. □

Theorem 4.4. *Every finite group satisfying a law*

$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l),$$

where $kn - ml = \pm 1$, *is abelian.*

Proof. Assume, on the contrary, that there exist finite nonabelian groups satisfying a law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$, where $kn - ml = \pm 1$. Let G be such a group of minimal order. Then every proper subgroup of G and every quotient group of G is abelian. By the result of Miller and Moreno [17], G is metabelian and by Schmidt's theorem ([26], 9.1.9) there exist primes p and q such that G has order $p^r q^s$ and a Sylow p-subgroup P of G is normal and a Sylow q-subgroup Q is cyclic. Moreover, the center of G is trivial. Otherwise $G/Z(G)$ is abelian, so G is nilpotent and by Theorem 2.6 it is abelian, contrary to assumption. Subgroups $P^p Q$ and PQ^q are proper, so they are abelian and P^p, Q^q lie in $Z(G) = \{1\}$. Hence, P is an elementary abelian p-group and Q is cyclic of order q. Therefore G has order $p^r q$.

If c is an element of order q and d is an element of order p, then $G = \langle c, d\rangle$. Indeed, if $H = \langle c, d\rangle \neq G$ then H is abelian. Hence, d belongs to $Z(G) = \{1\}$, a contradiction.

Every non-trivial element g of G has order p or q. If g does not belong to P then g^q lies in P, and $g^{pq} = 1$. If $g^q \neq 1$ then by the previous statement $G = \langle g^p, g^q\rangle$. Hence G is abelian, a contradiction.

Let a and b be elements of order q, such that $\langle a\rangle \cap \langle b\rangle = \{1\}$. Then $ab \neq ba$ since otherwise $\langle a\rangle\langle b\rangle$ is a subgroup of order q^2 which is impossible. So G is generated by a and b.

Now we show that there exist elements $a, b \in G$ of such that $G = \langle a^k b^l, a^m b^n\rangle$. Let a, b belong to different Sylow q-subgroups. We have two possibilities here.

The first is when at least one of k, l, m, n is divisible by q, suppose $q|k$. Then q does not divide m and l since $kn-ml = \pm 1$. Hence, we have $a^k b^l = b^l =: c$ and $a^m b^n = a^m c^r$ for some r. Thus $\langle a^k b^l, a^m b^n\rangle = \langle c, a^m c^r\rangle = \langle c, a^m\rangle = \langle c, a\rangle = G$.

The second is when none of k, l, m, n is divisible by q. There exists r not divisible by q such that $a^m b^{rn}$ belongs to P. So $a^m b^{rn} \neq 1$ has order p. Moreover $a^k b^{rl}$ does not belong to P. Indeed if in G/P we have $a^k P = b^{-rl} P$ then $a^{mk} P = b^{-rml} P$ and $b^{-rnk} P = b^{-rml} P$. Hence $b^{r(nk-ml)} P = P$, so $q|r$, a contradiction. Hence $a^k b^{rl}$ has order q and $a^m b^{rn}$ has order p. Thus $G = \langle a^k b^{rl}, a^m b^{rn}\rangle$ and G is abelian, contrary to assumption. □

Corollary 4.5. *Every residually finite group satisfying a law*

$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$$

where $kn - ml = \pm 1$ is abelian.

Proof. This follows directly from Theorem 4.4. □

Corollary 4.6. *Every solvable group satisfying a law*

$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l),$$

where $kn - ml = \pm 1$ is abelian.

Proof. By a well-known Hall's result a finitely generated metabelian group is residually finite (cf. [10]). So, by Theorem 4.4, every metabelian group satisfying a law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ is abelian. So, solvable groups satisfying such identity must also be abelian. □

Corollary 4.7. *Every group in the class $\mathcal{C}$ satisfying a law*

$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$$

where $kn - ml = \pm 1$ is abelian.

Proof. This follows from Corollaries 2.9 and 4.6. □

Example 4.8 ([29], Proposition 3.4.1). We show that for every integer k the pair $(x^2 y^{2k+1}, xy^{k+1})$ is an abelian pair.

Let us consider the law $x^2 y^{2k+1} x y^{k+1} \equiv x y^{k+1} x^2 y^{2k+1}$. After cancelation we obtain a new form of this law: $xy^{2k+1}x \equiv y^{k+1}x^2 y^k$. Next, we use the automorphism $x \mapsto x^{-1}$ and $y \mapsto y^{-1}$, then invert both sides of this law and we get $xy^{2k+1}x \equiv y^k x^2 y^{k+1}$. The left sides of the two last laws are the same, so the right ones are also equal. Thus we obtain the law $y^{k+1}x^2y^k \equiv y^k x^2 y^{k+1}$. After cancelation we get $yx^2 \equiv x^2y$. It can be deduced from [12] that every 2-generator group satisfying the law $[[x, y], [x^2, y]] \equiv 1$ is metabelian. In particular, if a 2-generator group satisfies $x^2y \equiv yx^2$ then it is metabelian. But by Corollary 4.6 every metabelian group satisfying the law $x^2y^{2k+1}xy^{k+1} \equiv xy^{k+1}x^2y^{2k+1}$ is abelian.

Lemma 4.9. *Let k, l, m, n be nonzero integers such that $\begin{vmatrix} k & l \\ m & n \end{vmatrix} = \pm 1$. Then the law*

$$(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$$

is equivalent to a positive law

$$(x^{k'} y^{l'})(x^{m'} y^{n'}) \equiv (x^{m'} y^{n'})(x^{k'} y^{l'}),$$

where $k', l', m', n' > 0$.

Proof. Without loss of generality, we can assume that at most two numbers of k, l, m, n are negative.

Let us assume that exactly one number is negative. Then

$$\pm 1 = \begin{vmatrix} k & l \\ m & n \end{vmatrix} = kn - lm = \pm(|k||n| + |l||m|).$$

So $|k||n| + |l||m| = 1$ and at least one of k, l, m, n equals zero, contrary to the hypotheses.

If two of k, l, m, n are negative then we have three cases.

1. $k, m < 0$ and $l, n > 0$. Then we apply the automorphism $x \mapsto x^{-1}$, $y \mapsto y$ to the law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ and we get the law $(x^{-k} y^l)(x^{-m} y^n) \equiv (x^{-m} y^n)(x^{-k} y^l)$ with all exponents positive. The case $l, n < 0$, $k, m > 0$ is analogous.

2. $k, l < 0$ and $m, n > 0$. Then we perform the following transformations:

$$\begin{aligned}(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l) &\longleftrightarrow x^m y^{n-l} x^{-k} \equiv y^{-l} x^{m-k} y^n \\ &\longleftrightarrow x^{m-k} y^{n-l} x^{-k} y^{-l} \equiv x^{-k} y^{-l} x^{m-k} y^{n-l},\end{aligned}$$

and we get the law with all exponents $m - k, n - l, -k, -l$ positive. The case $l, n < 0$, $k, m > 0$ is analogous.

3. $k, n < 0$ and $l, m > 0$. Then the substitution $x \mapsto x$, $y \mapsto y^{-1}$ reduces this case to the previous one. The case $l, m < 0$, $k, n > 0$ is similar. $\square$

We say that a group G is *locally graded* if every finitely generated subgroup of G has a proper subgroup of finite index.

THEOREM 4.10. *Let k, l, m, n be integers such that $\begin{vmatrix} k & l \\ m & n \end{vmatrix} = \pm 1$. Then every locally graded group satisfying the law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ is abelian.*

PROOF. A law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ is equivalent to a positive law. So, it follows from [5] (Theorem B) that every locally graded group satisfying a positive law is nilpotent-by-(locally finite of finite exponent). Hence every two generated locally graded group satisfying a law $(x^k y^l)(x^m y^n) \equiv (x^m y^n)(x^k y^l)$ is abelian-by-finite and by Theorem 4.4 is abelian-by-(finite abelian). So it is metabelian and by Corollary 4.6 it is abelian. □

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