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On BNA-normality and solvability of finite groups

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Abstract – Let G be a finite group. A subgroup H of G is called a BNA-subgroup if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$. In this paper, some interesting properties of BNA-subgroups are given and, as applications, the structure of the finite groups in which all minimal subgroups are BNA-subgroups have been characterized.

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1. Introduction

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [11]. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Recall that a subgroup H of G is said to be an *abnormal subgroup* if $x \in \langle H, H^x \rangle$ for all $x \in G$. There have been several researches on normal and abnormal subgroups. In 1974, A. Fattahi classified the finite groups with only normal and abnormal subgroups [7]. G. Ebert and S. Bauman in 1975 studied the finite groups whose subgroups are either subnormal or abnormal [6]. Cuccia and

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Liotta in 1982 showed that if G is a finite group and, for every minimal subgroup X of G, either $C_G(X)$ is subnormal or abnormal, then G is soluble [4]. G. J. Wood in [15] studied the finite soluble groups whose subgroups are pronormal. Recently, Liu and Li in [13] classified CLT-groups with normal or abnormal subgroups, etc.

The abnormality and the normality are two basic concepts in the theory of groups, which are dual concepts. Precisely speaking, G has only one subgroup, itself, that is both normal and abnormal in G. Each maximal subgroup of G is either normal or abnormal.

In general, for any group G and any subgroup H of G, the following inclusion holds:

$$H \leq \langle H, H^x \rangle \leq \langle H, x \rangle, \quad \text{for any } x \in G.$$

For brevity, we introduce the following definition.

Definition 1.1. Let G be a group. A subgroup H of G is called a BNA-subgroup of G if either $H^x = H$ or $x \in \langle H, H^x \rangle$ for all $x \in G$, H is also said to be BNA-normal in G.

Obviously, BNA-subgroups are in between normal subgroups and abnormal subgroups, all normal subgroups and all abnormal subgroups of G are BNA-subgroups of G. The following example shows that the concept of BNA-subgroups is meaningful and feasible.

Example 1.1. Let $G = S_4$, the symmetric group on 4 letters. Then

(1) the Sylow 3-subgroup of order 3 is not a BNA-subgroup of G;

(2) each cyclic subgroup of order 4 is a BNA-subgroup of G, but is neither normal nor abnormal in G.

In fact, let $P = \langle (123) \rangle$ and $x = (34)$. Obviously, $P^x \neq P$ and $\langle P, P^x \rangle = A_4$, but $x \notin A_4$. Hence, P is not a BNA-subgroup of G.

We know that S_4 has exactly three cyclic subgroups of order 4. Suppose that C is one of them. Then $\langle C, C^y \rangle = C$ or G for any $y \in G$. Therefore C is a BNA-subgroup of G (note that C is neither normal nor abnormal in G).

There has been an interest to investigate the structure of a group G under the assumption that minimal subgroups of G have some properties in G. Itô proved that if the center of a group G of order odd contains all minimal subgroups, then G is nilpotent. Later Buckley [3] proved that if G is a group of order odd whose minimal subgroups are normal in G, then the group G is supersoluble.

In this paper, as a generalization, we consider the finite groups all of whose minimal subgroups are BNA-subgroups. Our main result is as follows.

Main Theorem. *Suppose that all minimal subgroups of G are BNA-subgroups of G. Then the following statements hold*:

(1) *G is soluble*;

(2) *$G = TH$, where T is a Sylow 2-subgroup of G, H is a Hall $2'$-subgroup of G*;

(3) *G is p-supersoluble for each odd prime p dividing $|G|$*;

(4) *the derived subgroup $G' = T_1 \rtimes H_1$, where $T_1 \leq T$ and $H_1 \leq H$*;

(5) *the Fitting height of G is bounded by 4*;

(6) *for each odd prime p dividing $|G|$, the p-length of G is 1, that is,*

$$G = O_{p'pp'}(G).$$

2. Preliminaries

In this section, we state some lemmas which are useful for our main result.

Lemma 2.1. *Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that H is a BNA-subgroup of G. Then*

(1) *H is a BNA-subgroup of K*;

(2) *HN is a BNA-subgroup of G*;

(3) *HN/N is a BNA-subgroup of G/N*;

(4) *Any maximal subgroup of G is a BNA-subgroup of G.*

Proof. (1) Let x be an element of K such that $H \neq H^x$. Then $x \in \langle H, H^x \rangle$ by Definition 1.1 and so H is a BNA-subgroup of K.

(2) Let x be an element of G. If $H = H^x$, then $(HN)^x = HN$. If $x \notin N_G(H)$, then $x \in \langle H, H^x \rangle$. Since $\langle H, H^x \rangle \leq \langle HN, (HN)^x \rangle$, then $x \in \langle HN, (HN)^x \rangle$. Thus HN is a BNA-subgroup of G.

(3) It is easy to see $N_{G/N}(HN/N) \geq N_G(H)N/N$. If $xN \notin N_{G/N}(HN/N)$, then $x \notin N_G(H)$. As H is a BNA-subgroup of G, then $x \in \langle H, H^x \rangle$. Therefore $xN \in \langle HN/N, H^xN/N \rangle = \langle HN/N, (HN/N)^{xN} \rangle$. Definition 1.1 implies that HN/N is a BNA-subgroup of G/N.

(4) Definition 1.1 implies (4). □

Lemma 2.2. *Let $H \leq G$ and be a BNA-subgroup of G. Then*

(1) *the normal closure $H^G = H$ or $H^G = G$*;

(2) *if, in addition, H is subnormal in G, then H is normal in G.*

Proof. (1) If $H^G < G$, then there exists some element x of G such that $x \notin H^G$. Since $H^x \leq H^G$ for all $x \in G$, then $\langle H, H^x \rangle \leq H^G$ and it follows that x is not in $\langle H, H^x \rangle$. Thus, $x \in N_G(H)$ by Definition 1.1. Consequently, $G = H^G \cup N_G(H)$, which implies that $N_G(H) = G$. Therefore $H^G = H$.

(2) It follows from (1). □

Lemma 2.3. *Let H be a BNA-subgroup of G. Then*

(1) $N_G(H) \leq \langle H, H^x \rangle$ *whenever* $H^x \neq H$;

(2) *if* $N_G(H) \leq K \leq G$, *then* K *is an abnormal subgroup of* G.

Proof. (1) Suppose that $H^x \neq H$ for some $x \in G$. By Definition 1.1, we have $x \in \langle H, H^x \rangle$. Let n be an arbitrary element of $N_G(H)$, then $H^{nx} = H^x \neq H$. Similarly, we can get that $nx \in \langle H, H^{nx} \rangle = \langle H, H^x \rangle$. This implies $n \in \langle H, H^x \rangle$ and so $N_G(H) \leq \langle H, H^x \rangle$, as desired.

(2) Let x be an element of G, then we have $\langle H, H^x \rangle \leq \langle N_G(H), N_G(H^x) \rangle \leq \langle K, K^x \rangle$. If $x \in N_G(H)$, then $x \in \langle K, K^x \rangle$ holds obviously. If $x \notin N_G(H)$, Definition 1.1 implies that $x \in \langle H, H^x \rangle$. Therefore $x \in \langle K, K^x \rangle$ holds. □

Lemma 2.4. *Let G be a finite non-soluble group all of whose proper subgroups are soluble. Then $G/\Phi(G)$ is a minimal simple group, where $\Phi(G)$ is the Frattini subgroup of G.*

Proof. Let M be an arbitrary normal subgroup of G containing $\Phi(G)$. If $M \not\subseteq \Phi(G)$, then there exists a maximal subgroup H of G such that $G = MH$. By the hypotheses, H is soluble and hence $G/M \cong H/M \cap H$ is soluble. Because G is non-soluble, M is not soluble. We thus deduce that $M = G$ and so we can get that $G/\Phi(G)$ is a minimal simple group. □

Lemma 2.5 ([14]). *Let G be a minimal non-abelian simple group (a non-abelian simple group all of whose proper subgroups are soluble). Then G is one of the following groups:*

(1) PSL(3, 3);

(2) *the Suzuki group* $S_z(2^r)$ *where* r *is an odd prime*;

(3) PSL$(2, p)$ *where* p *is a prime with* $p > 3$ *and* $p^2 \neq 1 \mod (5)$;

(4) PSL$(2, 2^r)$ *where* r *is a prime*;

(5) PSL$(2, 3^r)$ *where* r *is an odd prime.*

Recall that a normal subgroup H of G is supersolvably embedded in G provided that every chief factor of G contained in H is cyclic.

LEMMA 2.6 ([1]). *Let H be a normal subgroup of G. Suppose that all subgroup of prime order and cyclic subgroups of order* 4(*if any*) *of H are normal in G. Then H is supersolvably embedded in G.*

LEMMA 2.7 ([2] or [11], P_{719}). *If the normal subgroup H of G* (*not necessary soluble*) *is supersolvably embedded in G, then $G/C_G(H)$ is supersoluble.*

LEMMA 2.8. *Let G be a p-soluble group for some odd prime $p \in \pi(G)$ and P a Sylow p-subgroup of G. If every subgroup of P of order p is normal in $N_G(P)$, then G is p-supersoluble.*

PROOF. This is a special case of Theorem 1.1 of [12]. □

LEMMA 2.9 ([11]). *If G is a p-supersoluble group, then G' is p-nilpotent. If G is a supersoluble group, then G' is nilpotent.*

3. Proof of main theorem

The proof of the main theorem will be finished by showing the following theorem.

THEOREM 3.1. *Suppose that all minimal subgroups of G are BNA-subgroups of G. Then G is soluble.*

In fact, we can show the following stronger result:

THEOREM 3.2. *Suppose that all minimal subgroups of G of order odd are BNA-subgroups of G. Then G is soluble.*

PROOF. Assume that the theorem is false and let G be a counterexample of minimal order. It follows from Lemma 2.1 that the condition is inherited by subgroups of G, so every proper subgroup of G is soluble by the choice of G. It follows by Lemma 2.4 that $G/\Phi(G)$ is a minimal simple group and so G is one of groups of Lemma 2.5.

(1) All minimal subgroups of $\Phi(G)$ of order odd are in $Z(G)$.

Suppose that some minimal subgroup X of $\Phi(G)$ of order odd is not in $Z(G)$. Then $C_G(X) < G$ and so $C_G(X)$ is soluble. Furthermore, X is subnormal in G and X is also a BNA-subgroup of G by the hypotheses of the theorem, it follows by Lemma 2.2 (2) that X is normal in G. So $C_G(X)$ is normal in G and hence $G/C_G(X)$ is cyclic. Thus we can get that G is soluble, a contradiction.

(2) Let H be a Hall $2'$-subgroup of $\Phi(G)$, then $H \leq Z(G)$.

By (1), all minimal subgroups of H are in $Z(G)$, so by Lemma 2.6, H is supersolvably embedded in G. It follows from Lemma 2.7 that $G/C_G(H)$ is supersoluble. If $C_G(H) < G$, then $C_G(H)$ is soluble and so G is soluble, a contradiction. Thus we have that $C_G(H) = G$, and so $H \leq Z(G)$.

(3) $\Phi(G)$ is a group of order odd and $\Phi(G) \leq Z(G)$.

Let Z be a Sylow 2-subgroup of $\Phi(G)$. It follows from Lemma 2.1 that G/Z satisfies the hypotheses of the theorem. If $Z \neq 1$, then G/Z is soluble and so G is soluble by the choice of G, a contradiction. Thus $Z = 1$, that is, $\Phi(G)$ is a $2'$-group and so $\Phi(G) \leq Z(G)$.

(4) $\Phi(G) = 1$, that is, G is a minimal simple group.

Since $Z(G)/\Phi(G) \trianglelefteq G/\Phi(G)$ and $G/\Phi(G)$ is a minimal simple groups, $\Phi(G) = Z(G)$. So G is a quasisimple group with the center of order odd. We claim that $\Phi(G) = 1$, that is, $Z(G) = 1$. It will suffice to show that the Schur multiplier of each of the minimal simple groups is a 2-group. Indeed, this is true by checking the list on the Schur multipliers of the known simple groups ([9], P_{302}).

(5) G can not be $\mathrm{PSL}(2, p), \mathrm{PSL}(2, 3^r)$ or $\mathrm{PSL}(3, 3)$.

Indeed, each of $\mathrm{PSL}(2, p), \mathrm{PSL}(2, 3^r)$ and $\mathrm{PSL}(3, 3)$ contains a subgroup which is isomorphic to A_4, the alternating group of degree 4. Let P be a Sylow 3-subgroup of A_4. It follows from Lemma 2.3 that $N_G(P) \leq \langle P, P^x \rangle$ for all $x \in G$ such that $x \notin N_G(P)$. In particular, let x be an arbitrary element of A_4 of order 2, then we have that $N_G(P) \leq A_4$. So $N_G(P) = P \leq C_G(P)$. Therefore P has to be a Sylow 3-subgroup of G. By Burnside theorem, we can get that G is 3-nilpotent, hence G would not be a non abelian simple group, a contradiction. Therefore we conclude that G is not any one of $\mathrm{PSL}(2, p), \mathrm{PSL}(2, 3^r)$ or $\mathrm{PSL}(3, 3)$.

(6) G can not be $\mathrm{PSL}(2, 2^r)$ or $S_z(2^r)$.

Suppose that $G \cong \mathrm{PSL}(2, 2^r)$ or $S_z(2^r)$. By ([8], P_{466}), we know that G is a Zassenhaus group of odd degree and the stabilizer $M = [T]H$ of a point is a Frobenius group with kernel T and with a complement H. For $\mathrm{PSL}(2, 2^r)$, the kernel T is an elementary 2-group of order 2^r and H is cyclic of order $2^r - 1$. For $S_z(2^r)$, T is a special 2-group of order 2^{2r} and H is cyclic of order $2^r - 1$. Let Q be a Sylow q-subgroup of H and L a minimal subgroup of Q for some prime q dividing $|H|$. It follows from Lemma 2.3 that $N_G(L) \leq \langle L, L^x \rangle \leq M$ for any 2-element x of M. As M is a Frobenius group, then $N_G(L)$ is a $2'$-subgroup of M and so $N_G(L) \leq H$ is cyclic. Thus $N_G(L) = C_G(L) = H$.

Since $C_G(Q) \leq N_G(Q) \leq N_G(L) = C_G(L) = H$, then Q is also a Sylow q-subgroup of G and so $C_G(Q) = N_G(Q) = H$. By Burnside theorem, G is q-nilpotent, hence G would not be a non abelian simple group, a contradiction. Therefore G can not be any one of $\mathrm{PSL}(2, 2^r)$ or $S_z(2^r)$ as well.

The proof of the theorem is now complete. □

Definition 3.1 ([5]). Let G be a finite group. The Fitting series

$$F_n(G),\ n = 0, 1, 2, \ldots$$

is defined inductively by $F_0(G) = 1$, $F_n(G)$ is the inverse image in G of $\mathrm{F}(G/F_{n-1}(G))$, for $n \geq 1$.

Evidently each $F_n(G)$ is a characteristic subgroup of G. If G is solvable, then there is some integer $h \geq 0$ such that $F_h(G) = G$. We call the least such integer h the Fitting height of G and denote it by $h(G)$.

Definition 3.2. ([10]) Let G be a finite p-soluble group for some prime p. Define the upper p-series

$$1 = P_0 \leq N_0 < P_1 < N_1 < P_2 < \cdots < P_l \leq N_l = G$$

inductively by the rule that N_k/P_k is the greatest normal p'-subgroup of G/P_k, and P_{k+1}/N_k the greatest normal p-subgroup of G/N_k. The number l, which is the least integer such that $N_l = G$, is called the p-length of G, and we denote it by l_p, or, if necessary, $l_p(G)$.

Recall that a finite p-group is said to be a PN-group if its subgroups of order p are normal.

Lemma 3.1 ([12], Lemma 1.4). *Let G be a finite p-soluble group for an odd prime p. If a Sylow p-subgroup of G is a PN-group, then the p-length $l_p(G) \leq 1$.*

Now we state and show the main theorem of this paper.

Theorem 3.3. *Suppose that all minimal subgroups of G are BNA-subgroups of G. Then the following statements hold*:

(1) *G is soluble*;

(2) *$G = TH$, where T is a Sylow 2-subgroup of G, H is a Hall $2'$-subgroup of G*;

(3) *G is p-supersoluble for each odd prime p dividing $|G|$*;

(4) *the derived subgroup* $G' = T_1 \rtimes H_1$*, where* $T_1 \leq T$ *and* $H_1 \leq H$;

(5) *the Fitting height of G is bounded by* 4;

(6) *for each odd prime* p *dividing* $|G|$*, the* p*-length of* G *is* 1*, that is,*

$$G = O_{p'pp'}(G).$$

Proof. (1) is Theorem 3.1.

(2) Let T be a Sylow 2-subgroup of G. Applying a well known theorem of P. Hall, we have that G possesses a Hall $2'$-subgroup H. Therefore (2) holds.

(3) For any odd prime p dividing $|G|$, let P be a Sylow p-subgroup of G. By Lemma 2.1, each subgroup X of P of order p is a BNA-subgroup of $N_G(P)$ and X is subnormal in $N_G(P)$. It follows from Lemma 2.2 that X is normal in $N_G(P)$. By (1), G is soluble, of course, is p-soluble. Thus we can apply Lemma 2.8 to see that G is p-supersoluble.

(4) Clearly, G is p-supersoluble for any odd primes p dividing $|G|$ by (3). Thus G' is p-nilpotent by Lemma 2.9 for all odd primes p of $\pi(G)$. Let $N(p)$ be the normal p-complement of G' for each odd prime p and set

$$T_0 = \bigcap_p N(p).$$

As $N(p)$ is a p'-group for each odd prime p, it is easy to see that T_0 must be a 2-group. Since $N(p)$ contains all Sylow 2-subgroups of G' for each odd prime p dividing $|G|$, T_0 is a Sylow 2-subgroup of G' and is normal in G. So $T_0 \leq T$. As G is soluble, of course, G' is also soluble, so there is a Hall $2'$-subgroup H_0 of G'. Then $H_0 \leq H^x$ for some $x \in G$. Thus $G' = T_0 \rtimes H_0 = T_0 \rtimes H_0^{x^{-1}}$. Consequently, $H_0^{x^{-1}}$ is supersoluble and (4) holds.

(5) We can get that $G' = T_1 \rtimes H_1$, where $T_1 \leq T$ and $H_1 \leq H$ by (4). Set $F_1 = T_1$, $F_2 = F_1(H_1)'$ and $F_3 = G'$. We can get that $[G'/T_1, G'/T_1] = [T_1H_1/T_1, T_1H_1/T_1] = [H_1T_1, H_1T_1]T_1/T_1$. Since $T_1 \trianglelefteq G$, it follows from Lemma 1.10 of Chapter 3 in [11] that $[H_1T_1, H_1T_1] = [H_1, T_1](T_1)'(H_1)'$. By Lemma 1.6 of Chapter 3 in [11], we have that $[H_1, T_1] \leq T_1$. Therefore $G/T_1 \trianglerighteq [G'/T_1, G'/T_1] = T_1(H_1)'/T_1$, so we have $F_2 \trianglelefteq G$. Thus F_1, F_2 and F_3 are all normal in G and we have a chain of normal subgroups

$$1 = F_0 \trianglelefteq F_1 \trianglelefteq F_2 \trianglelefteq F_3 \trianglelefteq F_4 = G.$$

It follows by Lemma 2.9 that $(H_1)'$ is nilpotent. Now, it is easy to check that F_i/F_{i-1} is nilpotent, $i = 1, 2, 3, 4$. Therefore we conclude that the Fitting height of G is at most 4 by Definition 3.1.

(6) For any odd prime p dividing $|G|$, let P be a Sylow p-subgroup of G. By Lemma 2.1, each subgroup X of P of order p is a BNA-subgroup of P. As X is subnormal in P, it follows by Lemma 2.2 that X is normal in P, so P is a PN-group. G is soluble, and of course, p-soluble. We apply Lemma 3.1 to see that $l_p(G) \leq 1$. If $l_p(G) = 0$, then G is a p'-group, a contradiction. Therefore $l_p(G) = 1$, and so $G = O_{p'pp'}(G)$.

The proof of the theorem is now complete. □

QUESTION. *Are there finite groups G that satisfy the condition of Theorem 3.3 with* $l_2(G) \geq 2$?

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REFERENCES

[1] M. ASAAD – S. LI, *On minimal subgroups of finite groups* II, *Comm. Algebra* **24** (1996), pp. 4603–4606.

[2] R. BAER, *Supersoluble immersion,* Can. J. Math. **11** (1959), pp. 353–369.

[3] J. BUCKLEY, *Finite groups whose minimal subgroups are normal,* Math. Z. **116** (1970), pp. 15–17.

[4] P. CUCCIA – M. LIOTTA, *A condition on the minimal subgroups of a finite group,* Boll. Un. Mat. Ital. A **1** (1982), pp. 303–308.

[5] E. C. DADE, *Carter subgroups and Fitting heights of finite solvable groups,* Illinois J. Math. **13** (1969), pp. 449–514.

[6] G. EBERT – S. BAUMAN, *A note on subnormal and abnormal chains,* J. Algebra **36** (1975), pp.frm-e287–293.

[7] A. FATTAHI, *Groups with only normal and abnormal subgroups,* J. Algebra **28** (1974), pp. 15–19.

[8] D. GORENSTEIN, *Finite groups,* Harper & Row, New York and London, 1968.

[9] D. GORENSTEIN, *Finite simple groups,* An introduction to their classification, Plenum Press, New York, 1982.

[10] P. HALL – G. HIGMAN, *On the p-length of p-soluble groups and reduction theorem for Burnside's problem,* Proc. London Math. Soc. **6** (1956), pp. 1–42.

[11] B. HUPPERT, *Endliche Gruppen* I, Springer, Berlin etc., 1967.

[12] S. Li – Q. Zhang, *A note on p-supersolvable groups,* Acta Math. Hung. **116** (2007), pp. 27–34.

[13] J. Liu – S. Li – J. He, *CLT-groups with normal or abnormal subgroups,* J. Algebra **362** (2012), pp. 99–106.

[14] J. G. Thompson, *Nonsoluble finite groups all of whose local subgroups are soluble* I, Bull. Amer. Math. Soc. **74** (1968), pp. 104–109.

[15] G. J. Wood, On pronormal subgroups of finite soluble groups, *Arch. Math.* **25** (1974), pp. 578–585.

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