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# On $\pi\mathfrak{F}$-supplemented subgroups of a finite group

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Abstract – Let $\mathfrak{F}$ be a class of groups and $G$ a finite group. A chief factor $H/K$ of $G$ is called *$\mathfrak{F}$-central in $G$* provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. A normal subgroup $N$ of $G$ is said to be *$\pi\mathfrak{F}$-hypercentral in $G$* if every chief factor of $G$ below $N$ of order divisible by at least one prime in $\pi$ is $\mathfrak{F}$-central in $G$. The $\pi\mathfrak{F}$-hypercentre of $G$ is the product of all the normal $\pi\mathfrak{F}$-hypercentral subgroups of $G$. In this paper, we study the structure of finite groups by using the notion of $\pi\mathfrak{F}$-hypercentre. New characterizations of some classes of finite groups are obtained.

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## 1. Introduction

Throughout this paper, all groups considered are finite. $G$ always denotes a group, $p$ denotes a prime, and $\pi$ denotes a non-empty subset of the set $\mathbb{P}$ of all primes. Moreover, $\pi(G)$ denotes the set of all prime divisors of $|G|$ and $\pi(\mathfrak{F}) = \bigcup\{\pi(G) | G \in \mathfrak{F}\}$, where $\mathfrak{F}$ is a non-empty class of finite groups.

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathfrak{F}$. A non-empty class $\mathfrak{F}$ of groups is called a *formation* if for every group $G$, every homomorphic image of $G/G^{\mathfrak{F}}$

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belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be (i) *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; (ii) *hereditary* (*normally hereditary*) if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$ ($H \trianglelefteq G \in \mathfrak{F}$, respectively). Note that the classes of all $p$-nilpotent groups and all supersolvable groups are both saturated and hereditary. In the sequel, we use $\mathfrak{U}$ to denote the class of all supersolvable groups.

For a class $\mathfrak{F}$ of groups, a chief factor $H/K$ of $G$ is called $\mathfrak{F}$-*central in* $G$ if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. Following [9], a normal subgroup $N$ of $G$ is said to be $\pi\mathfrak{F}$-*hypercentral in* $G$ if every chief factor of $G$ below $N$ of order divisible by at least one prime in $\pi$ is $\mathfrak{F}$-central in $G$. The symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-hypercentre of $G$, that is, the product of all normal $\pi\mathfrak{F}$-hypercentral subgroups of $G$. When $\pi = \mathbb{P}$ is the set of all primes, $Z_{\mathbb{P}\mathfrak{F}}(G)$ is called the $\mathfrak{F}$-*hypercentre* of $G$, and denoted by $Z_{\mathfrak{F}}(G)$. Clearly, for any non-empty set $\pi$ of primes, $Z_{\mathfrak{F}}(G) \leq Z_{\pi\mathfrak{F}}(G)$.

Applications of the $\pi\mathfrak{F}$-hypercentre are based on the following concept.

Definition 1.1. Let $\mathfrak{F}$ be a non-empty class of groups. A subgroup $H$ of $G$ is called $\pi\mathfrak{F}$-*supplemented* in $G$, if there exists a subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$, where $H_G$ is the maximal normal subgroup of $G$ contained in $H$.

In this paper, we will study the structure of finite groups by using the concept of $\pi\mathfrak{F}$-supplemented subgroup. Now characterizations of $p$-nilpotency and supersolvability of finite groups are obtained, and a series of known results are generalized.

All unexplained notations and terminologies are standard. The reader is referred to [6], [7], and [13].

## 2. Preliminaries

The following known results are helpful in our proof.

Lemma 2.1 ([9, Lemma 2.2] and [5, Lemma 2.8]). *Let $\mathfrak{F}$ be a saturated formation and $\pi \subseteq \pi(\mathfrak{F})$. Let $N \trianglelefteq G$ and $A \leq G$.*

(1) *$Z_{\pi\mathfrak{F}}(G)$ is $\pi\mathfrak{F}$-hypercentral in $G$.*

(2) $Z_{\pi\mathfrak{F}}(A)N/N \leq Z_{\pi\mathfrak{F}}(AN/N)$.

(3) *If $\mathfrak{F}$ is (normally) hereditary and $A$ is a (normal) subgroup of $G$, then $Z_{\pi\mathfrak{F}}(G) \cap A \leq Z_{\pi\mathfrak{F}}(A)$.*

LEMMA 2.2. *Let $\mathfrak{F}$ be a saturated formation and $H \leq K \leq G$.*

(1) *$H$ is $\pi\mathfrak{F}$-supplemented in $G$ if and only if there exists a subgroup $T$ of $G$ such that $G = HT$, $H_G \leq T$ and $(H/H_G) \cap (T/H_G) \leq Z_{\pi\mathfrak{F}}(G/H_G)$.*

(2) *Suppose that $H \trianglelefteq G$. Then $K/H$ is $\pi\mathfrak{F}$-supplemented in $G/H$ if and only if $K$ is $\pi\mathfrak{F}$-supplemented in $G$.*

(3) *Suppose that $H \trianglelefteq G$. Then for every $\pi\mathfrak{F}$-supplemented subgroup $E$ of $G$ satisfying $(|E|, |H|) = 1$, $EH/H$ is $\pi\mathfrak{F}$-supplemented in $G/H$.*

(4) *Suppose that $H$ is $\pi\mathfrak{F}$-supplemented in $G$. If $\mathfrak{F}$ is (normally) hereditary and $K$ is a (normal) subgroup of $G$, then $H$ is $\pi\mathfrak{F}$-supplemented in $K$.*

PROOF. (1) The sufficiency is clear. Now assume that $H$ is $\pi\mathfrak{F}$-supplemented in $G$. Then there exists a subgroup $T$ of $G$ such that $G = HT$ and that $(H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$. Let $T^* = TH_G$. Then $G = HT^*$, $H_G \leq T^*$ and we obtain $(H/H_G) \cap (T^*/H_G) = (H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$.

(2) First assume that $K/H$ is $\pi\mathfrak{F}$-supplemented in $G/H$. Then by (1), $G/H$ has a subgroup $T/H$ such that $G/H = (K/H)(T/H)$, $K_G/H \leq T/H$ and $((K/H)/(K_G/H)) \cap ((T/H)/(K_G/H)) \leq Z_{\pi\mathfrak{F}}((G/H)/(K_G/H))$. It follows that $(K/K_G) \cap (T/K_G) \leq Z_{\pi\mathfrak{F}}(G/K_G)$. Thus $K$ is $\pi\mathfrak{F}$-supplemented in $G$. Analogously, one can show that if $K$ is $\pi\mathfrak{F}$-supplemented in $G$, then $K/H$ is $\pi\mathfrak{F}$-supplemented in $G/H$.

(3) By (1), there exists a subgroup $T$ of $G$ such that $G = ET$, $E_G \leq T$ and $(E/E_G) \cap (T/E_G) \leq Z_{\pi\mathfrak{F}}(G/E_G)$. In view of (2), we only need to prove that $EH$ is $\pi\mathfrak{F}$-supplemented in $G$. Since $(|E|, |H|) = 1$, $H \leq T$, and so $EH \cap T = (E \cap T)H \leq ZH$, where $Z/E_G = Z_{\pi\mathfrak{F}}(G/E_G)$. Let $D = (EH)_G$. Then $(EH/D) \cap (TD/D) = (EH \cap T)D/D \leq ZD/D \leq Z_{\pi\mathfrak{F}}(G/D)$ by Lemma 2.1(2), and so $EH$ is $\pi\mathfrak{F}$-supplemented in $G$.

(4) By (1), $G$ has a subgroup $T$ such that both $G = HT$, and $H_G \leq T$, as well as $(H/H_G) \cap (T/H_G) \leq Z_{\pi\mathfrak{F}}(G/H_G)$. Let $T^* = K \cap T$. Since $K = HT^*$ and $(H/H_G) \cap (T^*/H_G) = (H \cap T)/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G) \cap (K/H_G) \leq Z_{\pi\mathfrak{F}}(K/H_G)$ by Lemma 2.1(3), $H$ is $\pi\mathfrak{F}$-supplemented in $K$. □

LEMMA 2.3 ([4, Lemma 2.12]). *Let $p$ be a prime divisor of $|G|$ with $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ for some integer $n \geq 1$. If $H \trianglelefteq G$ with $p^{n+1} \nmid |H|$ and $G/H$ is $p$-nilpotent, then $G$ is $p$-nilpotent. In particular, if $p^{n+1} \nmid |G|$, then $G$ is $p$-nilpotent.*

For any subgroup $H$ of $G$, a subgroup $T$ of $G$ is called a *supplement of $H$ in $G$* if $G = HT$.

Lemma 2.4 ([11, Lemma 2.12]). *Let $p$ be a prime divisor of $G$ such that $(|G|, p-1) = 1$. Suppose that $P$ is a Sylow $p$-subgroup of $G$ such that every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent.*

Lemma 2.5 ([8, Lemma 2.3]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group with a normal subgroup $E$ such that $G/E \in \mathfrak{F}$. If $E$ is cyclic, then $G \in \mathfrak{F}$.*

The following facts about the generalized Fitting subgroup are useful in our proof (see [14, Chapter X, Section 13] and [19, Lemmas 2.17–2.19]).

Lemma 2.6. (1) *If $N \trianglelefteq G$, then $F^*(N) = F^*(G) \cap N$.*

(2) $F^*(G) \neq 1$ *if* $G \neq 1$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; *if $F^*(G)$ is solvable, then*

$$F^*(G) = F(G).$$

(4) $F^*(G) = F(G)E(G)$, $[F(G), E(G)] = 1$, $F(G) \cap E(G) = Z(E(G))$ *and $E(G)/Z(E(G))$ is the direct product of simple non-abelian groups, where $E(G)$ is the layer of $G$.*

(5) $C_G(F^*(G)) \leq F(G)$.

(6) *If $P$ is a normal $p$-subgroup of $G$, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*

(7) *If $P$ is a normal $p$-subgroup of $G$ contained in $Z(G)$, then*

$$F^*(G/P) = F^*(G)/P.$$

## 3. Characterizations of $p$-nilpotent groups

Recall that a chain $H_0 = H \leq H_1 \leq \cdots \leq H_n = G$ is a maximal chain if each $H_i$ is a maximal subgroup of $H_{i+1}$ $(i = 0, 1, \ldots, n-1)$. The subgroup $H$ in such a series is an $n$-maximal subgroup of $G$. The following proposition is the main step in the proof of Theorem 3.2.

Proposition 3.1. *Let $p$ be a prime divisor of $|G|$ such that*

$$(|G|, (p-1)(p^2-1)\ldots(p^n-1)) = 1$$

*for some integer $n \geq 1$. If there exists a Sylow $p$-subgroup $P$ of $G$ such that every $n$-maximal subgroup (if exists) of $P$ is $p\mathfrak{U}$-supplemented in $G$, then $G$ is $p$-nilpotent.*

Proof. Suppose that the assertion is false and let $G$ be a counterexample of minimal order. Clearly, $p^{n+1}||G|$ by Lemma 2.3. We proceed via the following steps.

(1) $Z_{p\mathfrak{U}}(G) = 1$.

Proof. Suppose that $Z_{p\mathfrak{U}}(G) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $Z_{p\mathfrak{U}}(G)$. Clearly, either $N$ is a $p'$-group or $|N| = p$. By Lemmas 2.2(2) and (3), we see that $G/N$ satisfies the hypothesis of the proposition. Hence, $G/N$ is $p$-nilpotent by the choice of $G$. If $N$ is a $p'$-group, then $G$ is $p$-nilpotent, a contradiction. Thus $|N| = p$. As $(|G|, p-1) = 1$, we have that $N \leq Z(G)$, and so $G$ is $p$-nilpotent, a contradiction too. Thus (1) holds. △

(2) *If $O_p(G) \neq 1$, then $O_p(G)$ is a minimal normal subgroup of $G$ and $G = O_p(G) \rtimes M$, where $M$ is a $p$-nilpotent maximal subgroup of $G$.*

Proof. Let $N$ be a minimal normal subgroup of $G$ contained in $O_p(G)$. Then $N$ is abelian. Similarly as in the proof of (1), we can show that $G/N$ is $p$-nilpotent. Since the class of finite $p$-nilpotent groups is a saturated formation, $N$ is the unique minimal normal subgroup of $G$ contained in $O_p(G)$ and $N \nleq \Phi(G)$. It follows that $G = N \rtimes M$ for some maximal subgroup $M$ of $G$. Thus $M \cong G/N$ is $p$-nilpotent. Clearly, $O_p(G) \cap M \trianglelefteq G$. By the uniqueness of $N$, we have $O_p(G) \cap M = 1$, and so $O_p(G) = N(O_p(G) \cap M) = N$. Thus $O_p(G) = N$ is a minimal normal subgroup of $G$. △

(3) *The final contradiction.*

Proof. Let $P_n$ be any $n$-maximal subgroup of $P$. Then $(P_n)_G = 1$ or $O_p(G)$ by (2). If $(P_n)_G = O_p(G)$, then $G = O_p(G)M = P_nM$. Now assume that $(P_n)_G = 1$. Since $P_n$ is $p\mathfrak{U}$-supplemented in $G$, $G$ has a subgroup $T$ such that $G = P_nT$ and $P_n \cap T = 1$ by (1). Hence $T$ is $p$-nilpotent by Lemma 2.3. This shows that every $n$-maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$. Consequently, $G$ is $p$-nilpotent by Lemma 2.4. △

The final contradiction completes the proof. □

Theorem 3.2. *Let $p$ be a prime divisor of $|G|$ such that*

$$(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1,$$

*for some integer $n \geq 1$. Then $G$ is $p$-nilpotent if and only if $G$ has a normal subgroup $H$ such that $G/H$ is $p$-nilpotent, and for any Sylow $p$-subgroup $P$ of $H$, every $n$-maximal subgroup (if exists) of $P$ is $p\mathfrak{U}$-supplemented in $G$.*

Proof. The necessity is evident. We only need to prove the sufficiency. By Lemma 2.2(4), every $n$-maximal subgroup of $P$ is $p\mathfrak{U}$-supplemented in $H$. Hence $H$ is $p$-nilpotent by Proposition 3.1. Let $H_{p'}$ be the normal Hall $p'$-subgroup of $H$. Then obviously, $H_{p'} \trianglelefteq G$. By Lemma 2.2(3), $(G/H_{p'}, H/H_{p'})$ satisfies the hypothesis of the theorem. If $H_{p'} \neq 1$, then by induction, $G/H_{p'}$ is $p$-nilpotent, and so $G$ is $p$-nilpotent.

Hence we may assume that $H = P$. Let $K/P$ be the normal Hall $p'$-subgroup of $G/P$. By Lemma 2.2(4), every $n$-maximal subgroup of $P$ is $p\mathfrak{U}$-supplemented in $K$. Hence $K$ is $p$-nilpotent by Proposition 3.1, and so $K = P \times K_{p'}$, where $K_{p'}$ is a Hall $p'$-subgroup of $K$. This implies that $K_{p'}$ is a normal Hall $p'$-subgroup of $G$. Therefore, $G$ is $p$-nilpotent. □

Theorem 3.3. *Let $p$ be a prime divisor of $|G|$. Then $G$ is $p$-nilpotent if and only if there exists a normal subgroup $H$ of $G$ such that $G/H$ is $p$-nilpotent, and for any Sylow $p$-subgroup $P$ of $H$, one of the following holds*:

(1) $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ *for some integer* $n \geq 1$, $p^n > 2$ *and every subgroup $L$ of $P$ of order $p^n$ is $p\mathfrak{U}$-supplemented in $G$*;

(2) $p = 2$, *$P$ is abelian and every subgroup $L$ of $P$ of order* 2 *is* $2\mathfrak{U}$*-supplemented in $G$*;

(3) $p = 2$, *$P$ is non-abelian and every cyclic subgroup $L$ of $P$ of order* 2 *or* 4 *is* $2\mathfrak{U}$*-supplemented in $G$.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that this is false and let $(G, H)$ be a counterexample for which $|G|$ is minimal. We proceed via the following steps.

(1) $|P| \geq p^{n+1}$.

Proof. It follows from Lemma 2.3. △

(2) $G = P \rtimes Q$, *where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ a cyclic Sylow $q$-subgroup of $G$ ($p \neq q$), $P/\Phi(P)$ is a chief factor of $G$, and the exponent of $P$ is $p$ or* 4 (*when $P$ is a non-abelian* 2*-subgroup*).

Proof. Let $M$ be any maximal subgroup of $G$. Then by Lemma 2.2(4), $(M, M \cap H)$ satisfies the hypothesis of the theorem. The choice of $(G, H)$ implies that $M$ is $p$-nilpotent, and so $G$ is a minimal non-$p$-nilpotent group. Hence, by [13, IV, Satz 5.4] and [18, Theorem 1.1], $G = G_p \rtimes Q$, where $G_p = G^{\mathfrak{N}^p}$ is the normal Sylow $p$-subgroup of $G$ and $Q$ a cyclic Sylow $q$-subgroup of $G$ ($q \neq p$), $G_p/\Phi(G_p)$ is a chief factor of $G$, and the exponent of $G_p$ is $p$ or 4(when $G_p$ is a non-abelian 2-subgroup). Note that $G^{\mathfrak{N}^p} \leq H$. Therefore, $G_p = P$ and (2) holds. △

(3) *$P$ has a proper subgroup $L$ of order $p^n$ or* 4 (*when $P$ is a non-abelian* 2-*subgroup*) *such that $L \nleq \Phi(P)$ and $L$ is $p\mathfrak{U}$-supplemented in $G$.*

Proof. Take an element $x \in P \setminus \Phi(P)$ and let $E = \langle x \rangle$. Then $|E| = p$ or 4 (when $p = 2$ and $P$ is non-abelian) by (2). It follows that there exists a subgroup $L$ of $P$ of order $p^n$ or 4 (when $p = 2$, $n = 1$ and $P$ is nonablian, we may take $L = E$) such that $E \leq L$. By the hypothesis, $L$ is $p\mathfrak{U}$-supplemented in $G$. Moreover, if $L = P$, then $|P| = 4$ since $|P| \geq p^{n+1}$ by (1). This implies that $P$ is abelian, a contradiction. Thus $L < P$. △

(4) *The final contradiction.*

Proof. By (3), there exists a subgroup $T$ of $G$ such that $G = LT$ and $(L \cap T)L_G/L_G \leq Z_{p\mathfrak{U}}(G/L_G)$. Since $P/\Phi(P)$ is a chief factor of $G$ by (2), $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, and so $(P \cap T)\Phi(P) = \Phi(P)$ or $P$. If $(P \cap T)\Phi(P) = \Phi(P)$, then $P \cap T \leq \Phi(P)$, and thereby $P = P \cap LT = L(P \cap T) \leq L\Phi(P) \neq P$ unless $L = P$, a contradiction. We may, therefore, assume that $(P \cap T)\Phi(P) = P$. Then we get $P \leq T$, and so $T = G$. Thus $L/L_G \leq Z_{p\mathfrak{U}}(G/L_G)$. Since $P/\Phi(P)$ is a chief factor of $G$ by (2), $\Phi(P)L_G = \Phi(P)$ or $P$. If $\Phi(P)L_G = P$, then $L = P$, which contradicts (3). Therefore, $\Phi(P)L_G = \Phi(P)$, and so $L_G \leq \Phi(P)$. If $P/\Phi(P) \leq Z_{p\mathfrak{U}}(G/\Phi(P))$, then $|P/\Phi(P)| = p$, and so $P = L\Phi(P) = L$, a contradiction. Thus $Z_{p\mathfrak{U}}(G/\Phi(P)) \cap (P/\Phi(P)) = 1$ by (2). It follows from Lemma 2.1(2) that $L\Phi(P)/\Phi(P) \leq Z_{p\mathfrak{U}}(G/\Phi(P)) \cap (P/\Phi(P)) = 1$. This implies that $L \leq \Phi(P)$. △

The final contradiction completes the proof. □

## 4. Characterizations of supersolvable groups

In order to prove Theorem 4.2, we first establish the following proposition.

Proposition 4.1. *For any $p \in \pi(G)$, if every maximal subgroup of every non-cyclic Sylow $p$-subgroup $P$ of $G$ is $p\mathfrak{U}$-supplemented in $G$, then $G$ is a Sylow tower group of supersolvable type.*

Proof. Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of $G$. If $P$ is cyclic, then $G$ is $p$-nilpotent (see [17, (10.1.9)]). Otherwise, $G$ is $p$-nilpotent by Proposition 3.1. Let $V$ be the normal Hall $p'$-subgroup of $G$. Hence by Lemma 2.2(4), $V$ satisfies the hypothesis of the proposition. Therefore, by induction, we obtain that $G$ is a Sylow tower group of supersolvable type. □

Theorem 4.2. *$G$ is supersolvable if and only if $G$ has a normal subgroup $H$ such that $G/H$ is supersolvable, and every maximal subgroup of every non-cyclic Sylow $p$-subgroup of $H$ is $p\mathfrak{U}$-supplemented in $G$, for any prime $p \in \pi(H)$.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the result is false and let $(G, H)$ be a counterexample for which $|G|$ is minimal.

(1) *Let $q$ be the largest prime divisor of $|H|$ and $Q$ a Sylow $q$-subgroup of $H$. Then $Q \trianglelefteq G$.*

Proof. By Lemma 2.2(4) and Proposition 4.1, $H$ is a Sylow tower group of supersolvable type. This implies that $Q \trianglelefteq G$. △

(2) *$Q$ is a non-cyclic minimal normal subgroup of $G$.*

Proof. Let $N$ be a minimal normal subgroup of $G$ contained in $Q$, then $N$ is an elementary abelian group. By Lemmas 2.2(2) and (3), the hypothesis of the theorem holds for $(G/N, H/N)$. The choice of $G$ implies that $G/N \in \mathfrak{U}$. Since $\mathfrak{U}$ is a saturated formation, $N$ is the unique minimal normal subgroup of $G$ contained in $Q$ and $N \not\leq \Phi(G)$. It follows that $G$ has a maximal subgroup $M$ such that $G = N \rtimes M$. It is easy to see that $Q \cap M \trianglelefteq G$, and so $Q \cap M = 1$. Therefore, $Q = N(Q \cap M) = N$ is a minimal normal subgroup of $G$. If $Q$ is cyclic, then $G \in \mathfrak{U}$ by Lemma 2.5, which is impossible. Thus $Q$ is non-cyclic. △

(3) *The final contradiction.*

Proof. Let $Q_1$ be a maximal subgroup of $Q$. Then $(Q_1)_G = 1$ by (2). By the hypothesis, there exists a subgroup $T$ of $G$ such that $G = Q_1T$ and $Q_1 \cap T \leq Z_{q\mathfrak{U}}(G)$. Note that $Q \cap T \trianglelefteq G$. By (2), $Q \cap T = 1$ or $Q$. If $Q \cap T = 1$, then $Q = Q_1(Q \cap T) = Q_1$, a contradiction. Hence we may assume that $Q \cap T = Q$. Then $T = G$, and so $Q_1 \leq Z_{q\mathfrak{U}}(G) \cap Q$. Since $Q$ is a minimal normal subgroup of $G$, $Z_{q\mathfrak{U}}(G) \cap Q = 1$ or $Q$. It follows that either $Q_1 = 1$ or $Q \leq Z_{q\mathfrak{U}}(G)$. In both cases, we have that $Q$ is cyclic. △

The final contradiction completes the proof. □

The next proposition is useful in the proof of Theorem 4.4.

Proposition 4.3. *$G$ is supersolvable if and only if there exists a solvable normal subgroup $H$ of $G$ such that $G/H$ is supersolvable, and every maximal subgroup of every non-cyclic Sylow $p$-subgroup of $F(H)$ is $p\mathfrak{U}$-supplemented in $G$, for any $p \in \pi(F(H))$.*

Proof. The necessity is clear. We only need to prove the sufficiency. Suppose that this is false and let $(G, H)$ be a counterexample for which $|G|$ is minimal.

(1) $\Phi(G) \cap F(H) = 1$.

Proof. Assume that $\Phi(G) \cap F(H) \neq 1$, and let $P_1$ be a Sylow $p$-subgroup of $\Phi(G) \cap F(H)$ for some prime $p \in \pi(\Phi(G) \cap F(H))$. Then clearly, $P_1 \trianglelefteq G$. Note that $F(H/P_1) = F(H)/P_1$ by [6, Chapter A, Theorem 9.3(c)]. It is easy to see that $(G/P_1, H/P_1)$ satisfies the hypothesis of the proposition by Lemmas 2.2(2) and (3). Thus, the choice of $(G, H)$ implies that $G/P_1 \in \mathfrak{U}$, and so $G \in \mathfrak{U}$, a contradiction. Thus (1) holds. △

(2) $F(H) = N_1 \times N_2 \times \cdots \times N_t$, *where* $t \geq 1$ *is an integer, and* $N_i$ $(i = 1, 2, \ldots, t)$ *is a minimal normal subgroup of* $G$ *of prime order.*

Proof. Since $H \neq 1$ is solvable, $F(H) \neq 1$. By (1) and [13, Chapter III, Theorem 4.5], $F(H) = N_1 \times N_2 \times \cdots \times N_t$, where $N_i$ $(i = 1, 2, \ldots, t)$ is a minimal normal subgroup of $G$. Without loss of generality, we may assume that $P = N_1 \times N_2 \times \cdots \times N_s$ $(s \leq t)$ is a Sylow $p$-subgroup of $F(H)$. We claim that $|N_i| = p$ for any $i = 1, 2, \ldots, s$. Otherwise, without loss of generality, we may assume that $|N_1| > p$. Then $P$ is non-cyclic. Let $N_1^*$ be a maximal subgroup of $N_1$ and $P^* = N_1^* N_2 \ldots N_s$. Then $P^*$ is a maximal subgroup of $P$ and $(P^*)_G = N_2 \ldots N_s$. By the hypothesis and Lemma 2.2(1), there exists a subgroup $T$ of $G$ such that $G = P^*T$, $(P^*)_G \leq T$ and $(P^*/(P^*)_G) \cap (T/(P^*)_G) \leq Z_{p\mathfrak{U}}(G/(P^*)_G)$. Since $P \cap T \trianglelefteq G$ and $P/(P^*)_G$ is a chief factor of $G$, $P \cap T = (P^*)_G$ or $P$. If $P \cap T = (P^*)_G$, then $P \cap T \leq P^*$, and so $P = P^*(P \cap T) = P^*$, a contradiction. Hence we may assume that $P \cap T = P$. Then $T = G$. This implies that $P^*/(P^*)_G \leq Z_{p\mathfrak{U}}(G/(P^*)_G) \cap (P/(P^*)_G)$. Since $P^*/(P^*)_G \neq 1$, $P/(P^*)_G \leq Z_{p\mathfrak{U}}(G/(P^*)_G)$, and so $|N_1| = p$. This contradiction shows that (2) holds. △

(3) *The final contradiction.*

Proof. By (2), $G/C_G(N_i)$ is a cyclic group for any $1 \leq i \leq t$. Hence $G/C_G(F(H)) = G/(\bigcap_{i=1}^t C_G(N_i)) \in \mathfrak{U}$. Consequently, $G/F(H) \in \mathfrak{U}$. It follows from Theorem 4.2 that $G \in \mathfrak{U}$. △

The final contradiction completes the proof. □

Theorem 4.4. *$G$ is supersolvable if and only if there exists a normal subgroup $H$ of $G$ such that $G/H$ is supersolvable, and every maximal subgroup of every non-cyclic Sylow $p$-subgroup of $F^*(H)$ is $p\mathfrak{U}$-supplemented in $G$, for any prime $p \in \pi(F^*(H))$.*

Proof. The necessity is clear. We only need to prove the sufficiency. Suppose that the result is false and let $(G, H)$ be a counterexample for which $|G|$ is minimal.

(1) *$H = G$ and $F^*(G) = F(G) \neq 1$.*

Proof. By Lemma 2.2(4) and Theorem 4.2, $F^*(H) \in \mathfrak{U}$. Hence by Lemmas 2.6(2) and (3), $F^*(H) = F(H) \neq 1$. Obviously, $(H, H)$ satisfies the hypothesis of the theorem by Lemma 2.2(4). If $H < G$, then the choice of $(G, H)$ implies that $H \in \mathfrak{U}$. Hence $G \in \mathfrak{U}$ by Proposition 4.3, a contradiction. Thus $H = G$. △

(2) *Each proper normal subgroup of $G$ containing $F(G)$ is supersolvable.*

Proof. Let $F(G) \leq N \trianglelefteq G$ with $N < G$. Then by Lemmas 2.6(1) and (3), $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$, and so $F^*(G) = F^*(N)$. By Lemma 2.2(4), $(N, N)$ satisfies the hypothesis of the theorem. Therefore, $N \in \mathfrak{U}$ by the choice of $(G, H)$. △

(3) *$F(G)$ is elementary abelian and $C_G(F(G)) = F(G)$.*

Proof. Assume $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.6(6), $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Then $(G/\Phi(O_p(G)), G/\Phi(O_p(G)))$ satisfies the hypothesis of the theorem by Lemmas 2.2(2) and (3). The choice of $(G, H)$ implies that $G/\Phi(O_p(G)) \in \mathfrak{U}$, and so $G \in \mathfrak{U}$, a contradiction. Therefore, $\Phi(O_p(G)) = 1$ for any $p \in \pi(F(G))$, and thereby $F(G)$ is elementary abelian. By Lemma 2.6(5), we obtain that $C_G(F(G)) = F(G)$. △

(4) *There exists no normal subgroup of $G$ of prime order contained in $F(G)$.*

Proof. Suppose that $G$ has a normal subgroup $L$ contained in $F(G)$ such that $|L| = p$. Then clearly, $G/C_G(L)$ is cyclic and $F(G) \leq C_G(L)$. If $C_G(L) < G$, then $C_G(L) \in \mathfrak{U}$ by (2). It follows that $G$ is solvable, and so $G \in \mathfrak{U}$ by Proposition 4.3, a contradiction. Hence $C_G(L) = G$, and consequently $L \leq Z(G)$. Then by Lemma 2.6(7), $F^*(G/L) = F^*(G)/L$. It follows from Lemmas 2.2(2) and (3) that $(G/L, G/L)$ satisfies the hypothesis of the theorem. Therefore, $G/L \in \mathfrak{U}$ by the choice of $(G, H)$. Thus $G \in \mathfrak{U}$ by Lemma 2.5, a contradiction. Thus (4) holds. △

(5) *Let $P$ be a nontrivial Sylow $p$-subgroup of $F(G)$. Then $P$ is non-cyclic and $P \cap \Phi(G) \neq 1$.*

Proof. If $P$ is cyclic, then by (3), $P$ is elementary abelian, and so $|P| = p$, which contradicts (4). Hence $P$ is non-cyclic. Suppose that $P \cap \Phi(G) = 1$. Then by [13, Chapter III, Theorem 4.5], $P = R_1 \times R_2 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ are minimal normal subgroups of $G$. By discussing similarly as step (2) in Proposition 4.3, we have that $|R_i| = p$ for any $i = 1, 2, \ldots, t$, contrary to (4). Therefore, $P \cap \Phi(G) \neq 1$. △

(6) *There exists a unique normal subgroup $L$ of $G$ contained in $P$.*

Proof. In view of (5), let $L$ be a minimal normal subgroup of $G$ contained in $P \cap \Phi(G)$ and $E/L = E(G/L)$, where $E(G/L)$ is the layer of $G/L$. Then by Lemma 2.6(4), $F^*(G/L) = F(G)E/L$ and $[F(G), E] \leq L$. Let $N$ be a minimal normal subgroup of $G$ contained in $P$ such that $N \neq L$. Then $[N, E] \leq N \cap L = 1$, and so $E \leq C_G(N)$. If $C_G(N) < G$, then $E \leq C_G(N) \in \mathfrak{U}$ by (2). Consequently, $F^*(G/L) = F(G)/L$. Hence $(G/L, G/L)$ satisfies the hypothesis of the theorem by Lemmas 2.2(2) and (3). The choice of $(G, H)$ implies $G/L \in \mathfrak{U}$. This yields that $G \in \mathfrak{U}$, which is impossible. Hence $C_G(N) = G$, contrary to (4). Thus $L$ is the unique normal subgroup of $G$ contained in $P$. △

(7) *The final contradiction.*

Proof. By (3), $P$ is elementary abelian. Let $S$ be a complement of $L$ in $P$, $L^*$ be a maximal subgroup of $L$ and $P^* = L^*S$. Then $P^*$ is a maximal subgroup of $P$, and clearly $(P^*)_G = 1$ by (6). By the hypothesis, $P^*$ is $p\mathfrak{U}$-supplemented in $G$. Then there exists a subgroup $T$ of $G$ such that $G = P^*T$ and $P^* \cap T \leq Z_{p\mathfrak{U}}(G)$. If $P^* \cap T \neq 1$, then $L \leq Z_{p\mathfrak{U}}(G)$ by (6). Therefore, $|L| = p$. This contradiction shows that $P^* \cap T = 1$, and so $|P \cap T| \leq p$. If $P \cap T = 1$, then $P = P^*$, which is impossible. Thus $P \cap T \neq 1$. Since $P \cap T \trianglelefteq G$, $L \leq P \cap T$ by (6). This yields that $|L| = p$, which contradicts (4). △

The proof is thus completed. □

Theorem 4.5. *$G$ is supersolvable if and only if there exists a normal subgroup $H$ of $G$ such that $G/H$ is supersolvable, and every cyclic subgroup of $H$ of order $p$ or order 4 (if $H$ has a non-abelian Sylow 2-subgroup) is $p\mathfrak{U}$-supplemented in $G$, for any prime $p \in \pi(H)$.*

Proof. We only need to prove the sufficiency. Suppose that this is false and let $(G, H)$ be a counterexample with $|G| + |H|$ is minimal.

(1) $H = G^{\mathfrak{U}}$ *is a* $p$*-group for some prime* $p$*,* $H/\Phi(H)$ *is a chief factor of* $G$*, and the exponent of* $H$ *is* $p$ *or* 4 (*when* $H$ *is a non-abelian* 2*-subgroup*).

Proof. Obviously, $G^{\mathfrak{U}} \leq H$. If $G^{\mathfrak{U}} < H$, then $(G, G^{\mathfrak{U}})$ satisfies the hypothesis of the theorem, and so $G \in \mathfrak{U}$ by the choice of $(G, H)$, a contradiction. Thus $H = G^{\mathfrak{U}}$. Now let $M$ be any maximal subgroup of $G$. Then it is easy to check that the hypothesis of the theorem holds for $(M, M \cap H)$ by Lemma 2.2(4). Hence $M \in \mathfrak{U}$ by the choice of $(G, H)$. This shows that $G$ is a minimal non-supersolvable group. Consequently, $G$ is solvable by [17, (10.3.4)]. Now by [18, Theorem 1.1], $H$ is a $p$-group for some prime $p$, $H/\Phi(H)$ is a chief factor of $G$, and the exponent of $H$ is $p$ or 4 (when $H$ is a non-abelian 2-subgroup). △

(2) $|H/\Phi(H)| = p$.

Proof. If not, then we may take a subgroup $X/\Phi(H)$ of $H/\Phi(H)$ of order $p$ and an element $x \in X \setminus \Phi(H)$. Then $L = \langle x \rangle$ is a cyclic group of order $p$ or 4 (when $H$ is a non-abelian 2-subgroup) by (1), and $L\Phi(H) = X$. If $L \trianglelefteq G$, then $X \trianglelefteq G$, and so $H = X$ by (1). It follows that $H/\Phi(H)$ is cyclic. Thus $|H/\Phi(H)| = p$, a contradiction. Hence $L \ntrianglelefteq G$, and so $L_G \leq \Phi(H)$. By the hypothesis, there exists a subgroup $T$ such that $G = LT$ and $(L \cap T)L_G/L_G \leq Z_{p\mathfrak{U}}(G/L_G)$. By Lemma 2.1(2), $(L \cap T)\Phi(H)/\Phi(H) \leq Z_{p\mathfrak{U}}(G/\Phi(H)) \cap (H/\Phi(H))$. If $H/\Phi(H) \leq Z_{p\mathfrak{U}}(G/\Phi(H))$, then one has $|H/\Phi(H)| = p$, a contradiction. Thus $Z_{p\mathfrak{U}}(G/\Phi(H)) \cap (H/\Phi(H)) = 1$, and thereby $L \cap T \leq \Phi(H)$. This implies that $T < G$. Since $(H \cap T)\Phi(H) \trianglelefteq G$, $(H \cap T)\Phi(H) = H$ or $\Phi(H)$ by (1). If $(H \cap T)\Phi(H) = H$, then $H \leq T$, and so $T = G$, a contradiction. Therefore, $H \cap T \leq \Phi(H)$. It follows that $H = L(H \cap T) = L$, also a contradiction. Thus $|H/\Phi(H)| = p$. △

(3) *The final contradiction.*

Proof. In view of (2), $G/\Phi(H) \in \mathfrak{U}$ by Lemma 2.5, and thus $G \in \mathfrak{U}$. △

The final contradiction ends the proof. □

## 5. Some Applications

Recall that a subgroup $H$ of $G$ is said to be $\mathfrak{F}$-*supplemented* [8] *in* $G$, if there exists a subgroup $T$ of $G$ such that $G = HT$ and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. Moreover, many authors introduced various concepts, such as, $c$-normal subgroup (see [20]), $c$-supplemented subgroup (see [3]), $\mathfrak{U}_c$-normal subgroup (see [1]), $\mathfrak{F}$-$z$-supplemented subgroup (see [10]).

It is easy to see that, all these subgroups, whether they are $c$-normal, $c$-supplemented, $\mathfrak{U}_c$-normal, $\mathfrak{U}$-supplemented or $\mathfrak{U}$-$z$-supplemented, are all $\pi\mathfrak{U}$-supplemented subgroups for some set of primes $\pi$. However, a $\pi\mathfrak{U}$-supplemented subgroup is not necessarily a $\mathfrak{U}$-supplemented subgroup as the following example illustrates.

Example 5.1. Let $G = A_4$ and $H = \{1, (12)(34)\}$ be a subgroup of $G$ of order 2. Clearly, $H_G = 1$. It is easy to check that $Z_{\mathfrak{U}}(G) = 1$ and $Z_{3\mathfrak{U}}(G) = G$. Now we show that the subgroup $H$ is $3\mathfrak{U}$-supplemented, but not $\mathfrak{U}$-supplemented in $G$. In fact, if $H$ is $\mathfrak{U}$-supplemented in $G$, then there exists a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq Z_{\mathfrak{U}}(G) = 1$. Therefore, $|T| = 6$. But $A_4$ has no subgroup of order 6, a contradiction. Clearly, $H$ is $3\mathfrak{U}$-supplemented in $G$.

In the literature, one can find a large number of special cases of our theorems. We now list only a small part of them.

Corollary 5.2 ([12, Theorem 3.4]). *Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of $G$. If every maximal subgroup of $P$ is $c$-supplemented in $G$, then $G$ is $p$-nilpotent.*

Corollary 5.3 ([10, Theorem 3.2]). *Let $P$ be a Sylow $p$-subgroup of $G$, where $p$ is a prime divisor of $G$ with $(|G|, p-1) = 1$. If every maximal subgroup of $P$ is $\mathfrak{N}^p$-$z$-supplemented in $G$, where $\mathfrak{N}^p$ denotes the class of all $p$-nilpotent groups, then $G$ is $p$-nilpotent.*

Corollary 5.4 ([2, Lemma 3.1]). *Let $p$ be the smallest prime dividing $|G|$ and let $P$ be a Sylow $p$-subgroup of $G$. If all subgroups of $P$ of order $p$ or order 4 are $c$-normal in $G$, then $G$ is $p$-nilpotent.*

Corollary 5.5 ([15, Theorem 3.3]). *Let $N$ be a normal subgroup of $G$ such that $G/N$ is supersolvable, and $P_1$ is $c$-normal in $G$ for every Sylow subgroup $P$ of $N$ and every maximal subgroup $P_1$ of $P$. Then $G$ is supersolvable.*

Corollary 5.6 ([16, Theorem 2]). *Let $G$ be a solvable group. If $H$ is a normal subgroup of $G$ such that $G/H$ is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ are $c$-normal in $G$, then $G$ is supersolvable.*

Corollary 5.7 ([8, Corollary 3.1.1]). *$G$ is supersolvable if and only if every maximal subgroup of every non-cyclic Sylow subgroup of $G$ is $\mathfrak{U}$-supplemented in $G$.*

Corollary 5.8 ([10, Theorem 3.3]). *$G$ is supersolvable if and only if there exists a normal subgroup $N$ such that $G/N$ is supersolvable and every maximal subgroup of every Sylow subgroup of $N$ is $\mathfrak{U}$-$z$-supplemented in $G$.*

Corollary 5.9 ([3, Theorem 4.1]). *Let $K$ be the supersolvable residual $G^{\mathfrak{U}}$ of $G$. Suppose that every cyclic subgroup of $K$ of prime order or order 4 is $c$-supplemented in $G$. Then $G$ is supersolvable.*

Corollary 5.10 ([1, Corollary 1.5]). *$G$ is supersolvable if and only if all cyclic subgroups of $G$ with prime order or order 4 are $\mathfrak{U}_c$-normal in $G$.*

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## References

[1] A. Y. Alsheik Ahmad – J. J. Jaraden – A. N. Skiba, *On $\mathfrak{U}_c$-normal subgroups of finite groups,* Algebra Colloq. **14** (2007), no. 1, pp. 25–36.

[2] M. Asaad – M. Ezzat Mohamed, *On $c$-normality of finite groups,* J. Aust. Math. Soc. **78** (2005), no. 3, pp. 297–304.

[3] A. Ballester-Bolinches – Y. Wang – X. Guo, *$C$-supplemented subgroups of finite groups,* Glasg. Math. J. **42** (2000), no. 3, pp. 383–389.

[4] X. Chen – W. Guo, *On weakly $S$-embedded and weakly $\tau$-embedded subgroups,* Sibirsk. Mat. Zh. **54** (2013), no. 5, pp. 1162–1181. In Russian. English translation, Sib. Math. J. **54** (2013), no. 5, 931–945.

[5] X. Chen – W. Guo, *On the $\pi\mathfrak{F}$-norm and the $\mathfrak{H}$-$\mathfrak{F}$-norm of a finite group*, J. Algebra **405** (2014), pp. 213–231.

[6] K. Doerk – T. Hawkes, *Finite soluble groups,* de Gruyter Expositions in Mathematics, 4, Walter de Gruyter, Berlin, 1992.

[7] W. Guo, *The theory of classes of groups,* Translated from the 1997 Chinese original, Mathematics and its Applications, 505, Kluwer Academic Publishers Group, Dordrecht, and Science Press, Beijing, 2000.

[8] W. Guo, *On $\mathfrak{F}$-supplemented subgroups of finite groups,* Manuscripta Math. **127** (2008), no. 2, pp. 139–150.

[9] W. Guo – A. N. Skiba, *On the intersection of the $\mathfrak{F}$-maximal subgroups and the generalized $\mathfrak{F}$-hypercentre of a finite group,* J. Algebra **366** (2012), pp. 112–125.

[10] W. Guo – N. Tang – B. Li, *On $\mathfrak{F}$-$z$-supplemented subgroups of finite groups,* Acta Math. Sci. Ser. B Engl. Ed. **31** (2011), no. 1, pp. 22–28.

[11] W. Guo – F. Xie – B. Li, *Some open questions in the theory of generalized permutable subgroups,* Sci. China Ser. A **52** (2009), no. 10, pp. 2132–2144.

[12] X. Guo – K. P. Shum, *Finite $p$-nilpotent groups with some subgroups $c$-supplemented,* J. Aust. Math. Soc. **78** (2005), no. 3, pp. 429–439.

[13] B. Huppert, *Endliche Gruppen* I., Die Grundlehren der Mathematischen Wissenschaften, 134, Springer, Berlin etc., 1967.

[14] B. Huppert – N. Blackburn, *Finite groups* III, Grundlehren der Mathematischen Wissenschaften, 243, Springer, Berlin etc., 1982.

[15] D. Li – X. Guo, *The influence of $c$-normality of subgroups on the structure of finite groups,* J. Pure Appl. Algebra **150** (2000), no. 1, pp. 53–60.

[16] D. Li – X. Guo, *The influence of $c$-normality of subgroups on the structure of finite groups* II, Comm. Algebra **26** (1998), no. 6, pp. 1913–1922.

[17] D. J. S. Robinson, *A course in the theory of groups,* Graduate Texts in Mathematics, 80, Springer, New York and Berlin, 1982.

[18] V. N. Semenchuk, *Minimal non $\mathfrak{F}$-groups,* Algebra Logic **18** (1980), pp. 214–233.

[19] A. N. Skiba, *On weakly $s$-permutable subgroups of finite groups,* J. Algebra **315** (2007), no. 1, pp. 192–209.

[20] Y. Wang, *$C$-normality of groups and its properties,* J. Algebra **180** (1996), no. 3, pp. 954–965.

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