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Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group

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Abstract – We study the Berezin transform and the Stratonovich–Weyl correspondence associated with a holomorphic representation of the multi-dimensional Jacobi group.

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1. Introduction

This paper is part of a program to study Berezin transforms and Stratonovich–Weyl correspondences associated with holomorphic representations. The notion of Stratonovich–Weyl correspondence was introduced in [31] in order to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2n}$ and operators on $L^2(\mathbb{R}^n)$ (see [1] and [21]) to the general setting of a Lie group acting on a homogeneous space. Stratonovich–Weyl correspondences were systematically studied by J. M. Gracia-Bondìa, J. C. Vàrilly, and various co-workers, see in particular [23], [20], [18], and [22]. The following definition is taken from [22].

Definition 1.1. Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and μ a (suitably normalized) G-invariant measure on M. Then a Stratonovich–Weyl correspondence for the triple (G, π, M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on M satisfying the following properties:

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(1) W maps the identity operator of $\mathcal{H}$ to the constant function 1;

(2) REALITY: the function $W(A^*)$ is the complex-conjugate of $W(A)$;

(3) COVARIANCE: we have $W(\pi(g)\, A\, \pi(g)^{-1})(x) = W(A)(g^{-1} \cdot x)$;

(4) UNITARITY: we have

$$\int_M W(A)(x) W(B)(x)\, d\mu(x) = \mathrm{Tr}(AB).$$

In this context, M is generally a coadjoint orbit of G which is associated with π by the Kirillov–Kostant method of orbits [25]. For instance, consider the case when G is the $(2n+1)$-dimensional Heisenberg group H_n. Each non-degenerate coadjoint orbit M of G is then diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a Schrödinger representation π of H_n on $L^2(\mathbb{R}^n)$. In this case, the classical Weyl correspondence gives a Stratonovich–Weyl correspondence for the triple (H_n, π, M), [21] and [22].

In the case when G is a quasi-Hermitian Lie group and π is a unitary representation of G (on a Hilbert space $\mathcal{H}$) which is holomorphically induced from a unitary character of a compactly embedded subgroup K of G, we can apply an idea of [20] and we obtain a Stratonovich–Weyl correspondence by modifying suitably the Berezin correspondence S [14] (see also [2] and [3]).

More precisely, recall that S is an isomorphism from the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$ (endowed with the Hilbert-Schmidt norm) onto a space of square integrable functions on a homogeneous complex domain [32]. The map S satisfies (1), (2), and (3) of Definition 1.1 but not (4). A Stratonovich–Weyl correspondence W is then obtained by taking the isometric part in the polar decomposition of S, that is, $W := (SS^*)^{-1/2}S$. Let us mention that $B := SS^*$ is then the so-called Berezin transform which have been studied by many authors, see in particular [19], [27], [28], [32], and [33].

In [14], we considered the case when the Lie algebra $\mathfrak{g}$ of G is reductive. In this case, we proved that B can be extended to a class of functions which contains $S(d\pi(X_1 X_2 \ldots X_p))$ for $X_1, X_2, \ldots, X_p \in \mathfrak{g}$ and that the restrictions to each simple ideal of $\mathfrak{g}$ of the mappings $X \to S(d\pi(X))$ and $X \to W(d\pi(X))$ are proportional (see also [12] and [13]).

The case when $\mathfrak{g}$ is not reductive is more delicate. In [16] we investigated the case of the diamond group and, in [17], we studied B and W in the case of the Jacobi group.

The aim of the present paper is to generalize the results of [17] to the case of the multi-dimensional Jacobi group, which is technically more complicated. The multi-dimensional Jacobi group plays a central role in different areas of Mathematics and Physics and its holomorphic unitary representations were studied intensively, see [26], [9], [10], [4], and [6]. In particular, the metaplectic factorization should be used to reduce the study of the highest weight representations of a quasi-Hermitian Lie group to that of some generalized multi-dimensional Jacobi group [26]. Then the study of the case of the multi-dimensional Jacobi group can be considered as a first step towards the general case.

In this paper, we begin by some generalities on the multi-dimensional Jacobi group (Section 2) and its holomorphic representations (Section 3). Then we introduce the Berezin correspondence S, the Berezin transform B and the Stratonovich–Weyl correspondence W (Section 4). In Section 5, we show that, under some technical assumptions, the Berezin transform of $S(d\pi(X_1X_2\ldots X_p))$ is well-defined for each $X_1, X_2, \ldots, X_p \in \mathfrak{g}$. In Section 6, we identify a class of functions which is stable under B and contains $S(d\pi(X))$ for each $X \in \mathfrak{g}$. We also give an expression of $W(d\pi(X))$ in terms of some integrals of Hua's type (see [24]).

2. The multi-dimensional Jacobi group

The material of this section and of the following section is essentially taken from [21], Chapter 4, [26], Chapters VII and XII and [15].

Consider the symplectic form ω on $\mathbb{C}^n \times \mathbb{C}^n$ defined by

$$\omega((z,w),(z',w')) = \frac{i}{2}\sum_{k=1}^{n}(z_k w'_k - z'_k w_k).$$

for $z, w, z', w' \in \mathbb{C}^n$. The $(2n+1)$-dimensional real Heisenberg group is

$$H := \{((z,\bar{z}),c) \colon z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

endowed with the multiplication

$$((z,\bar{z}),c)\cdot((z',\bar{z}'),c') = \Big((z+z',\bar{z}+\bar{z}'), c+c'+\frac{1}{2}\omega((z,\bar{z}),(z',\bar{z}'))\Big).$$

Then the complexification H^c of H is

$$H^c := \{((z,w),c) \colon z,w \in \mathbb{C}^n, c \in \mathbb{C}\}$$

and the multiplication of H^c is obtained by replacing $(z,\bar{z})$ by (z,w) and $(z',\bar{z}')$ by (z',w') in the preceding equality. We denote by $\mathfrak{h}$ and $\mathfrak{h}^c$ the Lie algebras of H and H^c.

Now consider the group $S := \mathrm{Sp}(n,\mathbb{C}) \cap SU(n,n) \simeq \mathrm{Sp}(n,\mathbb{R})$, see [26], p. 501, and [21], p. 175. Then S consists of all matrices

$$h = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^\star - QQ^\star = I_n, \quad PQ^t = QP^t$$

and $S^c = \mathrm{Sp}(n,\mathbb{C})$.

The group S acts on H by

$$h \cdot ((z,\bar{z}),c) = (h(z,\bar{z}),c) = (Pz + Q\bar{z}, \bar{Q}z + \bar{P}\bar{z}, c)$$

where the elements of $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$ are considered as column vectors. Then we can form the semi-direct product $G := H \rtimes S$ called the multi-dimensional Jacobi group. The elements of G can be written as $((z,\bar{z}),c,h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in S$. The multiplication of G is thus given by

$$((z,\bar{z}),c,h)\cdot((z',\bar{z}'),c',h') = \Big((z,\bar{z})+h(z',\bar{z}'), c+c'+\frac{1}{2}\omega((z,\bar{z}),h(z',\bar{z}')), hh'\Big).$$

The complexification G^c of G is then the semi-direct product

$$G^c = H^c \rtimes \mathrm{Sp}(n,\mathbb{C})$$

whose elements can be written as $((z,w),c,h)$ where $z,w \in \mathbb{C}^n$, $c \in \mathbb{C}$, $h \in \mathrm{Sp}(n,\mathbb{C})$ and the multiplication of G^c is obtained by replacing $\bar{z}$ and $\bar{z}'$ by w and w' in the preceding formula.

We denote by $\mathfrak{s}$, $\mathfrak{s}^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of S, S^c, G and G^c. The Lie brackets of $\mathfrak{g}^c$ are given by

$$\begin{aligned}&[((z,w),c,A),((z',w'),c',A')]\\ &\quad = (A(z',w') - A'(z,w), \omega((z,w),(z',w')), [A,A']).\end{aligned}$$

Let θ denotes conjugation over the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X \in \mathfrak{g}^c$, we set $X^* = -\theta(X)$. We can easily verify that if $X = \big((z,w),c,\left(\begin{smallmatrix} A & B \\ C & -A^t\end{smallmatrix}\right)\big) \in \mathfrak{g}^c$ then we have

$$X^* = \left((-\bar{w},-\bar{z}),-\bar{c},\begin{pmatrix} \bar{A}^t & -\bar{C} \\ -\bar{B} & -\bar{A}\end{pmatrix}\right).$$

Also, we denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$ to G^c.

Let K be the subgroup of G consisting of all elements $\big((0,0),c,\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)\big)$ where $c \in \mathbb{R}$ and $P \in U(n)$. Then the Lie algebra $\mathfrak{k}$ of K is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ consisting of all elements of the form $((0,0),c,A)$ where A is diagonal is a compactly embedded Cartan subalgebra of $\mathfrak{g}$ [26], p. 250. Following [26], p. 532, we set

$$\mathfrak{p}^+ = \left\{\left((y,0),0,\begin{pmatrix}0 & Y\\ 0 & 0\end{pmatrix}\right): y\in\mathbb{C}^n, Y\in M_n(\mathbb{C}), Y^t=Y\right\}$$

and

$$\mathfrak{p}^- = \left\{\left((0,v),0,\begin{pmatrix}0 & 0\\ V & 0\end{pmatrix}\right): v\in\mathbb{C}^n, V\in M_n(\mathbb{C}), V^t=V\right\}.$$

Then we have the decomposition $\mathfrak{g}^c=\mathfrak{p}^+\oplus\mathfrak{k}^c\oplus\mathfrak{p}^-$.

Henceforth we denote by $a(y,Y)$ the element $\left((y,0),0,\left(\begin{smallmatrix}0 & Y\\ 0 & 0\end{smallmatrix}\right)\right)$ of $\mathfrak{p}^+$. Also, we denote by $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the above direct decomposition.

Let P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then we have

$$P^+ = \left\{\left((y,0),0,\begin{pmatrix}I_n & Y\\ 0 & I_n\end{pmatrix}\right): y\in\mathbb{C}^n, Y\in M_n(\mathbb{C}), Y^t=Y\right\}$$

and

$$P^- = \left\{\left((0,v),0,\begin{pmatrix}I_n & 0\\ V & I_n\end{pmatrix}\right): v\in\mathbb{C}^n, V\in M_n(\mathbb{C}), V^t=V\right\}.$$

In particular, we see that G is a group of the Harish-Chandra type [26], p. 507 (see also [30]), that is, the following properties are satisfied:

(1) $\mathfrak{g}^c=\mathfrak{p}^+\oplus\mathfrak{k}^c\oplus\mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^*=\mathfrak{p}^-$ and $[\mathfrak{k}^c,\mathfrak{p}^\pm]\subset\mathfrak{p}^\pm$;

(2) the multiplication map $P^+K^cP^-\to G^c, (z,k,y)\to zky$ is a biholomorphic diffeomorphism onto its open image;

(3) $G\subset P^+K^cP^-$ and $G\cap K^cP^-=K$.

We can easily verify that $g=\left((z_0,w_0),c_0,\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\right)\in G^c$ has a $P^+K^cP^-$-decomposition

$$g=\left((y,0),0,\begin{pmatrix}I_n & Y\\ 0 & I_n\end{pmatrix}\right)\cdot\left((0,0),c,\begin{pmatrix}P & 0\\ 0 & (P^t)^{-1}\end{pmatrix}\right)\cdot\left((0,v),0,\begin{pmatrix}I_n & 0\\ V & I_n\end{pmatrix}\right)$$

if and only if $\mathrm{Det}(D)\neq 0$ and, in this case, we have $y=z_0-BD^{-1}w_0$, $Y=BD^{-1}$, $v=D^{-1}w_0$, $V=D^{-1}C$, $P=A-BD^{-1}C=(D^t)^{-1}$ and $c=c_0-(1/4)i(z_0-BD^{-1}w_0)^tw_0$.

We denote by

$$\zeta\colon P^+K^cP^-\longrightarrow P^+,\qquad \kappa\colon P^+K^cP^-\longrightarrow K^c,\qquad \eta\colon P^+K^cP^-\longrightarrow P^-$$

the projections onto P^+-, K^c- and P^--components.

We can introduce an action (defined almost everywhere) of G^c on $\mathfrak{p}^+$ as follows. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, we define the element $g \cdot Z$ of $\mathfrak{p}^+$ by

$$g \cdot Z := \log \zeta(g \exp Z).$$

From the above formula for the $P^+K^cP^-$-decomposition, we deduce that the action of $g = \left((z_0, w_0), c_0, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\right) \in G^c$ on $a(y, Y) \in \mathfrak{p}^+$ is given by

$$g \cdot a(y, Y) = a(y', Y')$$

where $Y' := (AY + B)(CY + D)^{-1}$ and

$$y' := z_0 + Ay - (AY + B)(CY + D)^{-1}(w_0 + Cy).$$

This implies that

$$\mathcal{D} := G \cdot 0 = \{a(y, Y) \in \mathfrak{p}^+ \colon I_n - Y\bar{Y} > 0\} \cong \mathbb{C}^n \times \mathcal{B}.$$

where $\mathcal{B} := \{Y \in M_n(\mathbb{C}) : Y^t = Y,\ I_n - Y\bar{Y} > 0\}$.

Now we introduce a useful section $Z \to g_Z$ for the action of G on $\mathcal{D}$. Let $Z = a(y, Y) \in \mathcal{D}$. Define $g_Z := \left((z_0, \bar{z}_0), 0, \left(\begin{smallmatrix} P & Q \\ \bar{Q} & \bar{P} \end{smallmatrix}\right)\right) \in G$ as follows. We set

$$z_0 = (I_n - Y\bar{Y})^{-1}(y + Y\bar{y}), \quad P = (I_n - Y\bar{Y})^{-1/2}, \quad Q = (I_n - Y\bar{Y})^{-1/2}Y.$$

Then one has $g_Z \cdot 0 = Z$.

From the above formula for the action of G on $\mathcal{D}$, we can deduce the G-invariant measure μ on $\mathcal{D}$. Let μ_L be the Lebesgue measure on $\mathcal{D} \simeq \mathbb{C}^n \times \mathcal{B}$. Thus, we easily obtain that $d\mu(Z) = \operatorname{Det}(I_n - Y\bar{Y})^{-(n+2)}\, d\mu_L(y, Y)$, see for instance [5]. This result can be also deduced from the general formula for the invariant measure, see [26], p. 538.

In the rest of the paper, we fix the normalization of the Lebesgue measure as follows. For $y \in \mathbb{C}^n$, write $y = (a_1 + ib_1, a_2 + ib_2, \ldots, a_n + ib_n)$ with $a_j, b_j \in \mathbb{R}$ for $j = 1, 2, \ldots n$. Then we take the measure Lebesgue on $\mathbb{C}^n$ to be $dy := da_1 db_1 da_2 db_2 \ldots da_n db_n$. Similarly, writing $Y \in \mathcal{B}$ as $Y = (y_{kl})$, we denote by dY the Lebesgue measure on $\mathcal{B}$ defined by $dY := \prod_{kl} dy_{kl}$. Thus we set $d\mu_L(y, Y) := dy\, dY$.

Now we aim to compute the adjoint and coadjoint actions of G^c. First, we compute the adjoint action of G^c as follows. Let $g = (v_0, c_0, h_0) \in G^c$ where $v_0 \in \mathbb{C}^{2n}$, $c_0 \in \mathbb{C}$ and $h_0 \in S^c = \operatorname{Sp}(n, \mathbb{C})$ and $X = (w, c, U) \in \mathfrak{g}^c$ where $w \in \mathbb{C}^{2n}$, $c \in \mathbb{C}$ and $U \in \mathfrak{s}^c$. We set $\exp(tX) = (w(t), c(t), \exp(tU))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t = 0$ are w and c, we find that

$$\begin{aligned}\mathrm{Ad}(g)X &= \frac{d}{dt}(g\exp(tX)g^{-1})\Big|_{t=0}\\ &= \Big(h_0w - (\mathrm{Ad}(h_0)U)v_0, c + \omega(v_0, h_0w)\\ &\qquad - \frac{1}{2}\omega(v_0, (\mathrm{Ad}(h_0)U)v_0), \mathrm{Ad}(h_0)U\Big).\end{aligned}$$

On the other hand, let us denote by $\xi = (u, d, \varphi)$, where $u \in \mathbb{C}^{2n}$, $d \in \mathbb{C}$ and $\varphi \in (\mathfrak{s}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle \xi, (w, c, U)\rangle = \omega(u, w) + dc + \langle \varphi, U\rangle.$$

Moreover, for $u, v \in \mathbb{C}^{2n}$, we denote by $v \times u$ the element of $(\mathfrak{s}^c)^*$ defined by $\langle v \times u, U\rangle := \omega(u, Uv)$ for $U \in \mathfrak{s}^c$.

Let $\xi = (u, d, \varphi) \in (\mathfrak{g}^c)^*$ and $g = (v_0, c_0, h_0) \in G^c$. Then, by using the relation $\langle \mathrm{Ad}^*(g)\xi, X\rangle = \langle \xi, \mathrm{Ad}(g^{-1})X\rangle$ for $X \in \mathfrak{g}^c$, we obtain

$$\mathrm{Ad}^*(g)\xi = \Big(h_0u - dv_0, d, \mathrm{Ad}^*(h_0)\varphi + v_0 \times (h_0u - \frac{d}{2}v_0)\Big)$$

By restriction, we also get the formula for the coadjoint action of G. The following lemma will be needed later.

Lemma 2.1 ([15]). *The elements ξ_0 of $\mathfrak{g}^*$ fixed by K are the elements of the form $(0, d, \varphi_\lambda)$ where $d, \lambda \in \mathbb{R}$ and $\varphi_\lambda \in \mathfrak{s}^*$ is defined by* $\langle \varphi_\lambda, \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\rangle = i\lambda\,\mathrm{Tr}(A)$.

3. Holomorphic representations

The holomorphic representations of the multi-dimensional Jacobi group were studied by many authors, see in particular [26], [9], [10], [4], [5], and [6]. We follow here the general presentation of [26], Chapter XII (see also [14]).

Let χ be a unitary character of K. The extension of χ to K^c is also denoted by χ. We set $K_\chi(Z, W) := \chi(\kappa(\exp W^* \exp Z))^{-1}$ for Z, $W \in \mathcal{D}$ and $J_\chi(g, Z) := \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. We consider the Hilbert space $\mathcal{H}_\chi$ of all holomorphic functions f on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2\, K_\chi(Z, Z)^{-1} c_\chi d\mu(Z) < +\infty$$

where the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z, Z)^{-1}\, d\mu(Z).$$

We shall see that, under some hypothesis on χ, c_χ is well-defined and $\mathcal{H}_\chi \neq (0)$. In that case, $\mathcal{H}_\chi$ contains the polynomials [26], p. 546. Moreover, the formula

$$\pi_\chi(g) f(Z) = J_\chi(g^{-1}, Z)\, f(g^{-1}\cdot Z)$$

defines a unitary representation of G on $\mathcal{H}_\chi$ which is a highest weight representation [26], p. 540.

The space $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, if we set $e_Z(W) := K_\chi(W, Z)$ then we have we have the reproducing property $f(Z) = \langle f, e_Z\rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [26], p. 540. Here $\langle\cdot,\cdot\rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

Here we fix χ as follows. Let $\gamma \in \mathbb{R}$ and $m \in \mathbb{Z}$. Then, for each $k = \big((0,0), c, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)\big) \in K$, we set $\chi(k) := e^{i\gamma c}(\operatorname{Det} P)^m$.

We need the following lemma.

Lemma 3.1 ([24]). *Let $\lambda \in \mathbb{R}$. The integral*

$$J_n(\lambda) := \int_{\mathcal{B}} \operatorname{Det}(I_n - Y\bar{Y})^\lambda \, dY$$

is convergent if $\lambda > -1$ and in this case we have

$$J_n(\lambda) = \pi^{n(n+1)/2}\frac{\Gamma(2\lambda+3)\Gamma(2\lambda+5)\ldots\Gamma(2\lambda+2n-1)}{\Delta},$$

where

$$\Delta := (\lambda+1)(\lambda+2)\ldots(\lambda+n)\Gamma(2\lambda+n+2)\Gamma(2\lambda+n+3)\ldots\Gamma(2\lambda+2n).$$

Then we have the following result.

Proposition 3.2. (1) *Let $Z = a(y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We set*

$$E(y,v,Y,V) := 2y^t(I_n - \bar{V}Y)^{-1}\bar{v} + y^t(I_n-\bar{V}Y)^{-1}\bar{V}y + \bar{v}^t Y(I_n - \bar{V}Y)^{-1}\bar{v}.$$

Then we have

$$K_\chi(Z, W) = \operatorname{Det}(I_n - Y\bar{V})^m \exp\Big(\frac{\gamma}{4}E(y,v,Y,V)\Big).$$

(2) *We have $\mathcal{H}_\chi \neq (0)$ if and only if $\gamma > 0$ and $m + n + 1/2 < 0$. In this case, we also have $c_\chi^{-1} = (2\pi)^n\gamma^{-n}J_n(-m-n-3/2)$.*

(3) *For each* $g = \big((z_0, \bar{z}_0), c_0, \left(\begin{smallmatrix} P & Q \\ \bar{Q} & \bar{P} \end{smallmatrix}\right)\big) \in G$ *and each* $Z = a(y, Y) \in \mathcal{D}$, *we have*

$$J(g, Z) = e^{i\gamma c_0} \operatorname{Det}(\bar{Q}Y + \bar{P})^{-m} \exp\Big(\frac{\gamma}{4}\Big(z_0^t \bar{z}_0 + 2\bar{z}_0{}^t P y + y^t P^t \bar{Q} y \\ - (\bar{z}_0 + \bar{Q}y)^t (PY + Q)(\bar{Q}Y + \bar{P})^{-1}(\bar{z}_0 + \bar{Q}y)\Big)\Big)$$

Proof. We can verify (1) and (3) by computations based on the formula for κ given in Section 2. To prove (2), recall that, by [26], Theorem XII.5.6, we have $\mathcal{H}_\chi \neq (0)$ if and only if

$$I_\chi := \int_{\mathcal{D}} K_\chi(Z, Z)^{-1}\, d\mu(Z) < \infty.$$

Then we have to study the convergence of I_χ. By taking into account the expression of μ given in Section 2, we get

$$I_\chi = \int_{\mathcal{D}} \exp\Big(-\frac{\gamma}{4} E(y, y, Y, Y)\Big) \operatorname{Det}(I_n - Y\bar{Y})^{-m-n-2} d\mu_L(y, Y)$$

and, by making the change of variables $y \to (I_n - Y\bar{Y})^{1/2} y$ whose Jacobian is $\operatorname{Det}(I_n - Y\bar{Y})$, we find that

$$I_\chi = \int_{\mathbb{C}^n \times \mathcal{B}} \operatorname{Det}(I_n - Y\bar{Y})^{-m-n-1} \exp\Big(-\frac{\gamma}{4}(2y^t\bar{y} + y^t\bar{Y}y + \bar{y}^t Y \bar{y})\Big)\, dy\, dY.$$

But by [21], p. 258, we have

$$I_\chi = \Big(\frac{2\pi}{\gamma}\Big)^n \int_{\mathcal{B}} \operatorname{Det}(I_n - Y\bar{Y})^{-m-n-3/2}\, dY$$

for $\gamma > 0$. The result then follows from Lemma 3.1 □

Note that we can deduce from (3) of Proposition 3.2 an explicit but rather complicated expression for $\pi_\chi(g)$. Now we consider the derived representation $d\pi_\chi$.

Here we use the following notation. If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the right invariant vector field on L generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$.

By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

Lemma 3.3. *Let $X \in \mathfrak{g}^c$ and $g = z\,k\,y$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. We have*

(1) $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z)$;
(2) $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\, X))^+(k)$;
(3) $d\eta_g(X^+(g)) = (\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X))^+(y)$.

From this, we easily deduce the following proposition (see also [26], p. 515).

Proposition 3.4. *For $X \in \mathfrak{g}^c$, $f \in \mathcal{H}_\chi$ and $Z \in \mathcal{D}$, we have*

$$d\pi_\chi(X) f(Z) = d\chi(p_{\mathfrak{k}^c}(e^{-\mathrm{ad}\,Z}\, X))\, f(Z) - (df)_Z(p_{\mathfrak{p}^+}(e^{-\mathrm{ad}\,Z}\, X)).$$

In particular, we have

(1) *if $X \in \mathfrak{p}^+$ then $d\pi_\chi(X) f(Z) = -(df)_Z(X)$;*
(2) *if $X \in \mathfrak{k}^c$ then $d\pi_\chi(X) f(Z) = d\chi(X) f(Z) + (df)_Z([Z, X])$;*
(3) *if $X \in \mathfrak{p}^-$ then*

$$\begin{aligned} d\pi_\chi(X) f(Z) = (d\chi \circ p_{\mathfrak{k}^c})\Big(-[Z, X] + \frac{1}{2}[Z, [Z, X]]\Big) f(Z) \\ - (df_Z \circ p_{\mathfrak{p}^+})\Big(-[Z, X] + \frac{1}{2}[Z, [Z, X]]\Big). \end{aligned}$$

Now we need to introduce some notation. As usual, we write $Z \in \mathcal{D}$ as $Z = a(y, Y)$ where $y = (y_j)_{1\leq j\leq n} \in \mathbb{C}^n$ and $Y = (y_{kl})_{1\leq k,l\leq n} \in \mathcal{B}$. Define

$$\mathcal{I} := \{1, 2, \ldots, n\} \cup \{(k, l)\colon 1 \leq k, l \leq n\}$$

and consider $i \in \mathcal{I}$. Then we define ∂_i as follows. If $i \in \{1, 2, \ldots, n\}$ then ∂_i is the partial derivative with respect to y_i and if $i = (k, l)$ then ∂_i is the partial derivative with respect to y_{kl}. Moreover, we say that $P(Z)$ is a polynomial of degree $\leq q$ if $P(a(y, Y))$ is a polynomial of degree $\leq q$ in the variables y_j and y_{kl}.

From the preceding proposition we deduce the following result.

Proposition 3.5. *For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $d\pi_\chi(X_1 X_2 \ldots X_q)$ is a sum of terms of the form $P(Z)\partial_{i_1}\partial_{i_2}\ldots\partial_{i_r}$ where $r \leq q$, $i_1, i_2, \ldots, i_r \in \mathcal{I}$ and $P(Z)$ is a polynomial of degree $\leq 2q$.*

Proof. By Proposition 3.4 we see that, for each $X \in \mathfrak{g}^c$, $d\pi_\chi(X)$ is of the form $P^0(Z) + \sum_i P^i(Z)\partial_i$ where $P^0(Z)$, $P^i(Z)$ are polynomials of degree ≤ 2. The result then follows by induction on q. □

4. Generalities on the Stratonovich–Weyl correspondence

In this section, we review some general facts about the Berezin correspondence, the Berezin transform and the Stratonovich–Weyl correspondence.

First at all, recall that the Berezin correspondence on $\mathcal{D}$ is defined as follows. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. Then the Berezin symbol of A is the function $S_\chi(A)$ defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) := \frac{\langle A\, e_Z\,,\, e_Z\rangle_\chi}{\langle e_Z\,,\, e_Z\rangle_\chi}.$$

We can verify that each operator is determined by its Berezin symbol and that if an operator A has adjoint A^* then we have $S_\chi(A^*) = \overline{S_\chi(A)}$, see [7] and [8]. Moreover, for each operator A on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z for each $Z \in \mathcal{D}$ and each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi(\pi_\chi(g)^{-1}A\pi_\chi(g))(Z) = S_\chi(A)(g\cdot Z),$$

that is, S_χ is G-equivariant, see [14]. We have also the following result.

Proposition 4.1 ([14]). (1) *For $g \in G$ and $Z \in \mathcal{D}$, we have*

$$S_\chi(\pi_\chi(g))(Z) = \chi(\kappa(\exp Z^* g^{-1} \exp Z)^{-1}\kappa(\exp Z^* \exp Z)).$$

(2) *For $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have*

$$S_\chi(d\pi_\chi(X))(Z) = d\chi(p_{\mathfrak{k}^c}(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)\, X).$$

Let ξ be the linear form on $\mathfrak{g}^c$ defined by $\xi = -i\,d\chi$ on $\mathfrak{k}^c$ and $\xi = 0$ on $\mathfrak{p}^{\pm}$. Then we have $\xi(\mathfrak{g}) \subset \mathbb{R}$ and the restriction ξ_χ of ξ to $\mathfrak{g}$ is an element of $\mathfrak{g}^*$. In the notation of Section 2 we have $\xi_\chi = (0, \gamma, -m\varphi_0)$ where $\varphi_0 \in \mathfrak{s}^*$ is defined by $\langle \varphi_0, \left(\begin{smallmatrix} P & Q \\ \bar{Q} & \bar{P}\end{smallmatrix}\right)\rangle = i\,\mathrm{Tr}(P)$.

We denote by $\mathcal{O}(\xi_\chi)$ the orbit of ξ_χ in $\mathfrak{g}^*$ for the coadjoint action of G. This orbit is said to be associated with π_χ by the Kostant–Kirillov method of orbits, see [25] and [14]. Moreover, we have the following result.

Proposition 4.2 ([14]). (1) *For each $Z \in \mathcal{D}$, let*

$$\Psi_\chi(Z) := \mathrm{Ad}^*(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z))\,\xi_\chi.$$

Then, for each $X \in \mathfrak{g}^c$ and each $Z \in \mathcal{D}$, we have

$$S(d\pi_\chi(X))(Z) = i\,\langle \Psi_\chi(Z), X\rangle.$$

(2) *For each* $g \in G$ *and each* $Z \in \mathcal{D}$, *we have* $\Psi_\chi(g\cdot Z) = \mathrm{Ad}^*(g)\,\Psi_\chi(Z)$.

(3) *The map* Ψ_χ *is a diffeomorphism from* $\mathcal{D}$ *onto* $\mathcal{O}(\xi_\chi)$.

In order to make the expression of Ψ_χ more explicit, we introduce the following notation. For $\varphi \in \mathfrak{s}^*$, let $\alpha(\varphi)$ be the unique element of $\mathfrak{s}$ such that $\langle \varphi, X\rangle = \mathrm{Tr}(\alpha(\varphi)X)$ for each $X \in \mathfrak{s}$. In particular, one has $\alpha(\varphi_0) = \frac{1}{2}\begin{pmatrix} iI_n & 0 \\ 0 & -iI_n\end{pmatrix}$. Moreover, for $u = (x, \bar{x}) \in \mathbb{C}^{2n}$ and $u = (y, \bar{y}) \in \mathbb{C}^{2n}$ we have

$$\theta(v\times u) = \frac{1}{2}\begin{pmatrix} -iy\bar{x}^t & iyx^t \\ -i\bar{y}\bar{x}^t & i\bar{y}x^t\end{pmatrix}.$$

Note also that θ intertwines Ad^* and Ad. Then we have the following result.

Proposition 4.3 ([15]). *The map* $\psi_\chi\colon \mathcal{D} \to \mathcal{O}(\xi_\chi)$ *is given by*

$$\psi_\chi(a(y,Y)) = (-d(y_1,\bar{y}_1), \gamma, \varphi(y,Y))$$

where $y_1 = (I_n - Y\bar{Y})^{-1}(y + Y\bar{y})$ *and*

$$\begin{aligned}\varphi(y,Y) :=& -m\,\mathrm{Ad}^*\begin{pmatrix} (I_n - Y\bar{Y})^{-1/2} & (I_n - Y\bar{Y})^{-1/2}Y \\ (I_n - \bar{Y}Y)^{-1/2}\bar{Y} & (I_n - \bar{Y}Y)^{-1/2}\end{pmatrix}\varphi_0 \\ & - \frac{\gamma}{2}(y_1,\bar{y}_1)\times(y_1,\bar{y}_1).\end{aligned}$$

Moreover, we have

$$\alpha(\varphi(y,Y)) = -\frac{\gamma}{4}\begin{pmatrix} -iy_1\bar{y}_1^t & iy_1y_1^t \\ -i\bar{y}_1\bar{y}_1^t & i\bar{y}_1y_1^t\end{pmatrix} - \frac{m}{2}i\begin{pmatrix} A(Y) & B(Y) \\ -\overline{B(Y)} & -\overline{A(Y)}\end{pmatrix}.$$

where

$$\begin{aligned} A(Y) :=&(I_n + Y\bar{Y})(I_n - Y\bar{Y})^{-1/2}(I_n - \bar{Y}Y)^{-1/2};\\ B(Y) :=& -2Y(I_n - \bar{Y}Y)^{-1/2}(I_n - Y\bar{Y})^{-1/2}.\end{aligned}$$

Now we recall briefly the construction of the Stratonovich–Weyl correspondence [20], [13], and [14]. Denote by $\mathcal{L}_2(\mathcal{H}_\chi)$ the space of all Hilbert-Schmidt operators on $\mathcal{H}_\chi$ and by μ_χ the G-invariant measure on $\mathcal{D}$ defined by $d\mu_\chi(Z) = c_\chi d\mu(Z)$. Then the map S_χ is a bounded operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ into $L^2(\mathcal{D}, \mu_\chi)$ which is one-to-one and has dense range [29], [32]. Moreover, the Berezin transform is the operator on $L^2(\mathcal{D}, \mu_\chi)$ defined by $B_\chi := S_\chi S_\chi^*$. We can easily verify that we have the following integral formula for B_χ:

$$B_\chi F(Z) = \int_{\mathcal{D}} F(W)\,\frac{|\langle e_Z, e_W\rangle|_\chi^2}{\langle e_Z, e_Z\rangle_\chi \langle e_W, e_W\rangle_\chi}\,d\mu_\chi(W) \tag{4.1}$$

(see [7], [32], and [33] for instance).

Let ρ be the left-regular representation of G on $L^2(\mathcal{D},\mu_\chi)$. As a consequence of the equivariance property for S_χ, we see that B_χ commute with $\rho(g)$ for each $g\in G$.

Consider the polar decomposition of S_χ:

$$S_\chi=(S_\chi S_\chi^*)^{1/2}W_\chi=B_\chi^{1/2}W_\chi,$$

where $W_\chi:=B_\chi^{-1/2}S_\chi$ is a unitary operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ onto $L^2(\mathcal{D},\mu_\chi)$. Note that, by (2) of Proposition 4.2, the measure $\mu_0:=(\Psi_\chi^{-1})^*(\mu_\chi)$ is a G-invariant measure on $\mathcal{O}(\xi_\chi)$. The following proposition is then immediate.

Proposition 4.4. 1) *The map $W_\chi:\mathcal{L}_2(\mathcal{H}_\chi)\to L^2(\mathcal{D},\mu_\chi)$ is a Stratonovich–Weyl correspondence for the triple $(G,\pi_\chi,\mathcal{D})$, that is, we have*

(1) $W_\chi(A^*)=\overline{W_\chi(A)}$;

(2) $W_\chi(\pi_\chi(g)\,A\,\pi_\chi(g)^{-1})(Z)=W_\chi(A)(g^{-1}\cdot Z)$;

(3) W_χ *is unitary.*

2) *Similarly, the map $\mathcal{W}_\chi\colon\mathcal{L}_2(\mathcal{H}_\chi)\to L^2(\mathcal{O}(\xi_\chi),\mu_0)$ defined by*

$$\mathcal{W}_\chi(A)=W_\chi(A)\circ\Psi_\chi^{-1}$$

is a Stratonovich–Weyl correspondence for the triple $(G,\pi_\chi,\mathcal{O}(\xi_\chi))$.

Note that we have relaxed here (1) of Definition 1.1 which is not adapted to the present setting since I is not Hilbert-Schmidt. However, this requirement should be hold in some generalize sense, see for instance [22].

5. Extension of the Berezin transform

The aim of this section is to extend the Berezin transform to a class of functions which contains $S_\chi(d\pi_\chi(X))$ for each $X\in\mathfrak{g}^c$, in order to define and study $W_\chi(d\pi_\chi(X))$. This question was already investigated in [14] in the case of a reductive Lie group and in [17] in the case of the Jacobi group.

For $Z,\,W\in\mathcal{D}$, we set $l_Z(W):=\log\eta(\exp Z^*\exp W)\in\mathfrak{p}^-$.

Lemma 5.1. (1) *For each $Z,\,W\in\mathcal{D}$ and $V\in\mathfrak{p}^+$, we have*

$$\frac{d}{dt}\,e_Z(W+tV)\Big|_{t=0}=-e_Z(W)\,(d\chi\circ p_{\mathfrak{k}^c})\Big([l_Z(W),V]+\frac{1}{2}[l_Z(W),[l_Z(W),V]]\Big).$$

(2) *For each* Z, $W \in \mathcal{D}$ *and* $V \in \mathfrak{p}^+$, *we have*

$$\frac{d}{dt}\, l_Z(W+tV)\Big|_{t=0} = p_{\mathfrak{p}^-}\Big([l_Z(W),V] + \frac{1}{2}[l_Z(W),[l_Z(W),V]]\Big).$$

(3) *For each* $i_1, i_2, \ldots, i_q \in \mathcal{I}$ *and* $Z \in \mathcal{D}$, *the function* $(\partial_{i_1}\partial_{i_2}\ldots\partial_{i_q} e_Z)(W)$ *is of the form* $e_Z(W)Q(l_Z(W))$ *where* Q *is a polynomial on* $\mathfrak{p}^-$ *of degree* $\leq 2q$.

(4) *For each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, *the function* $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))(Z)$ *is a sum of terms of the form* $P(Z)Q(l_Z(Z))$ *where* P *and* Q *are polynomials of degree* $\leq 2q$.

Proof. The proof of this lemma is similar to those of Lemma 6.2 of [14] and Lemma 5.2 of [17]. Note that the proof of (1) is essentially based on Lemma 3.3, that (3) is a consequence of (1) and (2) and, finally, that (4) follows from (3) and Proposition 3.5. □

We can then establish the main result of this section.

Proposition 5.2. *If* $q < (1/4)(-m-2n)$ *then for each* $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, *the Berezin transform of* $S_\chi(d\pi_\chi(X_1X_2\ldots X_q))$ *is well-defined.*

Proof. First, we fix $Z \in \mathcal{D}$ and we make the change of variables $W \to g_Z \cdot W$ in (4.1). Then we obtain

$$(B_\chi F)(Z) = \int_{\mathcal{D}} F(g_Z \cdot W)\langle e_W, e_W\rangle_\chi^{-1}\, d\mu_\chi(W).$$

We take $F = S_\chi(d\pi_\chi(X_1X_2\ldots X_q)$ and we set

$$Y_k := \mathrm{Ad}(g_Z^{-1})X_k$$

for $k = 1, 2, \ldots, q$. Then, by G-invariance of S_χ, we have

$$F(g_Z \cdot W) = S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$$

for each $W \in \mathcal{D}$. Recall that, by the preceding lemma, the function

$$S_\chi(d\pi_\chi(Y_1Y_2\ldots Y_q))(W)$$

is a sum of terms of the form $P(W)Q(l_W(W))$ where P and Q are polynomials of degree $\leq 2q$. Then we have to prove that, for each $q < (1/4)(-m-2n)$ and each polynomials P and Q of degree $\leq 2q$, the integral

$$I := \int_{\mathcal{D}} P(W)Q(l_W(W))\langle e_W, e_W\rangle_\chi^{-1}\, d\mu_\chi(W)$$

is convergent.

First, we note that if $W=a(y,Y)$ then

$$l_W(W)=\left((0,-(I_n-\overline{Y}Y)^{-1}(\bar{y}+\overline{Y}y)),0,\begin{pmatrix}0&0\\-(I_n-\overline{Y}Y)^{-1}\overline{Y}&0\end{pmatrix}\right).$$

Thus we have

$$\begin{aligned}I=c_\chi\int_{\mathcal{D}}&P(y,Y)Q(-(I_n-\overline{Y}Y)^{-1}(\bar{y}+\overline{Y}y),-(I_n-\overline{Y}Y)^{-1}\overline{Y})\\&\exp\Big(-\frac{\gamma}{4}(2y^t(I_n-\overline{Y}Y)^{-1}\bar{y}\\&\qquad+y^t(I_n-\overline{Y}Y)^{-1}\overline{Y}y+\bar{y}^tY(I_n-\overline{Y}Y)^{-1}\bar{y})\Big)\\&\operatorname{Det}(I_n-Y\overline{Y})^{-m-n-2}\,d\mu_L(y,Y).\end{aligned}$$

As in the proof of Proposition 3.2, we make the change of variables

$$y\longmapsto(I_n-Y\overline{Y})^{1/2}y$$

and we find that

$$\begin{aligned}I=c_\chi\int_{\mathcal{D}}&P((I_n-Y\overline{Y})^{1/2}y,Y)\\&Q(-(I_n-\overline{Y}Y)^{-1/2}(\bar{y}+\overline{Y}y),-(I_n-\overline{Y}Y)^{-1}\overline{Y})\\&\exp\Big(-\frac{\gamma}{4}(2y^t\bar{y}+y^t\overline{Y}y+\bar{y}^tY\bar{y})\Big)\operatorname{Det}(I_n-Y\overline{Y})^{-m-n-1}\,d\mu_L(y,Y).\end{aligned}$$

Now we make the following remarks.

(1) Since P is a polynomial of degree $\leq 2q$ and $\mathcal{B}$ is bounded, there exists a constant $C_0>0$ such that

$$|P((I_n-Y\overline{Y})^{1/2}y,Y)|\leq C_0\sum_{r\leq 2q}|y|^r$$

for each $(y,Y)\in\mathbb{C}^n\times\mathcal{B}$.

(2) By using the classical formula for the inverse of a matrix, for each $Y\in\mathcal{B}$ we have

$$(I_n-\overline{Y}Y)^{-1}=\operatorname{Det}(I_n-\overline{Y}Y)^{-1}C(I_n-\overline{Y}Y)^t$$

where $C(A)$ denotes the cofactor matrix of a matrix A. From this we deduce that there exists a constant $C_0'>0$ such that

$$\begin{aligned}&|Q(-(I_n-\overline{Y}Y)^{-1}(I_n-\overline{Y}Y)^{1/2}(\bar{y}+\overline{Y}y),-(I_n-\overline{Y}Y)^{-1}\overline{Y})|\\&\quad\leq C_0'\operatorname{Det}(I_n-Y\overline{Y})^{-2q}\sum_{r\leq 2q}|y|^r\end{aligned}$$

for each $(y,Y)\in\mathbb{C}^n\times\mathcal{B}$.

(3) For each $(y, Y) \in \mathbb{C}^n \times \mathcal{B}$, we have

$$2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^tY\bar{y} = 2(y^ty + \mathrm{Re}(y^t\overline{Y}y)) \geq 2(1 - \|Y\|)|y|^2.$$

Here $\|\cdot\|$ denotes the operator norm corresponding to the Hermitian norm on $\mathbb{C}^n$.

By using these remarks, we can reduce the study of the convergence of I to that of the integral

$$I' := \int_{\mathcal{D}} \mathrm{Det}(I_n - Y\overline{Y})^{-2q-m-n-1}|y|^{4q}e^{-(\gamma/2)|y|^2(1-\|Y\|)}\,d\mu_L(y, Y).$$

We set

$$I(Y) := \int_{\mathbb{C}^n} |y|^{4q}e^{-\frac{\gamma}{2}|y|^2(1-\|Y\|)}\,dy$$

and, passing to spherical coordinates, we see that there exists some constants $C, C' > 0$ such that, for each $Y \in \mathcal{B}$, we have

$$I(Y) = C\int_0^{+\infty} x^{4q+2n-1}e^{-(\gamma/2)(1-\|Y\|)x^2}\,dx = C'(1 - \|Y\|)^{-2q-n}.$$

Then we have to study the integral

$$I'' := \int_{\mathcal{B}} \mathrm{Det}(I_n - Y\overline{Y})^{-2q-m-n-1}(1 - \|Y\|)^{-2q-n}\,dY.$$

Now denote by $\lambda_s(Y\overline{Y})$ the maximum of the eigenvalues of $Y\overline{Y}$ and recall that $\|Y\|^2 = \lambda_s(Y\overline{Y})$. Then we have

$$\mathrm{Det}(I_n - Y\overline{Y}) \leq 1 - \lambda_s(Y\overline{Y}) = 1 - \|Y\|^2 \leq 2(1 - \|Y\|)$$

for each $Y \in \mathcal{B}$. Thus we obtain

$$\mathrm{Det}(I_n - Y\overline{Y})^{-2q-m-n-1}(1 - \|Y\|)^{-2q-n} \leq 2^{2q+n}\,\mathrm{Det}(I_n - Y\overline{Y})^{-4q-m-2n-1}$$

for each $Y \in \mathcal{B}$. But by Lemma 3.1, we see that $J_n(-4q - m - 2n - 1)$ hence I'' converges if $q < \frac{1}{4}(-m - 2n)$. This ends the proof. $\square$

6. Stratonovich–Weyl symbols of derived representation operators

Here we assume that $-m > 2n + 4$. Then, by Proposition 5.2, $B_\chi(S_\chi(d\pi_\chi(X)))$ is well-defined for each $X \in \mathfrak{g}^c$. We aim to define also $W_\chi(d\pi_\chi(X))$ for $X \in \mathfrak{g}^c$. To this goal, we first introduce a space of functions on $\mathcal{D}$ which is stable under B_χ and contains $S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}^c$.

Recall that, by Proposition 4.2 we have $S_\chi(d\pi_\chi(X))(Z) = i\xi(\mathrm{Ad}(g_Z^{-1})X)$ for each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$. This leads us to introduce the vector space $\mathcal{S}$ generated by the functions $Z \to \phi_0(\mathrm{Ad}(g_Z^{-1})X)$ where $X \in \mathfrak{g}^c$ and ϕ_0 is an element of $(\mathfrak{g}^c)^*$ which is $\mathrm{Ad}^*(K)$-invariant. Such elements ϕ_0 were determined in [15], see Lemma 2.1 above. The following proposition is analogous to Proposition 6.2 of [17].

Proposition 6.1. *Let $\phi\colon \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ be a function such that*

(i) *for each $Z \in \mathcal{D}$, the map $X \to \phi(Z, X)$ is linear;*

(ii) *for each $X \in \mathfrak{g}^c$, $g \in G$ and $Z \in \mathcal{D}$, we have $\phi(g\cdot Z, X) = \phi(Z, \mathrm{Ad}(g^{-1})X)$.*

Then

(1) *the element ϕ_0 of $(\mathfrak{g}^c)^*$ defined by $\phi_0(X) := \phi(0, X)$ is fixed by K;*

(2) *for each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have*

$$\phi(Z, X) = \phi_0(\mathrm{Ad}(g_Z^{-1})X)$$

and

$$\begin{aligned}\phi(Z, X) &= \phi_0(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X)\\ &= (\phi_0 \circ p_{\mathfrak{k}^c})(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X);\end{aligned}$$

(3) *for each $X \in \mathfrak{g}^c$, the function $\psi\colon \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ given by*

$$\psi(\cdot, X) = B_\chi(\phi(\cdot, X))$$

is well-defined and satisfies (i) *and* (ii);

(4) *the vector space $\mathcal{S}$ is generated by all the functions $Z \to \phi(Z, X)$ for ϕ as above and $X \in \mathfrak{g}^c$. Moreover, $\mathcal{S}$ is stable under B_χ.*

Proof. (1) By (ii), for each $k \in K$ and $X \in \mathfrak{g}^c$, we have

$$\begin{aligned}(\mathrm{Ad}^*(k)\phi_0)(X) &= \phi_0(\mathrm{Ad}(k^{-1})X) = \phi(0, \mathrm{Ad}(k^{-1})X)\\ &= \phi(k\cdot 0, X) = \phi(0, X) = \phi_0(X).\end{aligned}$$

Then ϕ_0 is fixed by K.

(2) The first assertion follows from (ii). To prove the second assertion, recall that by [15], there exists $k_Z \in K$ such that $g_Z = \exp(-Z^*)\zeta(\exp Z^* \exp Z)k_Z^{-1}$. Then we have

$$\begin{aligned}\phi(Z, X) &= \phi_0(\mathrm{Ad}(k_Z\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X)\\ &= \phi_0(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X)\end{aligned}$$

and, noting that $\phi_0|_{\mathfrak{p}^\pm} = 0$ by Lemma 2.1, we can conclude that

$$\phi(Z, X) = (\phi_0 \circ p_{\mathfrak{k}^c})(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X).$$

(3) By using the same arguments as in the proof of Proposition 5.2, we can verify that, for each $X \in \mathfrak{g}^c$, the Berezin transform of $\phi(\cdot, X)$ is well-defined. The second assertion follows from the fact that B_χ commutes to the $\rho(g)$, $g \in G$.

(4) This follows from the preceding statements. □

Now we need the following lemmas.

Lemma 6.2. *For each $Y \in \mathcal{B}$, we have*

$$\begin{aligned} I_1(Y) &:= \int_{\mathbb{C}^n} y^t \bar{y} \exp\Big(-\frac{\gamma}{4}\big(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^t Y \bar{y}\big)\Big)\, dy \\ &= \frac{2}{\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n \mathrm{Det}(I_n - Y\overline{Y})^{-1/2}\,\mathrm{Tr}((I_n - Y\overline{Y})^{-1}) \\ I_2(Y) &:= \int_{\mathbb{C}^n} y^t \overline{Y} y \exp\Big(-\frac{\gamma}{4}\Big(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^t Y \bar{y}\Big)\Big)\, dy \\ &= \frac{2}{\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n \mathrm{Det}(I_n - Y\overline{Y})^{-1/2}(n - \mathrm{Tr}((I_n - Y\overline{Y})^{-1})). \end{aligned}$$

Proof. For $s \in [0, 1]$ and $Y \in \mathcal{B}$, let us introduce

$$J_s(Y) = \int_{\mathbb{C}^n} \exp\Big(-\frac{\gamma}{4}(2y^t\bar{y} + sy^t\overline{Y}y + s\bar{y}^t Y\bar{y})\Big)\, dy.$$

By [21], p. 258, we have

$$J_s(Y) = \Big(\frac{2\pi}{\gamma}\Big)^n \mathrm{Det}(I_n - s^2 Y\overline{Y})^{-1/2}.$$

Then, by computing the derivative of $J_s(Y)$ at $s = 1$, we get

$$\begin{aligned} &\int_{\mathbb{C}^n} (y^t\overline{Y}y + \bar{y}^t Y\bar{y}) \exp\Big(-\frac{\gamma}{4}(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^t Y\bar{y})\Big)\, dy \\ &\quad = -\frac{4}{\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n \mathrm{Det}(I_n - Y\overline{Y})^{-1/2}\,\mathrm{Tr}((I_n - Y\overline{Y})^{-1} Y\overline{Y}). \end{aligned}$$

Thus we have

$$I_2(Y) + \overline{I_2(Y)} = -\frac{4}{\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n \mathrm{Det}(I_n - Y\overline{Y})^{-1/2}\,\mathrm{Tr}(-I_n + (I_n - Y\overline{Y})^{-1}). \tag{6.1}$$

On the other hand, by integrating by parts, we get

$$J_1(Y) = -\int_{\mathbb{C}^n} y_k \frac{\partial}{\partial_k}\Big(\exp\Big(-\frac{\gamma}{4}(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^tY\bar{y})\Big)\Big)\,dy$$
$$\frac{\gamma}{4}\int_{\mathbb{C}^n} y_k(2\bar{y}_k + 2e_k^t\overline{Y}y)\exp\Big(-\frac{\gamma}{4}(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^tY\bar{y})\Big)dy$$

for each $k = 1, 2, \ldots, k$. By summing up over k, we obtain

$$nJ_1(Y) = \frac{\gamma}{2}(I_1(Y) + I_2(Y)). \tag{6.2}$$

This last equation implies that $I_2(Y)$ is real since $J_1(Y)$ and $I_1(Y)$ are real. Consequently, (6.1) gives the desired value for $I_2(Y)$ hence (6.2) provides the desired value for $I_1(Y)$. □

The following lemma gives a useful expression for $K_\chi(Z, Z)$ which will be used in the proof of Proposition 6.4 .

Lemma 6.3. *For each $Z = a(y, Y)$, let $z_0 := (I_n - Y\overline{Y})^{-1}(y + Y\bar{y})$. Then we have*

$$K_\chi(Z, Z) = \exp\Big(\frac{\gamma}{4}(2z_0^t\bar{z}_0 - z_0^t\overline{Y}z_0 - \bar{z}_0{}^tY\bar{z}_0)\Big)\operatorname{Det}(I_n - Y\overline{Y})^m.$$

Proof. The result follows from Proposition 3.2 by a routine computation. Alternatively, by [14], Lemma 4.1, we have

$$\langle e_Z, e_Z\rangle_\chi = \langle e_{g_Z\cdot 0}, e_{g_Z\cdot 0}\rangle_\chi = \langle \overline{J(g_Z, 0)}\pi(g_Z)e_0, \overline{J(g_Z, 0)}\pi(g_Z)e_0\rangle_\chi$$
$$= |J(g_Z, 0)|^2 = |\chi(\kappa(g_Z))|^2$$

and, by taking into account the expressions of χ and g_Z, we then recover the desired formula for $K_\chi(Z, Z)$. □

Let us introduce the following integral of Hua's type:

$$K_n(\lambda) := \int_{\mathcal{B}} \operatorname{Tr}((I_n - Y\overline{Y})^{-1})\operatorname{Det}(I_n - Y\overline{Y})^\lambda dY.$$

Since the maximum of the eigenvalues of $(I_n - Y\overline{Y})^{-1}$ is $(1 - \lambda_s(Y\overline{Y}))^{-1}$, we have

$$\operatorname{Tr}((I_n - Y\overline{Y})^{-1}) \le n(1 - \lambda_s(Y\overline{Y}))^{-1} \le n\operatorname{Det}(I_n - Y\overline{Y})^{-1}$$

and then we see that $K_n(\lambda)$ converges for $\lambda > 2$ since $J_n(\lambda)$ converges for $\lambda > 1$, see Lemma 3.1.

Also, we denote by ϕ^1 and ϕ^2 the elements of $\mathcal{S}$ defined by $\phi_0^1 = (0, 1, 0)$ and $\phi_0^2 = (0, 0, \varphi_0)$. We are now in position to establish the following proposition.

Proposition 6.4. *Let*

$$\mu_n := \frac{c_\chi}{n\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big);$$

$$\nu_n := -1 + \frac{2c_\chi}{n}\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big).$$

Let $\phi \in \mathcal{S}$ defined by $\phi_0 = (0, d, \lambda\varphi_0)$ with $d, \lambda \in \mathbb{C}$. Let $\psi \in \mathcal{S}$ such that $\psi(\cdot, X) = B_\chi(\phi(\cdot, X))$ for each $X \in \mathfrak{g}^c$. Then we have $\psi_0 = (0, d, d\mu_n + \lambda\nu_n)$.

Proof. We will use the formula

$$\psi_0(X) = \int_{\mathcal{D}} \phi_0(\mathrm{Ad}(g_Z^{-1})X))K_\chi(Z,Z)^{-1}c_\chi d\mu(Z)$$

in order to compute the Berezin transforms $\psi^1(\cdot, X)$ and $\psi^2(\cdot, X)$ of $\phi^1(\cdot, X)$ and $\phi^2(\cdot, X)$.

We write $\psi_0^1 = (0, d_1, \lambda_1\varphi_0)$ with $d_1, \lambda_1 \in \mathbb{C}$. Let $H_1 := ((0,0), 1, 0)$. Then we have $\mathrm{Ad}(g_Z^{-1})H_1 = H_1$ hence $\phi_0^1(\mathrm{Ad}(g_Z^{-1})H_1) = 1$ for each $Z \in \mathcal{D}$. This gives

$$\psi_0^1(H_1) = \int_{\mathcal{D}} K_\chi(Z,Z)^{-1}c_\chi d\mu(Z) = 1.$$

On the other hand, we also have $\psi_0^1(H_1) = d_1$. Then we find $d_1 = 1$.

Now, let $H_2 := \big((0,0), 0, \big(\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\big)\big)$. Then, for each $Z \in \mathcal{D}$, we have

$$\mathrm{Ad}(g_Z^{-1})H_2 = \left(\star,\ \frac{i}{2}z_0^t\bar{z}_0,\ \begin{pmatrix} (I_n - Y\bar{Y})^{-1}(I_n + Y\bar{Y}) & \star \\ \star & \star \end{pmatrix}\right)$$

where, as usual, $z_0 = (I_n - Y\bar{Y})^{-1}(y + Y\bar{y})$. Consequently, we have

$$\phi_0^1(\mathrm{Ad}(g_Z^{-1})H_2) = \frac{i}{2}z_0^t\bar{z}_0.$$

Thus, by Lemma 6.3, we get

$$\psi_0^1(H_2) = \frac{ic_\chi}{2}\int_{\mathcal{D}} z_0^t\bar{z}_0 \exp\Big(-\frac{\gamma}{4}(2z_0^t\bar{z}_0 - z_0^t\bar{Y}z_0 - \bar{z}_0{}^tY\bar{z}_0)\Big) \\ \mathrm{Det}(I_n - Y\bar{Y})^{-m-n-2}\,dy\,dY$$

and we make the change of variables

$$y = z_0 - Y\bar{z}_0$$

whose Jacobian is $\operatorname{Det}(I_n - Y\overline{Y})$. Hence, by using Lemma 6.2, we obtain

$$\psi_0^1(H_2) = \frac{i c_\chi}{\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big).$$

On the other hand, it is clear that $\psi_0^1(H_2) = i\lambda_1 n$. Finally, we find that

$$\lambda_1 = \frac{c_\chi}{n\gamma}\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big) = \mu_n.$$

Similarly, we write $\psi_0^2 = (0, d_2, \lambda_2\varphi_0)$. Since we have $\phi_0^2(\operatorname{Ad}(g_Z^{-1})H_1) = 0$ for each $Z \in \mathcal{D}$, we first obtain $d_2 = \psi_0^2(H_1) = 0$. Moreover, for each $Z = a(y, Y) \in \mathcal{D}$, we also have

$$\begin{aligned}\phi_0^2(\operatorname{Ad}(g_Z^{-1})H_2) &= i\operatorname{Tr}(I_n - Y\overline{Y})^{-1}(I_n + Y\overline{Y})\\ &= i(-n + 2\operatorname{Tr}\big((I_n - Y\overline{Y})^{-1}\big)).\end{aligned}$$

Then, changing variables $y \to (I_n - Y\overline{Y})^{1/2}y$, we get

$$\begin{aligned}\psi_0^2(\operatorname{Ad}(g_Z^{-1})H_2) = -in + 2ic_\chi \int_{\mathcal{B}\times\mathbb{C}^n} &\exp\Big(-\frac{\gamma}{4}\big(2y^t\bar{y} + y^t\overline{Y}y + \bar{y}^t Y\bar{y}\big)\Big)\\ &\operatorname{Tr}((I_n - Y\overline{Y})^{-1})\operatorname{Det}(I_n - Y\overline{Y})^{-m-n-1}\,dy\,dY.\end{aligned}$$

Thus, by using [21], p. 248, we obtain

$$\psi_0^2(\operatorname{Ad}(g_Z^{-1})H_2) = -in + 2ic_\chi\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big).$$

Also, we have $\psi_0^2(\operatorname{Ad}(g_Z^{-1})H_2) = i\lambda_2 n$. This gives

$$\lambda_2 = -1 + \frac{2c_\chi}{n}\Big(\frac{2\pi}{\gamma}\Big)^n K_n\Big(-m-n-\frac{3}{2}\Big) = \nu_n.$$

This finishes the proof. □

Recall that c_χ can be expressed in terms of the Hua's integral $J_n(-m-n-3/2)$ which can be explicitly computed, see Proposition 3.2 and Lemma 3.1. However, it seems difficult to compute $K_n(-m-n-3/2)$ similarly.

Now we give the matrix of B_χ in a suitable basis of $\mathcal{S}$. First, we consider the basis of $\mathfrak{g}^c$ consisting of the elements

$$X_i = ((e_i, 0), 0, 0),$$

$$Y_j = ((0, e_j), 0, 0),$$

$$F_{ij} = \left((0,0), 0, \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}\right),$$

$$G_{ij} = \left((0,0), 0, \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}\right),$$

$$H_1 = ((0,0), 1, 0),$$

$$A_{ij} = \left((0,0), 0, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}\right),$$

for $i, j = 1, 2, \ldots, n$, E_{ij} denoting the $n \times n$ complex matrix whose ij-th entry is 1 and all of whose other entries are 0.

Note that $\phi^2(\cdot, X_i) = \phi^2(\cdot, Y_j) = \phi^2(\cdot, H_1) = 0$. Then, from the preceding proposition, we easily deduce the following result.

Corollary 6.5. *The functions $\phi^1(\cdot, X_i)$, $\phi^1(\cdot, Y_j)$, $\phi^1(\cdot, H_1)$, $\phi^1(\cdot, F_{ij})$, $\phi^1(\cdot, G_{ij})$, $\phi^1(\cdot, A_{ij})$, $\phi^2(\cdot, F_{ij})$, $\phi^2(\cdot, G_{ij})$ and $\phi^2(\cdot, A_{ij})$ form a basis for $\mathcal{S}$ in which B_χ has matrix*

$$\begin{pmatrix} I_{2n+1} & O & O \\ O & I_{3n^2} & O \\ O & \mu_n I_{3n^2} & \nu_n I_{3n^2} \end{pmatrix}.$$

Recall that for each $X \in \mathfrak{g}^c$, we have $S_\chi(d\pi_\chi(X)) \in \mathcal{S}$. Consequently, we see that $W_\chi(d\pi_\chi(X)) = B_\chi^{-1/2}(S_\chi(d\pi_\chi(X)))$ is well-defined. Moreover, we have the following proposition.

Proposition 6.6. *For each $X \in \mathrm{Span}_{\mathbb{C}}\{H_1, X_i, Y_j,\ 1 \le i, j \le n\}$, we have $W_\chi(d\pi_\chi(X)) = S_\chi(d\pi_\chi(X))$. For each $X \in \mathrm{Span}_{\mathbb{C}}\{F_{ij}, G_{ij}, A_{ij},\ 1 \le i, j \le n\}$, we have*

$$W_\chi(d\pi_\chi(X)) = S_\chi(d\pi_\chi(X)) + i(1 - \nu_n^{-1/2})\Big(\frac{\gamma\mu_n}{1-\nu_n} + m\Big)\phi^2(\cdot, X).$$

Proof. For each $X \in \mathfrak{g}^c$ we have

$$S_\chi(d\pi_\chi(X)) = d\chi(\mathrm{Ad}(g_Z^{-1})X) = i\gamma\phi^1(\cdot, X) - im\phi^2(\cdot, X).$$

Now, by using the preceding corollary, we see that the matrix of $B_\chi^{-1/2}$ with respect to the above basis of $\mathcal{S}$ is

$$\begin{pmatrix} I_{2n+1} & O & O \\ O & I_{3n^2} & O \\ O & -\dfrac{\mu_n \nu_n^{-1/2}}{1+\nu_n^{1/2}} I_3 & \nu_n^{-1/2} I_{3n^2} \end{pmatrix}.$$

This implies that for $X \in \{H_1, X_i, Y_j, 1 \le i, j \le n\}$, we have $W_\chi(d\pi_\chi(X)) = S_\chi(d\pi_\chi(X))$ and, for $X \in \{F_{ij}, G_{ij}, A_{ij}, 1 \le i, j \le n\}$, we have

$$W_\chi(d\pi_\chi(X)) = i\gamma\Big(\phi^1(\cdot, X) - \frac{\mu_n \nu_n^{-1/2}}{1+\nu_n^{1/2}}\phi^2(\cdot, X)\Big) - im\nu_n^{-1/2}\phi^2(\cdot, X).$$

Hence the result follows. □

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