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Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type equation

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Abstract – This paper is concerned with the existence and multiplicity to $p(x)$-Kirchhoff type problem of the following form

$$\begin{cases} -M\left(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence and multiplicity of solutions for the problem.

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1. Introduction

In this paper, we are concerned with the following problem

$$(1)\qquad \begin{cases} -M\Big(\displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $M : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-*Laplacian,* and becomes p-*Laplacian* when $p(x) \equiv p$ (a constant). An essential difference between them is that the p-Laplacian operator is $(p-1)$-homogeneous, that is, $\triangle_p(\lambda u) = \lambda^{p-1}\triangle_p u$ for every $\lambda > 0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. Problems involving the $p(x)$-Laplace operator have been intensively studied. Lebesgue and Sobolev spaces with variable exponent have been used in the last decades to model various phenomena. Chen, Levine, and Rao [6] proposed a framework for image restoration based on a variable exponent Laplacian. Another application that uses nonhomogeneous Laplace operators is related to the modeling of electrorheological fluids. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories.

(1) is called a *nonlocal problem* because of the presence of the term M, which implies that the equation in (1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particularly interesting. Nonlocal differential equations are also called *Kirchhoff type* equations because Kirchhoff [20] has investigated an equation of the form

$$(2)\qquad \rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0,$$

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of (2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx$, and hence the equation is no longer a pointwise identity. The parameters in (2) have the following meanings: L is the length of the string, h is the area of the cross-section,

E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. Lions [22] has proposed an abstract framework for the Kirchhoff type equations. After the work of Lions [22], various equations of Kirchhoff type have been studied extensively, see e.g. [4, 5] and [8]–[13]. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [8, 9, 12, 13]) and $p(x)$-Laplacian (see [10, 11, 19]). For the physical and biological meaning of the nonlocal coefficients we refer the readers to [7, 22, 23, 24] and the references therein. In [1] and [2] the authors considered (1) where $f(x,u) = \lambda|u|^{q(x)-2}u$ and $f(x,u) = \lambda(x)|u|^{q(x)-2}u$, respectively. Combining the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle, they proved that under suitable conditions (1) has multiple solutions. Our main purpose is to consider the perturbed problem (1) in general case, we obtain conditions for the existence of infinitely many solutions.

2. Notations and preliminaries

In this section, we recall some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$. Denote

$$C_+(\overline{\Omega}) = \{h(x);\ h(x) \in C(\overline{\Omega}),\ h(x) > 1, \text{for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x);\ x \in \overline{\Omega}\}, \quad h^- = \min\{h(x);\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)} dx \le 1\Big\}.$$

Then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a Banach space, cf. [21].

Proposition 2.1 ([14]). (i) *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

(ii) *If* $p_1(x),\ p_2(x) \in C_+(\overline{\Omega}),\ p_1(x) \le p_2(x)$, *for all* $x \in \overline{\Omega}$, *then we have a continuos embedding* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

An important role in manipulating the generalized Lebesgue space is played by the $p(x)$-modular of the $L^{p(x)}(\Omega)$ space, which is the mapping

$$\rho_{p(x)}\colon L^{p(x)}(\Omega) \longrightarrow \mathbb{R}$$

defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Proposition 2.2 ([15]). *For* $u \in L^{p(x)}(\Omega)$ *and* $u_n \subset L^{p(x)}(\Omega)$,

(1) $|u|_{p(x)} < 1$ (*resp.* $= 1, > 1$) $\iff$ $\rho_{p(x)}(u) < 1$ (*resp.* $= 1, > 1$),

(2) *for* $u \neq 0$, $|u|_{p(x)} = \lambda \iff \rho_{p(x)}(\frac{u}{\lambda}) = 1$,

(3) *if* $|u|_{p(x)} > 1$, *then* $|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}$,

(4) *if* $|u|_{p(x)} < 1$, *then* $|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}$,

(5) $|u_n - u|_{p(x)} \to 0$ (*resp.* $\to \infty$) $\iff$ $\rho_{p(x)}(u_n - u) \to 0$ (*resp.* $\to \infty$),

since $p^+ < \infty$.

The space $W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u(x)|_{p(x)}.$$

Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3 ([17]). *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \le p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Lemma 2.4 (See [16]). *Denote*

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad \textit{for all } u \in X,$$

then $I(u) \in C^1(X, \mathbb{R})$ *and the derivative operator* I' *of* I *is*

$$\langle I'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \textit{for all } u, v \in X,$$

and

(1) *I is a convex functional,*

(2) *$I'\colon X \to X^*$ is a bounded homeomorphism and strictly monotone operator,*

(3) *I' is a mapping of type (S_+), namely*

$$(u_n \rightharpoonup u \quad \textit{and} \quad \limsup_{n\to+\infty} I'(u_n)(u_n - u)) \leq 0 \implies u_n \to u,$$

(4) *I is weakly lower semi-continuous.*

In this paper, we denote by $X = W_0^{1,p(x)}(\Omega)$; $X^* = (W_0^{1,p(x)}(\Omega))^*$, the dual space and $\langle\cdot,\cdot\rangle$, the dual pair and let "$\rightharpoonup$" represent weak convergence. For simplicity, we use c_i to denote the general nonnegative or positive constant (the exact value may change from line to line).

3. Existence and multiplicity of weak solutions

In this section, we will state and prove our main results on problem (1). We introduce the following assumptions on the functions M and f:

(M_1) there exist $m_2 \geq m_1 > 0$ and $\alpha > 1$ such that for all $t \in \mathbb{R}^+$, $m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1}$;

(F_0) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist a constant $c_1 \geq 0$ such that

$$|f(x,t)| \leq c_1(1 + |t|^{\gamma(x)-1}),$$

for all $(x,t) \in \Omega\times\mathbb{R}$ where $\gamma(x) \in C_+(\overline{\Omega})$ and $\gamma(x) < p^*(x)$ for all $x \in \overline{\Omega}$;

(F_1) there exist $M > 0$, $\theta > \alpha p^+$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M$,

$$0 < \theta F(x,t) \leq t f(x,t),$$

where α comes from (M_1) above;

(F_2) $f(x,t) = o(|t|^{\alpha p^+ -1})$ as $t \to 0$ uniformly with respect to $x \in \Omega$, where $\gamma^- > \alpha(p^+)^\alpha > \alpha p^+$, α comes from (M_1);

(F_3) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$;

(F_4) $f(x,t) \geq c_2|t|^{\xi(x)-1}$ as $t \to 0$, where $\xi \in C_+(\overline{\Omega})$ and $p^+ < \xi^- \leq \xi^+ < \alpha p^-$ for a.e. $x \in \Omega$, where α comes from (M_1) .

Definition 3.1. We say that $u \in X$ is a *weak solution* of (1) if

$$M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u)v\,dx,$$

for all $v \in X$.

The Euler–Lagrange functional associated to (1) is given by

$$J(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big) - \int_\Omega F(x,u)\,dx, \tag{3}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$ and $F(x,t) = \int_0^t f(x,s)\,ds$. It should be noticed that under the condition (F_0) the functional J is of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned}\langle J'(u), v\rangle = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx\\ - \int_\Omega f(x,u)v\,dx,\end{aligned}$$

for all $u,\ v \in X$, then we know that the weak solution of (1) corresponds to the critical point of the functional J.

Theorem 3.2. *If* (M_1) *holds and* f *satisfies*

$$|f(x,t)| \le c_3(1+|t|^\nu) \tag{4}$$

where $1 \le \nu < \alpha p^-$, *then* (1) *has a weak solution.*

Proof. From (4) we have $|F(x,t)| \le c_3(|t| + |t|^\nu)$. We can write

$$\begin{aligned} J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big) - \int_\Omega F(x,u)\,dx\\ &\ge \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)^\alpha - c_3\int_\Omega |u|dx - c_3\int_\Omega |u|^\nu\,dx\\ &\ge \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_3\|u\| - c_3\|u\|^\nu \longrightarrow +\infty \quad \text{as } \|u\| \to +\infty.\end{aligned}$$

Due to the condition ($\mathbf{M_1}$) and Proposition 2.3, it is easy to verify that J is weakly lower semi continuous. So J has a minimum point u in X and u is a weak solution of (1). □

Definition 3.3. We say that J *satisfies the* (PS) *condition in* X if any sequence (u_n) such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence, where (PS) means *Palais–Smale*.

LEMMA 3.4. *If* (F_0), (F_1), (M_1), *and*

$$\theta m_1 (p^-)^{\alpha-1} > \alpha m_2 (p^+)^\alpha, \tag{5}$$

hold, then J satisfies the (PS) *condition.*

PROOF. Suppose that $(u_n) \subset X$, $|J(u_n)| \leq c_4$ and $J'(u_n) \to 0$. Then

$$\begin{aligned}
c_4 + \|u_n\| &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)} dx\Big) - \int_\Omega F(x,u_n)\,dx \\
&\quad - \frac{1}{\theta}\Big[M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)} dx\Big)\int_\Omega |\nabla u_n|^{p(x)} dx \\
&\qquad - \int_\Omega f(x,u_n)u_n\,dx\Big] \\
&\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)^\alpha \\
&\quad - \frac{m_2}{\theta}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)^{\alpha-1}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big) \\
&\quad + \int_\Omega \Big[\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n)\Big]\,dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\Big(\int_\Omega |\nabla u_n|^{p(x)}\,dx\Big)^\alpha - \frac{m_2}{\theta(p^-)^{\alpha-1}}\Big(\int_\Omega |\nabla u_n|^{p(x)}\,dx\Big)^\alpha \\
&\geq c_5\Big(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}\Big)\|u_n\|^{\alpha p^-}.
\end{aligned}$$

Hence, $(\|u_n\|)$ is bounded. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $J'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Thus we have

$$\begin{aligned}
&\langle J'(u_n), u_n - u\rangle \\
&\quad = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \\
&\qquad - \int_\Omega f(x,u_n)(u_n - u)\,dx \longrightarrow 0.
\end{aligned}$$

From (F_0), Proposition 2.1 and Proposition 2.3, we can easily get that

$$\int_\Omega f(x,u_n)(u_n - u)\,dx \longrightarrow 0 \quad \text{as } n \to \infty.$$

Therefore, we have as $n \to \infty$,

$$(6)\qquad M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx \longrightarrow 0.$$

Since (u_n) is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx \longrightarrow t_0 \geq 0 \quad \text{as } n \to \infty.$$

If $t_0 = 0$ then (u_n) converges strongly to $u = 0$ in X and the proof is finished. If $t_0 > 0$ then since the function M is continuous, we get

$$M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right) \longrightarrow M(t_0) \geq 0 \quad \text{as } n \to \infty.$$

Thus, by (M_1), for sufficiently large n, we have

$$(7)\qquad 0 < c_6 \leq M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\right) \leq c_7.$$

From (6) and (7), we deduce that

$$(8)\qquad \lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)\,dx = 0.$$

Using Lemma 2.4(3), we have $u_n \to u$ strongly in X as $n \to \infty$ and the functional J satisfies the (PS) condition. □

THEOREM 3.5. *If M satisfies* (M_1), *f satisfies* (F_0)–(F_2) *and relation* (5) *holds, then* (1) *has a nontrivial weak solution.*

PROOF. Let us show that J satisfies the conditions of mountain pass lemma [3]. By Lemma 3.4, J satisfies (PS) condition in X. Since $\alpha p^+ < \alpha(p^+)^\alpha < \gamma^- \leq \gamma(x) < p^*(x)$, $X \hookrightarrow L^{\alpha p^+}(\Omega)$, then there exists $c_8 > 0$ such that

$$|u|_{\alpha p^+} \leq c_8\|u\|, \quad \text{for all } u \in X.$$

Let $\epsilon > 0$ be small enough such that $\epsilon c_8^{\alpha p^+} < m_1/2\alpha(p^+)^\alpha$. By assumptions (F_0) and (F_2), we have

$$(9)\qquad F(x,t) \leq \epsilon|t|^{\alpha p^+} + c(\epsilon)|t|^{\gamma(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

In view of ($\mathbf{M_1}$) and (9), we have

$$\begin{aligned} J(u) &\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)^\alpha - \epsilon\int_\Omega |u|^{\alpha p^+}\,dx - c(\epsilon)\int_\Omega |u|^{\gamma(x)}\,dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - \epsilon c_8^{\alpha p^+}\|u\|^{\alpha p^+} - c_9\|u\|^{\gamma^-} \\ &\geq \frac{m_1}{2\alpha(p^+)^\alpha}\|u\|^{\alpha p^+} - c_9\|u\|^{\gamma^-}. \end{aligned}$$

Therefore, there exist $r > 0$ and $\delta > 0$ such that $J(u) \geq \delta > 0$ for every $\|u\| = r$. From (F_1) it follows that

$$F(x,t) \geq c_{10}|t|^\theta - c_{11},$$

for all $x \in \Omega$ and $|t| \geq M$. For $\omega \in X\backslash\{0\}$ and $t > 1$ we have

$$\begin{aligned} J(t\omega) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\omega|^{p(x)}\,dx\Big) - \int_\Omega F(x,t\omega)\,dx \\ &\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^+}\Big(\int_\Omega |\nabla\omega|^{p(x)}\,dx\Big)^\alpha \\ &\quad - c_{10}t^\theta\int_\Omega |\omega|^\theta\,dx - c_{11}|\Omega| \longrightarrow -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

due to $\theta > \alpha p^+$. Since $J(0) = 0$, J satisfies the conditions of mountain pass lemma[3]. So J admits at least one nontrivial critical point. □

THEOREM 3.6. *If M satisfies* (M_1) *and f satisfies* (F_0), (F_1), (F_3), *and $\gamma^- > \alpha(p^+)^\alpha > \alpha p^+$, then problem* (1) *has a sequence of weak solutions* $(\pm u_k)$ *such that $J(\pm u_k) \to +\infty$ as $k \to \infty$.*

THEOREM 3.7. *If M satisfies* (M_1) *and f satisfies* (F_0), *and* (F_2)–(F_4), *then problem* (1) *has a sequence of weak solutions* $(\pm v_k)$ *such that $J(\pm v_k) < 0$, $J(\pm v_k) \to 0$ as $k \to \infty$.*

We will use the following fountain theorem and the dual fountain theorem to prove Theorem 3.6 and Theorem 3.7, respectively.

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [26]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1,2,\ldots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1,2,\ldots\}},$$

and

$$\langle e_i, e_j^*\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denote the duality product between X and X^*. We define

$$X_j = \text{ span } \{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

LEMMA 3.8 (fountain theorem [25]). *Assume that*

(A1) *X is a Banach space and $J \in C^1(X, \mathbb{R})$ is an even functional.*

If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(A2) $\inf\{J(u): u \in Z_k,\ \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$,

(A3) $\max\{J(u): u \in Y_k,\ \|u\| = \rho_k\} \leq 0$,

(A4) *J satisfies the* (PS) *condition for every* $c > 0$,

then J has an unbounded sequence of critical points.

LEMMA 3.9 ([18]). *If $\gamma(x) \in C_+(\overline{\Omega})$ for all $x \in \Omega$, denote*

$$\gamma_k = \sup\{|u|_{L^{\gamma(x)}(\Omega)};\ \|u\| = 1,\ u \in Z_k\},$$

then $\lim_{k\to\infty} \gamma_k = 0$.

LEMMA 3.10 (dual fountain theorem see [25]). *Assume* (A1) *is satisfied and there is $k_0 > 0$ so that, for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that*

(B1) $a_k = \inf\{J(u): u \in Z_k,\ \|u\| = \rho_k\} \geq 0$,

(B2) $b_k = \max\{J(u): u \in Y_k,\ \|u\| = r_k\} < 0$,

(B3) $d_k = \inf\{J(u): u \in Z_k,\ \|u\| \leq \rho_k\} \to 0$ *as* $k \to +\infty$,

(B4) *J satisfies the* $(\mathrm{PS})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$;

then J has a sequence of negative critical values converging to 0.

DEFINITION 3.11. We say that J *satisfies the* $(\mathrm{PS})_c^*$ *condition* (with respect to (Y_n)) if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$, contain a subsequence converging to a critical point of J.

LEMMA 3.12. *Assume that the conditions in Theorem 3.7 hold, then J satisfies the* $(\mathrm{PS})_c^*$ *condition.*

PROOF. Suppose $(u_{n_j}) \subset X$ such that

$$n_j \longrightarrow +\infty, \quad u_{n_j} \in Y_{n_j},$$

and

$$(J|_{Y_{n_j}})'(u_{n_j}) \longrightarrow 0.$$

Similar to the process of verifying the (PS) condition in the proof of Lemma 3.4, we can get the boundedness of $\|u_{n_j}\|$. Going if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
&\lim_{n_j \to +\infty} \langle J'(u_{n_j}), u_{n_j} - u\rangle \\
&= \lim_{n_j \to +\infty} \langle J'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \lim_{n_j \to +\infty} \langle J'(u_{n_j}), v_{n_j} - u\rangle \\
&= \lim_{n_j \to +\infty} \langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\rangle \\
&= 0.
\end{aligned}$$

As J' is of type (S_+), we can conclude $u_{n_j} \to u$, furthermore we have

$$J'(u_{n_j}) \longrightarrow J'(u).$$

Let us prove $J'(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}
\langle J'(u), \omega_k\rangle &= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \langle J'(u_{n_j}), \omega_k\rangle \\
&= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \langle (J|_{Y_{n_j}})'(u_{n_j}), \omega_k\rangle.
\end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle J'(u), \omega_k\rangle = 0, \quad \text{for all } \omega_k \in Y_k,$$

so $J'(u) = 0$, this show that J satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$. □

PROOF OF THEOREM 3.6. According to (F_3) and Lemma 3.4, J is an even functional and satisfies the (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that A_2 and A_3 hold. Thus, Theorem 3.6 follows from the fountain theorem.

(A_2). For any $u \in Z_k$,

$$\|u\| = r_k = (c_{12}\gamma^+\gamma_k^{\gamma^+} m_1^{-1})^{\frac{1}{\alpha p^- - \gamma^+}}$$

and we have

$$\begin{aligned}
J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) - \int_\Omega F(x,u)dx \\
&\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^\alpha - c_{12}\int_\Omega |u|^{\gamma(x)}dx - c_{12}\int_\Omega |u|dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_{12}|u|_{\gamma(x)}^{\gamma(\zeta)} - c_1\|u\| \quad \text{(where } \zeta \in \Omega) \\
&\geq \begin{cases} \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_{12} - c_{12}\|u\| & (\text{if } |u|_{\gamma(x)} \leq 1) \\ \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_{12}\gamma_k^{\gamma^+}\|u\|^{\gamma^+} - c_{12}\|u\| - c_{13} & (\text{if } |u|_{\gamma(x)} > 1) \end{cases} \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}\|u\|^{\alpha p^-} - c_{12}\gamma_k^{\gamma^+}\|u\|^{\gamma^+} - c_{12}\|u\| - c_{13} \\
&= m_1\Big(\frac{1}{\alpha(p^+)^\alpha} - \frac{1}{\gamma^+}\Big)r_k^{\alpha p^-} - c_{10}.
\end{aligned}$$

Since $\gamma_k \to 0$, $r_k \to \infty$, $\gamma^+ \geq \gamma^- > \alpha(p^+)^\alpha$, we know $J(u) \to \infty$ as $k \to \infty$.

(A_3). From (F_1), we have $F(x,t) \geq c_{10}|t|^\theta - c_{11}$. Therefore, for any $\omega \in Y_k$ with $\|\omega\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned}
J(t\omega) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\omega|^{p(x)}dx\Big) - \int_\Omega F(x,t\omega)dx \\
&\leq \frac{m_2}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\omega|^{p(x)}dx\Big)^\alpha - c_{10}\int_\Omega |t\omega|^\theta dx - c_{11} \\
&\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^+}\Big(\int_\Omega |\nabla \omega|^{p(x)}dx\Big)^\alpha - c_{10}t^\theta\int_\Omega |\omega|^\theta dx - c_{11}.
\end{aligned}$$

By $\theta > \alpha p^+$ and $\dim Y_k < \infty$, it is easy to see that $J(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$. $\square$

Proof of Theorem 3.7. From (F_3) and Lemma 3.12, we know that J satisfies both (A_1) and (B_4). We prove (B_1)–(B_3).

(B_1) For any $v \in Z_k$, $\|v\| = 1$ and $0 < t < 1$, we have

$$\begin{aligned}
J(tv) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla tv|^{p(x)}dx\Big) - \int_\Omega F(x,tv)dx \\
&\geq \frac{m_1}{\alpha}\Big(\int_\Omega \frac{1}{p(x)}|\nabla tv|^{p(x)}dx\Big)^\alpha - \int_\Omega F(x,tv)dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}t^{\alpha p^+}\Big(\int_\Omega |\nabla v|^{p(x)}dx\Big)^\alpha \\
&\quad - \epsilon t^{\alpha p^+}\int_\Omega |v|^{\alpha p^+}p^+dx - c_1 t^{\gamma^-}\int_\Omega |v|^{\gamma(x)}dx \\
&\geq \frac{m_1}{\alpha(p^+)^\alpha}t^{\alpha p^+}\|v\|^{\alpha p^+} - \epsilon t^{\alpha p^+}||v||^{\alpha p^+} \\
&\quad - \begin{cases} c_1\gamma_k^{\gamma^-}t^{\gamma^-}\|v\|^{\gamma^-} & (\text{if } |u|_{\gamma(x)} \leq 1) \\ c_1\gamma_k^{\gamma^+}t^{\gamma^-}\|v\|^{\gamma^+} & (\text{if } |u|_{\gamma(x)} > 1) \end{cases} \\
&\geq \frac{m_1}{2\alpha(p^+)^\alpha}t^{\alpha p^+} - \begin{cases} c_1\gamma_k^{\gamma^-}t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} \leq 1) \\ c_1\gamma_k^{\gamma^+}t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} > 1). \end{cases}
\end{aligned} \tag{10}$$

Since $\gamma^- > \alpha p^+$, taking $\rho_k = t$ small enough and sufficiently large k, for $v \in Z_k$ with $\|v\| = 1$, we have $J(tv) \geq 0$. So for sufficiently large k

$$\inf_{u\in Z_k,\ \|u\|=\rho_k} J(u) \geq 0,$$

i.e. (B_1) is satisfied.

(B_2) For $v \in Y_k$, $\|v\| = 1$ and $0 < t < \rho_k < 1$, we have

$$\begin{aligned}
J(tv) &= \widehat{M}\Bigg(\int_\Omega \frac{1}{p(x)}|\nabla tv|^{p(x)}dx\Bigg) - \int_\Omega F(x,tv)dx \\
&\leq \frac{m_2}{\alpha(p^-)^\alpha}t^{\alpha p^-}\Bigg(\int_\Omega |\nabla v|^{p(x)}dx\Bigg)^\beta - c_{13}t^{\xi^+}\int_\Omega |v|^{\xi(x)}dx.
\end{aligned}$$

Condition $\xi^+ < \alpha p^-$ implies that there exists a $r_k \in (0, \rho_k)$ such that $J(tv) < 0$ when $t = r_k$. Hence, we get

$$b_k := \max_{u\in Y_k,\ \|u\|=r_k} J(u) < 0,$$

so B_2 is satisfied.

(B_3) Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf\{J(u)\colon u \in Z_k,\ \|u\| \le \rho_k\} \le b_k = \max\{J(u)\colon u \in Y_k,\ \|u\| = r_k\} < 0.$$

From (10), for $v \in Z_k$, $\|v\| = 1$, $0 \le t \le \rho_k$ and $u = tv$, we have

$$J(u) = J(tv) \ge \frac{m_1}{2\alpha(p^+)^\alpha} t^{\alpha p^+} - \begin{cases} c_1 \gamma_k^{\gamma^-} t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} \le 1) \\ c_1 \gamma_k^{\gamma^+} t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} > 1) \end{cases}$$
$$\ge - \begin{cases} c_1 \gamma_k^{\gamma^-} t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} \le 1) \\ c_1 \gamma_k^{\gamma^+} t^{\gamma^-} & (\text{if } |u|_{\gamma(x)} > 1), \end{cases}$$

hence, $d_k \to 0$, i.e. (**B**$_3$) is satisfied.

Theorem 3.7 follows from the dual fountain theorem. □

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