Rend. Sem. Mat. Univ. Padova, Vol. 136 (2016), 137–155
DOI 10.4171/RSMUP/136-10

Stabilization for Iwasawa modules in $\mathbb{Z}_p$-extensions

Andrea Bandini (∗) – Fabio Caldarola (∗∗)

Abstract – Let K/k be a $\mathbb{Z}_p$-extension of a number field k with layers k_n. Let $i_{n,m}$ be the map induced by inclusion between the p-parts of the class groups of k_n and k_m ($m \geqslant n$). We study the capitulation kernels $H_{n,m} := \ker(i_{n,m})$ and $H_n := \bigcup_{m \geqslant n} H_{n,m}$ to give some explicit formulas for their size and prove stabilization properties for their orders and p-ranks. We also briefly investigate stabilization properties for the cokernel of $i_{m,n}$ and for the kernels of the norm maps and point out their relations with the nullity of the Iwasawa invariants for K/k.

Mathematics Subject Classification (2010). 11R23; 11R29.

Keywords. $\mathbb{Z}_p$-extensions, Iwasawa modules, Iwasawa invariants.

1. Introduction

Let K/k be a $\mathbb{Z}_p$-extension (p a prime number) of a number field k, with Galois group Γ and whose layers we denote by k_n. Assume that all ramified primes in K/k are totally ramified in K/k_{n_0}. We denote by A_n the p-part of the ideal class group of k_n and, for any $m \geqslant n$, we let $N_{m,n}$ (resp. $i_{n,m}$) be the map $A_m \to A_n$ (resp. $A_n \to A_m$) induced by the natural norm (resp. inclusion). Put

$$X(K) := \varprojlim_n A_n \quad \text{and} \quad A(K) := \varinjlim_n A_n$$

(defined via norms and inclusions respectively). By class field theory there is a (canonical) isomorphism $X(K) \simeq \mathrm{Gal}(L(K)/K)$, where $L(K)$ is the maximal

(∗) *Indirizzo dell'A.*: Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma (PR), Italy.
E-mail: andrea.bandini@unipr.it

(∗∗) *Indirizzo dell'A.*: Dipartimento di Matematica e Informatica, Università della Calabria, via P. Bucci, Cubo 31B, 87036 Arcavacata di Rende (CS), Italy.
E-mail: caldarola@mat.unical.it

abelian unramified pro-p-extension of K. Let

$$\Lambda := \varprojlim_{n} \mathbb{Z}_p[\mathrm{Gal}(k_n/k)] = \mathbb{Z}_p[[\Gamma]]$$

be the Iwasawa algebra associated with K/k. Via the usual action of conjugation $X(K)$ and $A(K)$ can be considered as Λ-modules, moreover there exist Λ-submodules Y_n of $X(K)$ and elements $\nu_{n,m} \in \Lambda$ such that $Y_m = \nu_{n,m} Y_n$ and $X(K)/Y_n \simeq A_n$ for any $n \geqslant n_0$ (see [15, Chapter 13]). Many results relating the Λ-module structure of $X(K)$ with the $\ker(i_{n,m})$ and the $\ker(N_{m,n})$ have been proved since the beginning of Iwasawa theory (see, e.g., [8]). In particular the relation between the finiteness of $X(K)$ and the groups $H_{n,m} := \ker(i_{n,m})$ and $H_n := \bigcup_{m \geqslant n} H_{n,m}$ has been exploited in [7] and [12] (and generalized to $\mathbb{Z}_p^d$-extensions in, for example, [1], [2] and [9]). We consider this type of relations together with the phenomenon of stabilization.

We say that the order (resp. the p-rank) of the modules $\{M_n\}_{n\in\mathbb{N}}$ *stabilizes* if there is an index q such that $|M_n| = |M_q|$ (resp. $\mathrm{rk}_p(M_n) = \mathrm{rk}_p(M_q)$) for all $n \geqslant q$. It is customary for Iwasawa modules to stabilize at the very first step in which they do not vary: in particular one can prove (see, e.g., [3] and [5])

Theorem 1.1. *The following hold*:

(a) *if for some $n \geqslant n_0$ one has $|A_n| = |A_{n+1}|$, then $Y_n = 0$ and $X(K) \simeq A_n$*;

(b) *if for some $n \geqslant n_0$ one has $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$, then $Y_n \subseteq pX(K)$ and $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_m)$ for any $m \geqslant n$.*

In this paper we provide a description of $H_{n,m}$ in terms of the maximal finite submodule D of $X(K)$ (see Proposition 3.3, derived from [12, Proposition])

$$H_{n,m} \simeq \ker\{\nu_{n,m} \colon D/(Y_n \cap D) \longrightarrow D/(Y_m \cap D)\} \quad \text{and} \quad H_n \simeq D/(Y_n \cap D).$$

With that we can prove statements like the ones of Theorem 1.1 for the modules H_n (see Theorem 3.7).

The following theorem enables us to give examples of finite Iwasawa modules for which the $H_{n,m}$ have "delayed" stabilization (cf. Theorem 3.11 and Example 3.12).

Theorem 1.2. *Let $r \geqslant n_0$ be the least index such that $|H_r| = |H_{r+1}|$. If $r > n \geqslant n_0$, there exist an index $h(n)$ (cf. Definition 3.9) such that*

$$\begin{aligned} 1 = |H_{n,n}| &\leqslant |H_{n,n+1}| \leqslant |H_{n,n+2}| \leqslant \cdots \leqslant |H_{n,r}| \\ &= |H_{n,r}| < |H_{n,r+1}| < |H_{n,r+2}| < \cdots < |H_{n,h(n)}| \\ &= |H_{n,h(n)}| = |H_{n,h(n)+1}| = |H_{n,h(n)+2}| = \cdots = |D/D \cap Y_n|. \end{aligned}$$

Other examples arise from the non-abelian theory discussed by M. Ozaki in [14], since the setting is quite different we shall present them in another paper (see [4]).

We recall that Iwasawa proved the following formula for the orders of the A_n

$$|A_n| = p^{\mu(K/k)p^n+\lambda(K/k)n+\nu}, \quad n \gg 0,$$

where the *Iwasawa invariants* $\mu(K/k)$ and $\lambda(K/k)$ depend on the Λ-module structure of $X(K)$. In the final section we provide relations between the triviality of these invariants and the stabilization of the modules H_n, $\operatorname{coker}(i_{n,m})$ and $\ker(N_{m,n})$ (see Theorems 4.2 and 4.4).

2. Notations and preliminaries

We quickly describe the basic objects of Iwasawa theory we are going to work with and list a few results which will be used in the next sections (comprehensive references are [15, Chapter 13] and [11, Chapter V]).

Let $\Gamma := \operatorname{Gal}(K/k) \simeq \mathbb{Z}_p$ and choose a topological generator γ of Γ. The map $\gamma \to 1+T$ provides a noncanonical isomorphism between $\mathbb{Z}_p[[\Gamma]]$ (the Iwasawa algebra of Γ) and $\mathbb{Z}_p[[T]]$, and we shall always identify them with our Λ. Let k_n be the n-th layer of K (i.e., the fixed field of $\overline{\langle\gamma^{p^n}\rangle}$): we will assume that all primes which ramify in K/k are totally ramified in K/k_{n_0} (sometimes it is useful to take a minimal n_0 but it is not really necessary in the proofs). For any $n \geqslant n_0$ we let Y_n be the Λ-submodule of $X := X(K)$ such that $X/Y_n \simeq A_n$ (it is the closure of the module generated by the commutators and the inertia subgroups of $\operatorname{Gal}(L(K)/k_n)$). For any $m \geqslant n \geqslant n_0$ one has $\nu_{n,m}Y_n = Y_m$, where

$$\nu_{n,m} = \frac{\nu_m}{\nu_n} = \frac{(1+T)^{p^m}-1}{(1+T)^{p^n}-1} = 1 + (1+T)^{p^n} + \cdots + ((1+T)^{p^n})^{p^{m-n}-1}$$

is a distinguished polynomial (irreducible if $m = n+1$). Moreover, for any $m \geqslant n \geqslant 0$, the $\nu_{n,m}$ verify the following formula (see, e.g., [5, Lemma])

$$\text{(1)} \qquad \nu_{n,m} = \frac{(((1+T)^{p^n}-1)+1)^{p^{m-n}}-1}{(1+T)^{p^n}-1} \equiv p^{m-n} \pmod{T\nu_n}.$$

A homomorphism φ between Λ-modules will be called pseudo-isomorphism if it has finite kernel and cokernel. If $\varphi\colon M \to N$ is a pseudo-isomorphism, we write $M \sim_\Lambda N$ and say that M and N are pseudo-isomorphic; being pseudo-isomorphic is an equivalence relation between finitely generated torsion Λ-modules.

If M is a finitely generated Λ-module, there exists an exact sequence

$$0 \longrightarrow D(M) \longrightarrow M \stackrel{\varphi}{\longrightarrow} E(M) \longrightarrow B(M) \longrightarrow 0, \tag{2}$$

where φ is a pseudo-isomorphism, $D(M)$ is the maximal finite submodule of M, $B(M)$ is finite and $E(M)$ is an elementary Λ-module, i.e.,

$$E(M) \simeq \Lambda^s \oplus \Big(\bigoplus_{i=1}^{u} \Lambda/(g_i^{e_i})\Big)$$

with $s, u \in \mathbb{N}$, the g_i are irreducible distinguished polynomials and the e_i are positive integers (all uniquely determined by M). The *characteristic polynomial* of M is defined to be

$$f_M(T) := \begin{cases} 0 & \text{if } s \neq 0, \\ \displaystyle\prod_{i=1}^{u} g_i^{e_i} & \text{if } s = 0. \end{cases}$$

Note that a Λ-module M is finite (i.e., $M \sim_\Lambda 0$) if and only if $f_M(T) \in \Lambda^*$. The Iwasawa invariants of the extension K/k are related to $f_{X(K)}(T)$: indeed $\mu(K/k)$ is the exact power of p dividing $f_{X(K)}(T)$ and $\lambda(K/k) = \deg(f_{X(K)}(T))$.

For any finitely generated $\mathbb{Z}_p$-module M we let $\mathrm{rk}_p(M)$ denote the p-rank of M (i.e., $\mathrm{rk}_p(M) = \dim_{\mathbb{F}_p}(M/pM)$).

Remark 2.1. We recall a general fact on modules over a commutative ring R. Let $N \subseteq M$ be R-modules such that $|M/N|$ is finite and let $\mathfrak{a} = (a_1, \dots, a_u)$ be a finitely generated ideal of R. Then, using induction on u, it is easy to see that $|\mathfrak{a}M/\mathfrak{a}N| \leqslant |M/N|^u$. Note, furthermore, that the given bound is sharp: take for example $M = R = \Lambda$ and $N = \mathfrak{a} = (p, T)$.

Proposition 2.2. *Let $\varphi\colon M \to N$ be a pseudo-isomorphism of Λ-modules, $\mathfrak{a}$ an ideal in Λ, τ a nonzero element of Λ and $M[\tau]$ (resp. $N[\tau]$) the kernel of $\tau\colon M \to M$ (resp. of $\tau\colon N \to N$). Then we have canonical pseudo-isomorphisms*

(a) $\varphi|_{M[\tau]}\colon M[\tau] \to N[\tau]$;

(b) $\varphi|_{\mathfrak{a}M}\colon \mathfrak{a}M \to \mathfrak{a}N$;

(c) $\bar{\varphi}\colon M/\tau M \to N/\tau N$ *(where $\bar{\varphi}$ is induced by φ).*

Proof. For (a) and (c) just consider the snake lemma sequences associated to the diagrams

$$\begin{tikzcd}
0 \arrow[r] & M[\tau] \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\varphi"] & \tau M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N[\tau] \arrow[r] & N \arrow[r] & \tau N \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \tau M \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\varphi"] & M/\tau M \arrow[r] \arrow[d, "\bar{\varphi}"] & 0 \\
0 \arrow[r] & \tau N \arrow[r] & N \arrow[r] & N/\tau N \arrow[r] & 0
\end{tikzcd}$$

with easy estimates between the cardinalities of kernels and cokernels. For (b) use also the previous remark. $\square$

Corollary 2.3. *Let M be a finitely generated torsion Λ-module and τ a nonzero element of Λ, then*

$$\gcd(\tau, f_M) = 1 \iff M[\tau] \sim_\Lambda 0 \iff M/\tau M \sim_\Lambda 0\,.$$

Moreover, if any of the previous conditions holds, then, for any submodule N of M, the induced map $\bar{\tau}\colon M/N \to \tau M/\tau N$ has finite kernel.

Proof. The statements are obvious for an elementary torsion Λ-module E (where one actually finds $E[\tau] = 0$ and $E/N \simeq \tau E/\tau N$). For a general module M just consider a pseudo-isomorphism $\varphi\colon M \to E(M)$ and apply the previous proposition. $\square$

3. Stabilization of the capitulation kernels

We now consider capitulation of ideals and study the maps $i_{n,m}\colon A_n \to A_m$ induced by inclusion (in particular their kernels denoted by $H_{n,m}$). We provide a description in terms of $D := D(X(K))$ (the maximal finite submodule of $X(K)$ appearing in the sequence (2)) in the spirit of the results of [12] for $H_n = \bigcup_{m\geqslant n} H_{n,m}$.

Definition 3.1. We put

(a) for all $m \geqslant n \geqslant 0$, $D_{n,m} := \nu_{n,m} D$;

(b) for all $n \geqslant n_0$, $D_n := D \cap Y_n$.

The following lemma shows that the D_n behave well with respect to the usual Iwasawa relations.

Lemma 3.2. *For all $m \geqslant n \geqslant n_0$, we have $\nu_{n,m} D_n = D_m = D_{n,m} \cap Y_m$.*

Proof. The relation $\nu_{n,m} D_n \subseteq D_m$ is trivial. Now take $z \in D_m - \nu_{n,m} D_n$, i.e., a $y \in Y_n$ such that

$$z = \nu_{n,m} y \in D - \nu_{n,m} D_n,$$

and note that $y \notin D_n$ yields $y \notin D$. Since $|X/\nu_{n,m} X| \leqslant |X/Y_m| = |A_m|$ is finite and $X/\nu_{n,m} X \sim_\Lambda E(X)/\nu_{n,m} E(X)$ (by Proposition 2.2 (c)), Corollary 2.3 shows that $\nu_{n,m} \colon E(X) \to E(X)$ is injective. Hence the induced map $\nu_{n,m} \colon X/D \to X/D$ is injective as well: this contradicts our choice of $y \notin D$ with $\nu_{n,m} y \in D$.

For the last equality, observe that

$$\nu_{n,m} D_n \subseteq \nu_{n,m} Y_n = Y_m \implies \nu_{n,m} D_n \subseteq \nu_{n,m} D \cap Y_m \subseteq D_m,$$

and we have already seen that the two extremities of the chain are equal. □

Since $\nu_{n,m} \in (p, T)$ (the maximal ideal of Λ), Nakayama's lemma and Lemma 3.2 show that, for $n \geqslant n_0$, $D_{n+1} \neq D_n$ unless $D_n = 0$. Moreover, since D is finite, there exists an r such that $D_r = 0$. The isomorphisms $X/Y_m \simeq A_m$ induce embeddings $D/D_m \hookrightarrow A_m$ for all $m \geqslant n_0$ and, in particular, we can embed D into A_r.

The following is a reinterpretation of [12, Proposition] which we shall repeatedly use.

Proposition 3.3. *With the above notations we have*

$$H_{n,m} \simeq \ker\{\nu_{n,m} \colon D/D_n \longrightarrow D/D_m\} \tag{3}$$

and

$$H_n \simeq D + Y_n/Y_n \simeq D/D_n \tag{4}$$

for all $m \geqslant n \geqslant n_0$. Moreover, if $D_n = 0$, then $H_{n,m} \simeq D[p^{m-n}]$ (where $D[p^{m-n}]$ is the submodule of the p^{m-n}-torsion elements of D).

Proof. From the well-known commutative diagram

$$\begin{tikzcd} A_m \arrow[r, "\simeq"'] & X/Y_m \\ A_n \arrow[u, "i_{n,m}"] \arrow[r, "\simeq"'] & X/Y_n \arrow[u, "\nu_{n,m}"'] \end{tikzcd} \tag{5}$$

we have that $H_{n,m} \simeq \ker\{\nu_{n,m} : X/Y_n \longrightarrow X/Y_m\}$. Now consider the diagram

$$\begin{tikzcd} D+Y_n/Y_n \arrow[r, hook] \arrow[d, "\nu^{(1)}_{n,m}"] & X/Y_n \arrow[r, two heads] \arrow[d, "\nu_{n,m}"] & X/D+Y_n \arrow[d, "\nu^{(2)}_{n,m}"] \\ D+Y_m/Y_m \arrow[r, hook] & X/Y_m \arrow[r, two heads] & X/D+Y_m \end{tikzcd}.$$

Let $\alpha \in X$ be such that $\nu_{n,m}\alpha \in D+Y_m$, i.e., $\alpha \pmod{D+Y_n} \in \ker(\nu^{(2)}_{n,m})$. Then there exist $d \in D$ and $y_m \in Y_m$ such that $\nu_{n,m}\alpha = d+y_m$ and, since $Y_m = \nu_{n,m}Y_n$, $\nu_{n,m}\alpha = d + \nu_{n,m}y_n$ for some $y_n \in Y_n$. Therefore $\nu_{n,m}(\alpha - y_n) = d$ and $\alpha - y_n \in \ker\{\nu_{n,m} : X \to X/D\}$. The injectivity of the map $\nu_{n,m} : X/D \to X/D$ yields $\alpha - y_n \in D$, i.e., $\alpha \in D + Y_n$, which means $\nu^{(2)}_{n,m}$ is injective. Hence

$$H_{n,m} \simeq \ker(\nu_{n,m}) \simeq \ker\{\nu^{(1)}_{n,m} : D+Y_n/Y_n \longrightarrow D+Y_m/Y_m\}$$

and the isomorphism $D/D_i = D/D\cap Y_i \simeq D+Y_i/Y_i$ concludes the proof of (3).

The second isomorphism follows easily (note that $\nu_{n,m}D = 0$ for large enough m).

For the final statement use (1) to get $\nu_{n,m} = p^{m-n} + g(T)T\nu_n$ (for some $g(T) \in \Lambda$). If $D_n = 0$, then

$$T\nu_n D = \nu_{n_0,n}\nu_{n_0}TD \subseteq \nu_{n_0,n}(Y_{n_0} \cap D) = \nu_{n_0,n}D_{n_0} = D_n = 0,$$

by Lemma 3.2. Hence $\nu_{n,m}$ acts as multiplication by p^{m-n} on D. □

Corollary 3.4. *For all $m \geqslant n \geqslant n_0$ we have*

(a) $|H_{n,m}| = \dfrac{|D|\cdot|D_m|}{|D_n|\cdot|D_{n,m}|} = |D+Y_n/D_{n,m}+Y_m| \cdot \dfrac{|A_n|}{|A_m|}$;

(b) *if $D \neq 0$ and $n \geqslant n_0$, then $i_n : A_n \to A = A(K)$ is injective if and only if $n = n_0$ and D is contained in Y_{n_0}.*

Proof. (a) For the first equality just note that $\mathrm{im}(\nu_{n,m}) = D_{n,m}/D_m$ and use Proposition 3.3. For the second one note that $\mathrm{im}(\nu^{(1)}_{n,m}) = D_{n,m}+Y_m/Y_m$ yields

$$|H_{n,m}| = \frac{|D+Y_n/Y_n|}{|D_{n,m}+Y_m/Y_m|}.$$

Using the exact sequences

$$Y_n/Y_m \hookrightarrow D+Y_n/Y_m \twoheadrightarrow D+Y_n/Y_n, \quad Y_n/Y_m \hookrightarrow X/Y_m \twoheadrightarrow X/Y_n$$

and

$$D_{n,m}+Y_m/Y_m \hookrightarrow D+Y_n/Y_m \twoheadrightarrow D+Y_n/D_{n,m}+Y_m$$

(recalling that $|X/Y_i| = |A_i|$ for any $i \geqslant n_0$), one gets

$$\begin{aligned}\frac{|D+Y_n/Y_n|}{|D_{n,m}+Y_m/Y_m|} &= \frac{|D+Y_n/Y_m|}{|Y_n/Y_m|}\cdot\frac{|D+Y_n/D_{n,m}+Y_m|}{|D+Y_n/Y_m|}\\ &= \frac{|D+Y_n/D_{n,m}+Y_m|}{|Y_n/Y_m|}\\ &= |D+Y_n/D_{n,m}+Y_m|\cdot\frac{|A_n|}{|A_m|}.\end{aligned}$$

(b) By Proposition 3.3, $H_n = 0$ implies $D \subseteq Y_n \subseteq Y_{n_0}$, so $D = D_n = D_{n_0}$. Now as remarked before Proposition 3.3, since D is not zero, we obtain $n = n_0$. The converse is trivial. □

Note that from the last assertion it follows that if $D \neq 0$ and $n > n_0$, then there are at least $p-1$ ideal classes in A_n which capitulate in some A_m.

Corollary 3.5. *For any $\mathbb{Z}_p$-extension K/k, the following are equivalent*:

(a) *X does not contain any nontrivial finite submodule*;

(b) $H_{n_0+1} = 0$;

(c) *$i_{n,m}\colon A_n \to A_m$ is injective for all $m \geqslant n \geqslant n_0$.*

Proof. (a) $\Longrightarrow$ (c) follows from Proposition 3.3, (c) $\Longrightarrow$ (b) is obvious and (b) $\Longrightarrow$ (a) is given by Corollary 3.4 (b). □

The following corollary generalizes [5, Proposition].

Corollary 3.6. *Let K/k be a $\mathbb{Z}_p$-extension, assume that $A_n \neq 0$ and $i_{n,m}$ is injective for some $m > n \geqslant n_0$. Then $|A_m| \geqslant p^{m-n}|A_n|$.*

Proof. It suffices to prove that $Y_i \supsetneqq Y_{i+1}$ for any $n \leqslant i \leqslant m-1$ or, equivalently, $Y_{m-1} \neq 0$ (by Nakayama's lemma the modules Y_n and Y_{n+1} become equal only when they are zero). So we assume $Y_{m-1} = 0$ and look for a contradiction. Obviously

$$|A_n| = |i_{n,m-1}(A_n)| = |\nu_{n,m-1}X/Y_{m-1}| = |\nu_{n,m-1}X|$$

and

$$|A_n| = |i_{n,m}(A_n)| = |\nu_{n,m}X/Y_m| = |\nu_{n,m}X|.$$

This yields

$$\nu_{m-1,m}\nu_{n,m-1}X = \nu_{n,m-1}X$$

and, by Nakayama's lemma,

$$\nu_{n,m-1}X = 0.$$

Then $i_{n,m}$ is the zero map and this contradicts $A_n \neq 0$. □

We are now ready to state the stabilization result regarding the modules H_n.

THEOREM 3.7. *Assume $n \geqslant n_0$:*

(a) *if $|H_n|=|H_{n+1}|$, then $H_m \simeq H_n \simeq D$ for all $m \geqslant n$. In particular there exists $r \geqslant n_0$ such that*

(6) $$|H_{n_0}| < |H_{n_0+1}| < \cdots < |H_r| = |H_{r+1}| = \cdots = |D|;$$

(b) *if $\mathrm{rk}_p(H_n) = \mathrm{rk}_p(H_{n+1})$, then $\mathrm{rk}_p(H_m) = \mathrm{rk}_p(H_n) = \mathrm{rk}_p(D)$ for all $m \geqslant n$. In particular there exists $\tilde{r} \geqslant n_0$ such that*

(7) $$\mathrm{rk}_p(H_{n_0}) < \mathrm{rk}_p(H_{n_0+1}) < \cdots < \mathrm{rk}_p(H_{\tilde{r}}) = \mathrm{rk}_p(H_{\tilde{r}+1}) = \cdots = \mathrm{rk}_p(D).$$

PROOF. (a) Since $H_n \simeq D/D_n$, the hypothesis yields

$$D_n = D_{n+1} = \nu_{n,n+1}D_n$$

(by Lemma 3.2). Nakayama's lemma implies $D_n = 0$, so, for any $m \geqslant n$, $D_m = 0$ and $H_m \simeq H_n \simeq D$.

(b) The hypothesis yields

$$D/D_n + pD \simeq D/D_{n+1} + pD,$$

i.e.,

$$D_n + pD = D_{n+1} + pD.$$

Therefore

$$D_n + pD/pD = \nu_{n,n+1}(D_n + pD/pD),$$

and, from Nakayama's lemma, $D_n + pD/pD = 0$. Thus, for any $m \geqslant n$, $D_m \subseteq D_n \subseteq pD$ and $D/D_m + pD = D/D_n + pD = D/pD$, which is the claim. □

REMARK 3.8. From the proof of Theorem 3.7 it is easy to see that the orders (resp. the p-ranks) of the H_n stabilize exactly when $D_r = 0$ (resp. when $D_{\tilde{r}} \subseteq pD$).

The $H_{n,m}$ have a less regular (hence more interesting) behaviour with respect to stabilization, at least for small indices: to describe it we use the following

DEFINITION 3.9. For any $n \geqslant 0$, let $h(n) := \min\{z \geqslant n \ s.t.\ D_{n,z} = 0\}$.

PROPOSITION 3.10. *Let $|D| = p^\delta$ with $\delta \in \mathbb{N}$ and let $p^\varepsilon \in \mathbb{N}$ be the exponent of D (i.e., the minimum integer for which $p^\varepsilon D = 0$). Then*

(a) *for any $n \geqslant 0$, we have $h(n) - n \leqslant \delta$ and, for every $n \geqslant \delta$, $h(n) - n = \varepsilon$;*

(b) *for any $n \geqslant r$, we have $h(n) - n = \varepsilon$.*

PROOF. (a) The first statement follows from Nakayama's lemma: indeed, for any $n \geqslant 0$, $\nu_{n,n+\delta} D = 0$ (since $\nu_{n,m} P = P$ if and only if $P = 0$, the order of a nontrivial module must decrease of a factor at least p at any step, i.e., D vanishes after at most δ steps).

For the second statement, consider the action of $\Gamma = \mathrm{Gal}(K/k)$ over D and let p^ω be the cardinality of the greatest orbit in D. Then Γ^{p^ω} acts trivially on D, so, for all $n \geqslant \delta \geqslant \omega$, the element $\nu_{n,n+1} = 1 + \gamma^{p^n} + \cdots + \gamma^{(p-1)p^n}$ acts on D as multiplication by p. This implies both $\nu_{n,n+\varepsilon-1} D = p^{\varepsilon-1} D \neq 0$ and $\nu_{n,n+\varepsilon} D = p^\varepsilon D = 0$, i.e., $h(n) = n + \varepsilon$.

(b) The hypothesis $n \geqslant r$ yields $D_n = 0$, hence, as seen in the proof of Proposition 3.3, $\nu_{n,n+1} D = pD$. This immediately leads to $h(n) = n + \varepsilon$. $\square$

THEOREM 3.11. *Let $n \geqslant n_0$, then*

(a) *if $n < r$, one has*

$$\begin{aligned}
1 = |H_{n,n}| &\leqslant |H_{n,n+1}| \leqslant |H_{n,n+2}| \leqslant \cdots \leqslant |H_{n,r}| \\
&= |H_{n,r}| < |H_{n,r+1}| < |H_{n,r+2}| < \cdots < |H_{n,h(n)}| \\
&= |H_{n,h(n)}| \\
&= |H_{n,h(n)+1}| = |H_{n,h(n)+2}| = \cdots = |D/D_n|;
\end{aligned}$$

(b) *if $n \geqslant r$, one has $|H_{n,m}| = \frac{|D|}{|D_{n,m}|}$ for all $m \geqslant n$, $h(n) = n + \varepsilon$ and*

$$1 < |H_{n,n+1}| < \cdots < |H_{n,n+\varepsilon}| = |H_{n,n+\varepsilon+1}| = \cdots = |D|.$$

PROOF. Everything follows from Corollary 3.4 and Proposition 3.10. For the strict inequalities of the central line of (a) note that, since $\nu_{n,r+j} D_n = D_{r+j} = 0$ for any $j \geqslant 0$, one has

$$|H_{n,r+j}| = \frac{|D|}{|D_n| \cdot |D_{n,r+j}|}$$

and, by Lemma 3.2 and Nakayama's lemma, the $D_{n,r+j}$ stabilize (i.e., become 0) only at the level $D_{n,h(n)}$. □

EXAMPLE 3.12. By [13, Theorem 1], for any finite $\mathbb{Z}_p[[\Gamma]]$-module D there exists a field k whose cyclotomic $\mathbb{Z}_p$-extension k_{cyc}/k provides $X(k_{\mathrm{cyc}}) \simeq D$. Take $D \simeq \Lambda/(p^u, T)$ and let u_0 be such that $0 \leqslant u_0 \leqslant u$ and $D_0 = p^{u_0} D$ (enlarging the base field, if necessary, we can assume $n_0 = 0$). A little calculation shows that

$$|H_{n,m}| = \begin{cases} 1 & \text{if } 0 \leqslant n \leqslant m \leqslant u - u_0, \\ p^{m-u+u_0} & \text{if } n \leqslant u - u_0 \text{ and } u - u_0 < m \leqslant n + u, \\ p^{m-n} & \text{if } n > u - u_0 \text{ and } n \leqslant m \leqslant n + u, \\ p^u & \text{if } n > u - u_0 \text{ and } m > n + u. \end{cases}$$

Furthermore we can easily see that our parameters take the following values:

$$r = u - u_0, \quad \tilde{r} = 0$$

and

$$h(n) = n + u \qquad \text{for all } n \geqslant 0.$$

In particular, if $n < u - u_0$, the equation of Theorem 3.11 (a) becomes

$$1 = |H_{n,n}| = \cdots = |H_{n,u-u_0}| < |H_{n,u-u_0+1}| < \cdots < |H_{n,n+u}| = \cdots = |H_n|$$

and shows that the orders of the $H_{n,m}$ (unlike the ones of the other Iwasawa modules) can be constant for some indices, then increase and finally stabilize for good.

4. Stabilization and Iwasawa invariants

In this final section we deal with relations between the stabilization of Λ-modules and the triviality of Iwasawa invariants for K. To simplify notations we assume the following

Assumption 4.1. *All primes which ramify in K/k are totally ramified, i.e.,* $n_0 = 0$.

We remark that all statements can be proved for a general n_0 (substituting 0 with n_0 and 1 with $n_0 + 1$) with no relevant modifications.

Theorem 4.2. *The following are equivalent*:

(a) $\lambda(K/k) = \mu(K/k) = 0$;

(b) $\mathrm{im}(i_{n,m}) = \mathrm{im}(i_{n-1,m})$ *for some* $m \geqslant n \geqslant 1$;

(c) $\ker(N_{m,n}) = \ker(N_{m,n-1})$ *for some* $m \geqslant n \geqslant 1$;

(d) $\mathrm{rk}_p(H_n) = \mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ *for some* $n \geqslant 0$.

Proof. The first equivalences are just exercises with Λ-modules. For (b) $\Longrightarrow$ (a) the hypothesis yields $\nu_{n-1,m}X = \nu_{n,m}X$, hence one gets $\nu_{n,m}X = 0$, $Y_m = 0$, and $X \simeq A_m$. Conversely (a) $\Longrightarrow$ (b) because if X is finite, then $\nu_{n-1,m}X = \nu_{n-1,n}\nu_{n,m}X = 0$ for $m \gg 0$.

For (c) $\Longrightarrow$ (a) recall the commutative diagram

$$\begin{array}{ccc} A_m & \xrightarrow{\ \simeq\ } & X_m \simeq X/Y_m \\ \downarrow{\scriptstyle N_{m,n}} & & \downarrow{\scriptstyle \pi_{m,n}} \\ A_n & \xrightarrow{\ \simeq\ } & X_n \simeq X/Y_n \end{array} \tag{8}$$

(where $\pi_{m,n}$ is the projection), which, in particular, implies $\ker(N_{m,n}) \simeq Y_n/Y_m$. The hypothesis yields $Y_{n-1}/Y_m = Y_n/Y_m = \nu_{n-1,n}Y_{n-1}/Y_m$, i.e., one gets $Y_{n-1} = Y_n = \nu_{n-1,n}Y_{n-1}$. Hence $Y_{n-1} = 0$ and $X \simeq A_{n-1}$ (and $\ker(N_{b,a}) = 0$ for all $b \geqslant a \geqslant n-1$). The reverse arrow is similar to the previous one (if X is finite, the Y_n are finite as well).

For (a) $\Longrightarrow$ (d) just recall the well-known fact that the finiteness of X implies $A_n = H_n$ for any $n \geqslant 0$. For the reverse (and final) arrow consider the map $\psi := \pi \circ i$ given by the composition

$$D + Y_n \xhookrightarrow{\ i\ } X \xrightarrow{\ \pi\ } X/Y_n + pX.$$

Since $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ (i.e., $Y_n \subseteq pX$ by Theorem 1.1(b)), we have the equalities

$$\begin{aligned}\ker(\psi) &= (D + Y_n) \cap (pX + Y_n)\\ &= (D + Y_n) \cap pX\\ &= (D \cap pX) + Y_n\\ &= pD + Y_n,\end{aligned}$$

where the last equality comes from Corollary 2.3 (indeed if px is an element of $D \cap pX$ of order p^β, then $x \in \ker\{p^{\beta+1}\colon X \to X\}$ which is finite, hence contained in D, because $\mu(K/k) = 0$). So ψ induces an embedding

$$D + Y_n/pD + Y_n \overset{\bar\psi}{\hookrightarrow} X/Y_n + pX.$$

Now note that $D + Y_n/pD + Y_n \simeq D/(D \cap Y_n) + pD \simeq H_n/pH_n$ (by Proposition 3.3), hence the hypothesis $\mathrm{rk}_p(H_n) = \mathrm{rk}_p(A_n)$ implies that $\bar\psi$ is an isomorphism. Then $X = D + pX$ and eventually $X = D$. □

REMARK 4.3. Note that if (b), (c), or (d) are true for a certain suitable n, then they are true for every n (again because of $A_n = H_n$ for all $n \geqslant 0$).

We mention that one can find several other equivalences (mainly dealing with inverse images of norms and inclusions), but we decided to include only kernels, cokernels and images since they are more commonly used in the theory (see, e.g., [6] or [10]) and they give a full account of the techniques used in the proofs.

The following theorem only deals with the triviality of the μ-invariant which is related to the stabilization of the p-rank of kernels and cokernels of natural maps.

THEOREM 4.4. *The following are equivalent*:

(a) $\mu(K/k) = 0$;

(b) *for some* $m \geqslant n \geqslant 1$, $\mathrm{rk}_p(\ker(N_{m,n})) = \mathrm{rk}_p(\ker(N_{m+1,n}))$ (*equivalently,* $\mathrm{rk}_p(\ker(N_{m,n})) = \mathrm{rk}_p(\ker(N_{m,n-1}))$);

(c) *for some* $m \geqslant n \geqslant 1$, $\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{coker}(i_{n,m+1}))$ (*equivalently,* $\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{coker}(i_{n-1,m}))$);

(d) *for some* $m \geqslant n \geqslant 1$, $\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) = \mathrm{rk}_p(A_m)$.

Proof. The proof will be provided by the main results of the following two subsections. In particular,

- the equivalence (b) $\iff$ (a) follows from Theorem 4.7 and Proposition 4.9;
- the equivalence (c) $\iff$ (a) follows from Theorem 4.10 and Proposition 4.12;
- the equivalence (c) $\iff$ (d) follows from Theorem 4.10 and Remark 4.11. $\square$

4.1 – *The kernel of the norm map*

Since the extension K/k is totally ramified at some prime, the maps $N_{m,n}$ have trivial cokernels, hence we focus on the $\ker(N_{m,n})$ to show their link with the stabilization of the $\mathrm{rk}_p(A_n)$. Before studying the stabilization properties of the $\mathrm{rk}_p(\ker(N_{m,n}))$, we give a final result on the relation between $\ker(N_{m,n})$, H_m and the finiteness X (which we left out of Theorem 4.2 because it does not involve stabilization).

Proposition 4.5. *The following hold*

(a) *if $H_{n,m} = A_n$ for some $m \geqslant n \geqslant 1$, then $X \simeq A_m$;*

(b) *if $\ker(N_{n,n-1}) \subseteq H_{n,m}$ for some $m \geqslant n \geqslant 1$, then $X \simeq A_m$.*

Proof. (a) Note that $H_{n,m} = A_n$ implies $H_{1,m} = A_1$, so we only consider the case $n = 1$. From diagram (5) and the hypothesis, one gets

$$\nu_{1,m} X \subseteq Y_m = \nu_{0,1}\nu_{1,m} Y_0 \subseteq \nu_{0,1}\nu_{1,m} X.$$

Nakayama's lemma yields $\nu_{1,m} X = 0$, thus $Y_m = 0$ and $X \simeq A_m$.

(b) By diagram (8), the hypothesis yields $Y_{n-1}/Y_n \subseteq \ker(\nu_{n,m})$, i.e., one gets $\nu_{n,m} Y_{n-1} \subseteq Y_m$. Since $Y_m = \nu_{n,m} Y_n \subseteq \nu_{n,m} Y_{n-1}$, we have $\nu_{n,m} Y_{n-1} = Y_m$. Multiplying by $\nu_{n-1,n}$, one has $Y_m = \nu_{n-1,n} Y_m$, which yields $Y_m = 0$ and $X \simeq A_m$. $\square$

Having seen that $\ker(N_{m,n}) \subseteq H_m$ implies the finiteness of X, we point out that the reverse inclusion holds if and only if $D = 0$.

Proposition 4.6. *The following hold*

(a) *if $m \geqslant n \geqslant 1$, then $H_m \subseteq \ker(N_{m,n})$ if and only if $D = 0$;*

(b) *if $n \geqslant 0$, then $H_n \subseteq \ker(N_{n,0})$ if and only if $H_0 = 0$.*

Proof. Note that $H_m \subseteq \ker(N_{m,n}) \iff D = D_n$. Then (b) follows from (4), while, in case (a), $n \geqslant 1$ yields $D = D_n = D_{n-1}$ and finally $D = 0$. $\square$

Now we work on results involving p-ranks which appear in Theorem 4.4 .

THEOREM 4.7. *Let K/k be as above.*

(a) *If* $\mathrm{rk}_p(\ker(N_{n,l})) = \mathrm{rk}_p(\ker(N_{n,l-1}))$ *for some* $n \geqslant l \geqslant 1$, *then*

$$\mathrm{rk}_p(Y_l) = \mathrm{rk}_p(\ker(N_{m,l})) = \mathrm{rk}_p(\ker(N_{n,l}))$$

and

$$\mathrm{rk}_p(A_m) = \mathrm{rk}_p(A_n)$$

for all $m \geqslant n$;

(b) *if* $\mathrm{rk}_p(\ker(N_{n,l})) = \mathrm{rk}_p(\ker(N_{n+1,l}))$ *for some* $n \geqslant l \geqslant 0$, *then*

$$\mathrm{rk}_p(Y_l) = \mathrm{rk}_p(\ker(N_{m,l})) = \mathrm{rk}_p(\ker(N_{n,l}))$$

and

$$\mathrm{rk}_p(A_m) = \mathrm{rk}_p(A_n)$$

for all $m \geqslant n$.

PROOF. (a) Since $\ker\{\nu_{l-1,l} : Y_{l-1} \to Y_l/Y_n\} \supseteq \nu_{l,n} Y_{l-1}$ one has a surjective map

$$Y_{l-1}/Y_n \xrightarrow{\pi} Y_{l-1}/\nu_{l,n}Y_{l-1} \xrightarrow{\nu_{l-1,l}} Y_l/Y_n$$

(where π is the projection). This map $\upsilon := \nu_{l-1,l} \circ \pi$ induces a surjection $\bar{\upsilon}: Y_{l-1}/Y_n + pY_{l-1} \to Y_l/Y_n + pY_l$. By hypothesis

$$\mathrm{rk}_p(Y_{l-1}/Y_n) = \mathrm{rk}_p(\ker(N_{n,l-1})) = \mathrm{rk}_p(\ker(N_{n,l})) = \mathrm{rk}_p(Y_l/Y_n),$$

thus both $\bar{\upsilon}$ and $\bar{\pi}: Y_{l-1}/Y_n + pY_{l-1} \to Y_{l-1}/\nu_{l,n}Y_{l-1} + pY_{l-1}$ are isomorphisms. This means that $Y_n + pY_{l-1} = \nu_{l,n}Y_{l-1} + pY_{l-1}$ and, if we consider the quotient module $M := \nu_{l,n}Y_{l-1} + pY_{l-1}/pY_{l-1}$, we have that $\nu_{l-1,l}M = M$. Nakayama's lemma yields $M = 0$ and $\nu_{l,n}Y_{l-1} \subseteq pY_{l-1}$. Therefore $Y_n \subseteq pY_l$ and, in general, $Y_m \subseteq pY_l \subseteq pX$ for any $m \geqslant n$. Hence

$$\mathrm{rk}_p(\ker(N_{m,l})) = \dim_{\mathbb{F}_p}(Y_l/Y_m + pY_l) = \dim_{\mathbb{F}_p}(Y_l/pY_l) = \mathrm{rk}_p(Y_l)$$

and

$$\mathrm{rk}_p(A_m) = \dim_{\mathbb{F}_p}(X/Y_m + pX) = \dim_{\mathbb{F}_p}(X/pX)$$

for any $m \geqslant n$ (note that, in particular, the last equality implies $\mu(K/k) = 0$).

(b) From the hypothesis we have $Y_l/Y_n + pY_l \simeq Y_l/Y_{n+1} + pY_l$, then $Y_n + pY_l = Y_{n+1} + pY_l$. As in (a), letting $M := Y_n + pY_l/pY_l$, one gets $M = 0$, i.e., $Y_n \subseteq pY_l$ and, in general, $Y_m \subseteq pY_l \subseteq pX$ for any $m \geqslant n$. □

Remark 4.8. The previous statements cannot be reversed, i.e., the equality $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ for some $n \geqslant l \geqslant 0$ does not imply that one gets $\mathrm{rk}_p(\ker(N_{n,l})) = \mathrm{rk}_p(\ker(N_{n+1,l}))$. But with the following proposition we can give a bound for the delay of the stabilization of the $\mathrm{rk}_p(\ker(N_{m,l}))$.

Proposition 4.9. *Let* $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$. *Then, for any* $l \in \{0, 1, \ldots, n\}$, *we have*

$$\mathrm{rk}_p(\ker(N_{m,l})) = \mathrm{rk}_p(\ker(N_{n+\varepsilon_l,l})) \text{ for all } m \geqslant n + \varepsilon_l$$

(*where* p^{ε_l} *is the exponent of* A_l).

Proof. The hypothesis yields $Y_n \subseteq pX$. Now $\exp(A_l) = p^{\varepsilon_l}$ and (1) imply $\nu_{n,n+\varepsilon_l} X \subseteq Y_l$. Thus

$$Y_m = \nu_{n+\varepsilon_l,m} Y_{n+\varepsilon_l} \subseteq Y_{n+\varepsilon_l} = \nu_{n,n+\varepsilon_l} Y_n \subseteq \nu_{n,n+\varepsilon_l} pX \subseteq pY_l \ .$$

Since $\mathrm{rk}_p(\ker(N_{m,l})) = \mathrm{rk}_p(Y_l/Y_m + pY_l)$, the statement follows. □

4.2 – *The cokernel of the inclusion maps*

First note that, once n and m are fixed, we have increasing sequences

$$\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) \leqslant \mathrm{rk}_p(\mathrm{coker}(i_{n,m+1})) \leqslant \mathrm{rk}_p(\mathrm{coker}(i_{n,m+2})) \leqslant \cdots \tag{9}$$

and

$$\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) \leqslant \mathrm{rk}_p(\mathrm{coker}(i_{n-1,m})) \leqslant \mathrm{rk}_p(\mathrm{coker}(i_{n-2,m})) \leqslant \cdots . \tag{10}$$

Theorem 4.10. *If, for some* $m \geqslant n$, $\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{coker}(i_{n,m+1}))$ (*or* $\mathrm{rk}_p(\mathrm{coker}(i_{n,m})) = \mathrm{rk}_p(\mathrm{coker}(i_{n-1,m}))$ *with* $n \geqslant 1$), *then*

(a) $\mathrm{rk}_p(\mathrm{coker}(i_{l,q})) = \mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$ *for all* $l \leqslant n \leqslant m \leqslant q$;

(b) $\mathrm{rk}_p(A_q) = \mathrm{rk}_p(A_m) = \mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$ *for all* $q \geqslant m$.

Proof. (a) By hypothesis

$$X/\nu_{n,m}X + pX \simeq X/\nu_{n,m+1}X + pX,$$

hence

$$pX + \nu_{n,m}X = pX + \nu_{n,m+1}X$$

and

$$\nu_{m,m+1}(\nu_{n,m}X + pX/pX) = \nu_{n,m+1}X + pX/pX.$$

By Nakayama's lemma, $\nu_{n,m}X + pX/pX = 0$, thus $\nu_{n,m}X \subseteq pX$ and $\nu_{l,q}X \subseteq pX$ for any $l \leqslant n \leqslant m \leqslant q$.

(b) Just note that $\nu_{n,m}X \subseteq pX$ implies $Y_q \subseteq Y_m \subseteq pX$ for all $q \geqslant m$. □

REMARK 4.11. An immediate consequence of Theorem 4.10 (resp. of Theorem 4.7) and of Theorem 4.2 (d) is that if $\mathrm{rk}_p(\mathrm{coker}(i_{a,n}))$ (resp. $\mathrm{rk}_p(\ker(N_{n,a})))$ stabilizes and is equal to $\mathrm{rk}_p(H_n)$, then X is finite. Note also that the equality $\mathrm{rk}_p(A_m) = \mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$ is easily seen to imply $\nu_{n,m}X \subseteq pX$, i.e., the stabilization of $\mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$.

We can give a bound for the delay of the stabilization of the $\mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$ analogous to the one in Proposition 4.9 (the proof is similar).

PROPOSITION 4.12. *Let* $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$. *Then, for any* $l \in \{0, 1, \ldots, n\}$, *we have*

$$\mathrm{rk}_p(\mathrm{coker}(i_{l,m})) = \mathrm{rk}_p(\mathrm{coker}(i_{l,n+\varepsilon_l})) \quad \text{for all } m \geqslant n + \varepsilon_l \ .$$

4.3 – *A relation between* $\mathrm{coker}(i_{n,m})$ *and* $\ker(N_{m,n})$

We conclude with a special case in which the stabilization of the two modules is achieved at the same level (i.e., without the delay described in Propositions 4.9 and 4.12). If $|A_1^\Gamma| = |A_0|$ (resp. $A_0 = 0$, but limiting ourselves to $n \geqslant 1$), one has $Y_0 = TX$ (resp. $Y_0 = X$) and it is easy to see that the stabilization of the $\mathrm{rk}_p(\mathrm{coker}(i_{n,m}))$ yields stabilization of the $\mathrm{rk}_p(\ker(N_{m,n}))$ (i.e., $\nu_{n,m}X \subseteq pX$ implies $Y_m \subseteq pY_n$). To obtain a relation in the other direction one needs to assume also the maximality of $\mathrm{rk}_p(\ker(N_{m,n}))$.

THEOREM 4.13. *Assume* $|A_1^\Gamma| = |A_0|$ *or* $A_0 = 0$ *and* $n \geqslant 1$. *If one has* $\mathrm{rk}_p(A_m) = \mathrm{rk}_p(\ker(N_{m,n}))$, *then* $\mathrm{rk}_p(\mathrm{coker}(i_{n,q}))$ *stabilizes for* $q \geqslant m$. *Moreover* $\mathrm{rk}_p(A_q)$ *and* $\mathrm{rk}_p(\ker(N_{q,n}))$ *stabilize for* $q \geqslant m$ *too.*

PROOF. We give a proof only in the case $|A_1^\Gamma| = |A_0|$ because the other one is similar. Consider the map

$$\beta := \pi \circ \nu_n \circ T$$

pictured as

$$X \xrightarrow{T} Y_0 \xrightarrow{\nu_n} Y_n \xrightarrow{\pi} Y_n/Y_m + pY_n$$

where π is the canonical projection. Since $pX + \nu_{n,m}X \subseteq \ker(\beta)$, we can consider the map

$$X/Y_m + pX \longmapsto X/\nu_{n,m}X + pX \xrightarrow{\bar{\beta}} Y_n/Y_m + pY_n,$$

where $\bar{\beta}$ is induced by β and the first map is again a projection. By hypothesis

$$X/Y_m + pX \simeq Y_n/Y_m + pY_n,$$

hence, from the middle term, we get

$$Y_m + pX = \nu_{n,m}X + pX.$$

Moding out by pX one has

$$\begin{aligned} T(\nu_{n,m}X + pX/pX) &= \nu_{n,m}Y_0 + pX/pX \\ &\supseteq Y_m + pX/pX \\ &= \nu_{n,m}X + pX/pX. \end{aligned}$$

Nakayama's lemma yields $\nu_{n,m}X + pX/pX = 0$, i.e., $\nu_{n,m}X \subseteq pX$ and, in general, $\nu_{n,q}X \subseteq pX$ for any $q \geqslant m$. This implies

$$\mathrm{rk}_p(\mathrm{coker}(i_{n,q})) = \dim_{\mathbb{F}_p}(X/\nu_{n,q}X + pX) = \dim_{\mathbb{F}_p}(X/pX)$$

for any $q \geqslant m$. The final statement follows from Theorem 4.10. □

References

[1] A. Bandini, *Greenberg's conjecture for $\mathbb{Z}_p^d$-extensions,* Acta Arith. **108** (2003), no. 4, pp. 357–368.

[2] A. Bandini, *Greenberg's conjecture and capitulation in $\mathbb{Z}_p^d$-extensions,* J. Number Theory **122** (2007), pp. 121–134.

[3] A. Bandini, *A note on p-ranks of class groups in $\mathbb{Z}_p$-extensions,* JP J. Algebra Number Theory Appl. **9** (2007), pp. 95–103.

[4] A. Bandini – F. Caldarola, *Stabilization in non-abelian Iwasawa theory,* Acta Arith. **169** (2015), no. 4, 319–329.

[5] T. Fukuda, *Remarks on $\mathbb{Z}_p$-extensions of number fields,* Proc. Japan Acad. **70** Ser. A (1994), pp. 264–266.

[6] M. Grandet – J. F. Jaulent, *Sur la capitulation dans une $\mathbb{Z}_l$-extension,* J. Reine Angew. Math. **362** (1985), pp. 213–217.

[7] R. Greenberg, *On the Iwasawa invariants of totally real number fields,* Amer. J. Math. **98** (1976), pp. 263–284.

[8] K. Iwasawa, *On $\mathbb{Z}_l$-extensions of algebraic number fields,* Ann. of Math. (2) **98** (1973), pp. 246–326.

[9] A. Lannuzel – T. Nguyen Quang Do, *Conjectures de Greenberg et extensions pro-p-libres d'un corps de nombres,* Manuscr. Math. **102** (2000), pp. 187–209.

[10] M. Le Floc'h – A. Movahhedi – T. Nguyen Quang Do, *On capitulation cokernels in Iwasawa theory,* Amer. J. Math. **127** (4) (2005), pp. 851–877.

[11] J. Neukirch – A. Schmidt – K. Wingberg, *Cohomology of number fields,* 2nd ed., Grundlehren der Mathematischen Wissenschaften, 323, Springer, Berlin etc., 2008.

[12] M. Ozaki, *A note on the capitulation in $\mathbb{Z}_p$-extensions,* Proc. Japan Acad. **71** Ser. A (1995), pp. 218–219.

[13] M. Ozaki, *Construction of $\mathbb{Z}_p$-extensions with prescribed Iwasawa modules,* J. Math. Soc. Japan **56** (3) (2004), pp. 787–801.

[14] M. Ozaki, *Non-abelian Iwasawa theory of $\mathbb{Z}_p$-extensions,* J. Reine Angew. Math. **602** (2007), pp. 59–94.

[15] L. C. Washington, *Introduction to cyclotomic fields,* 2nd ed., Graduate Texts in Mathematics, 83. Springer, New York, N.Y., 1997.

Manoscritto pervenuto in redazione il 25 settembre 2014.