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On the Sylvester–Gallai theorem for conics

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Abstract – In the present note we give a new proof of a result due to Wiseman and Wilson [13] which establishes an analogue of the Sylvester–Gallai theorem valid for curves of degree two. The main ingredients of the proof come from algebraic geometry. Specifically, we use Cremona transformation of the projective plane and Hirzebruch inequality (1).

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1. Introduction

Configurations of lines and points are a classical subject of study and a source of interesting results in combinatorics, geometry and algebra. One of the most celebrated results in this area is the Sylvester–Gallai Theorem.

Theorem 1.1 (Sylvester–Gallai Theorem). *Let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a finite number of points in the real projective plane. Then*

a) either all points are collinear

b) or there exists a line passing through exactly two points in the set $P_1, \ldots, P_s$.

Remark 1.2. The above result is of course also valid in the affine (euclidean) real plane. We have chosen the projective setting since it allows a particularly transparent proof of Theorem 1.4.

A line as in part b) of the above Theorem is called an *ordinary line* for the set $\mathcal{P}$. It is natural to wonder about the minimal number of ordinary lines in the dependence on the number of points s. Melchior [10] showed that there are at least 3 such lines. It has been generalized to $\frac{3}{7}s$ by Kelly and Moser, [9, Theorem 3.6] and further improved by Csima and Sawyer [4, Theorem 2.15].

Theorem 1.3 (Kelly and Moser, Csima and Sawyer). *For a set of s non-collinear points in the real projective plane there are at least*

$$\frac{3}{7}s \quad \textit{ordinary lines.}$$

Moreover, if $s \neq 7$, then the number of ordinary lines is at least $\frac{6}{13}s$.

It has been conjectured by many authors that apart of the two cases constructed by Kelly and Moser, and Csima and Sawyer, the number of ordinary lines is bounded from below by $\frac{1}{2}s$. Recently, in a ground breaking paper [6], Green and Tao proved that this is indeed the case for large values of s.

There is a number of variations and generalizations of the Sylvester–Gallai Theorem, see e.g. [3], [2], [7]. In most generalizations only linear objects are considered. This is in contrast with the following remarkable result proved in [13] by Wiseman and Wilson.

Theorem 1.4 (a Sylvester Theorem for conic sections). *Let $\mathcal{P} = \{P_1, \dots, P_s\}$ be a finite number of points in the real projective plane. Then*

a) *either all points lie on a conic*

b) *or there exists a conic C passing through exactly five of the points in the set $\mathcal{P}$ determined by these 5 points (i.e. C is the unique conic passing through these 5 points).*

Remark 1.5. A conic as in part b) of the above Theorem is called an *ordinary conic* for the set $\mathcal{P}$. It is irrelevant whether this conic is singular or not. In fact it might happen that *all* ordinary conics for $\mathcal{P}$ are singular, see Example 4.1. The other extreme of all smooth ordinary conics is also possible, see Example 4.2.

The proof of this result presented in [13] is quite involved. The purpose of the present note is twofold. First, we provide a simpler and more streamlined proof of Theorem 1.4. Second, it seems that the result of Wiseman and Wilson has not attracted as much attention as it deserves, we want to change this state of matters. In fact we find the result quite appealing and opening an unexplored path of research, with high potential for substantial results. This fits well the philosophy presented in the recent survey by Tao [12], to the effect that there are more hidden connections between various aspects of combinatorics and algebraic geometry. In section 4 we discuss some of natural further generalizations and pose some questions which hopefully will sparkle more interest and research in this direction.

2. Tools from algebraic geometry

The main tools we use in the proof of Theorem 1.4 are the Cremona transformation and Hirzebruch inequality (1). In this section we recall briefly these useful notions.

We begin by the presentation of some basic properties of Cremona transformations. This part is valid over an arbitrary ground field. Let F, G, H be non-collinear points in the projective plane $\mathbb{P}^2$. Let h be the linear form defining the line determined by the points F and G and similarly: g by F, H and f by G and H. (By a slight abuse of notation we denote the lines by the same letters as their equations.) Then

$$\begin{aligned}&\mathbb{P}^2 \ni (x : y : z)\\ &\quad\overset{\varphi}{\longmapsto} (g(x,y,z)\cdot h(x,y,z) : f(x,y,z)\cdot h(x,y,z) : f(x,y,z)\cdot g(x,y,z)) \in \mathbb{P}^2\end{aligned}$$

is a birational automorphism of $\mathbb{P}^2$ (i.e. it is a $1:1$ map up to a codimension 1 subvariety). It is the *Cremona transformation based at the points* F, G and H. After a projective change of coordinates, one may assume that the points F, G and H are the fundamental points (i.e. $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$). Then the mapping φ has a simple form

$$\varphi\colon \mathbb{P}^2 \ni (x : y : z) \longmapsto (yz : xz : xy) \in \mathbb{P}^2.$$

The planes before and after Cremona are schematically depicted in Figure 1 below.

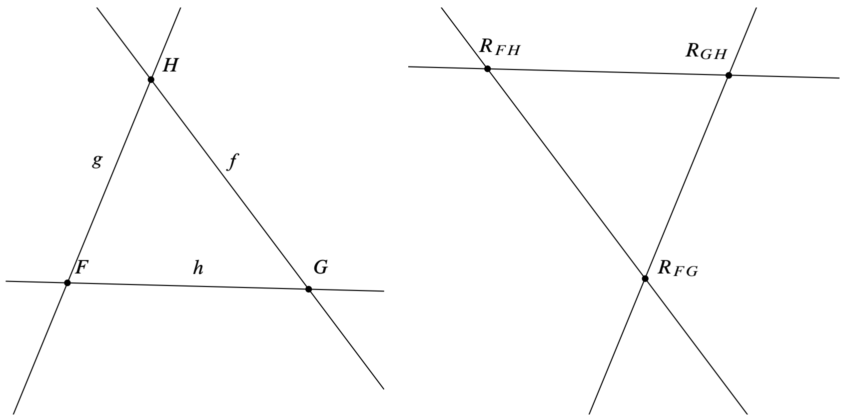

Figure 1. Cremona transformation

The next Proposition collects basic properties of the Cremona map, which are relevant in the sequel. We refer to Dolgachev's masterpiece [5] for proofs and background. By a slight abuse of the notation the line defined by f is denoted by f and similarly for g and h.

Proposition 2.1. a) *The Cremona map contracts the lines* f, g, h *to points* R_{GH}, R_{FH}, R_{FG} *respectively and it is* $1:1$ *away of them.*

b) *The inverse mapping (in the category of birational maps) is also a Cremona transformation based at points* $R_{FG} = \varphi(h)$, $R_{FH} = \varphi(g)$ *and* $R_{GH} = \varphi(f)$.

c) *Let* D *be an irreducible curve, different from the three contracted lines, of degree* d *with multiplicities* m_1 *at* F, m_2 *at* G *and* m_3 *at* H. *Then its image* $\varphi(D)$ *is an irreducible curve of degree* $2d - m_1 - m_2 - m_3$ *with multiplicities* $d - m_2 - m_3$ *at* R_{GH}, $d - m_1 - m_3$ *at* R_{FG} *and* $d - m_1 - m_2$ *at* R_{FG}.

Now we pass to an inequality proved by Hirzebruch in the complex setting, see [8, Section 3 and p. 140]. The inequality itself is based on a very deep result due to Miyaoka, Yau, and Bogomolov, [11]. Of course it remains valid for a configuration of real lines. Let $\mathcal{L}$ be an arrangement of d lines. For $k \geq 2$, let $t_k(\mathcal{L})$ denote the number of points where exactly k lines from $\mathcal{L}$ meet.

Theorem 2.2 (Hirzebruch inequality). *Let* $\mathcal{L}$ *be an arrangement of* d *lines in the complex (or real) projective plane* $\mathbb{P}^2$. *Then*

$$t_2(\mathcal{L}) + t_3(\mathcal{L}) \geq d + \sum_{k \geq 5} (k-4) t_k(\mathcal{L}), \tag{1}$$

provided $t_d = t_{d-1} = 0$.

In fact we will need the dual version of this inequality. To this end given a set of s points in the projective plane let $t_i(\mathcal{P})$ denote the number of lines determined by this set (i.e. by pairs of points in the set), which pass through exactly i points.

Theorem 2.3 (Dual Hirzebruch inequality). *Let* $\mathcal{P}$ *be a set of* s *distinct points. Assume that not all points are collinear and also not all but one point are collinear, then*

$$t_2(\mathcal{P}) + t_3(\mathcal{P}) \geq s + \sum_{k \geq 5} (k-4) t_k(\mathcal{P}), \tag{2}$$

There is a similar inequality

$$t_2(\mathcal{P}) \geq 3 + \sum_{k \geq 4} (k-3) t_k(\mathcal{P}) \quad \text{for } \mathcal{P} \subset \mathbb{P}^2(\mathbb{R}), \tag{3}$$

which was established by Melchior [10] using Euler formula applied to the partition of the *real* projective plane given by the arrangement of lines. In the argument given in the next section one could work with this inequality instead of (2). However, in the view of Problem 4.4 we prefer to work with a more general tool.

3. Proof of Theorem 1.4

We begin by the following very useful observation.

Lemma 3.1 (main cases). *Let $\mathcal{P} = \{P_1, \dots, P_s\}$ be a finite set of points in the real projective plane $\mathbb{P}^2(\mathbb{R})$. Then one of the following holds*:

a) *all points in $\mathcal{P}$ are collinear* (*i.e.* $t_s(\mathcal{P}) = 1$), *or*

b) *there is a line which contains exactly* 3 *points from the set* $\mathcal{P}$ (*i.e.* $t_3(\mathcal{P}) \geq 1$), *or*

c) *there is a pair of ordinary lines intersecting in a point from* $\mathcal{P}$.

Proof. If a) holds, then we are done. So suppose that the points in $\mathcal{P}$ are not collinear. If c) holds, then we are done again. So we are left with the situation when any two ordinary lines are disjoint (note that such configurations of points exist, see [2, Figure 7]). However then the number of ordinary lines is at most $\lfloor \frac{s}{2} \rfloor$. Inequality (2) implies then $t_3(\mathcal{P}) \geq 1$. □

Our proof of Theorem 1.4 splits into three cases distinguished in Lemma 3.1. If the set $\mathcal{P}$ consists of collinear points, then they are also contained in a conic and we are done. The next Lemma shows that the Theorem holds also in case b) of Lemma 3.1.

Lemma 3.2 (triple line). *Let $\mathcal{P}$ be a finite set in the real projective plane $\mathbb{P}^2(\mathbb{R})$. If there is a line L containing exactly* 3 *points from $\mathcal{P}$, then Theorem* 1.4 *holds.*

Proof. Let $\mathcal{P}' = \mathcal{P} \setminus L$. If the set $\mathcal{P}'$ is contained in a line M, then $\mathcal{P}$ is contained in the union $L \cup M$, hence in a conic. Otherwise, there exists an ordinary line M for $\mathcal{P}'$. In that case, we take also $C = L \cup M$. There are exactly 5 points from $\mathcal{P}$ on C and C is uniquely determined by these points, since neither L, nor M contains 4 points from $\mathcal{P}$. Note that it is irrelevant if the intersection point $L \cap M$ belongs to $\mathcal{P}$ or not. □

The rest of the proof deals with case c) of Lemma 3.1. The key idea here is to reduce the statement to Sylvester–Gallai theorem for lines applying the Cremona transformation based at the three points from $\mathcal{P}$ on the intersecting ordinary lines. The argument splits into several cases.

Let FG and FH be ordinary lines for $\mathcal{P}$ (intersecting in the point F). If their union contains the whole set $\mathcal{P}$, then we are done. So we assume that this is not the case. We denote by $\mathcal{P}''$ all points in $\mathcal{P}$ contained in the union of the three lines determined by the points F, G and H. In particular we have $F, G, H \in \mathcal{P}''$.

We call the residual set $\mathcal{P}'$, i.e. $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}''$. If $\mathcal{P}'$ is empty, then $\mathcal{P}$ is contained in the union of the line GH and any line through the point F, hence in a conic. So we assume that the set $\mathcal{P}'$ is non-empty.

Let φ be the Cremona transformation based on the points F, G and H and let $\mathcal{R} = \varphi(\mathcal{P}')$. In particular $\mathcal{R}$ is non-empty.

Case 1. We assume that all points in $\mathcal{R}$ are collinear, contained in a line L. If the line L is not uniquely determined, i.e. if there is just one point in $\mathcal{R}$, then we take L as in Subcase 1.b.

Subcase 1.a. The line L omits the points R_{FG}, R_{FH} and R_{GH} (so in particular there are at least two points in $\mathcal{R}$). Then by Proposition 2.1 c) the preimage of L under φ is a smooth conic D passing through the points F, G and H. The set $\mathcal{P}$ is contained in the union of the four curves (3 lines and the conic D) indicated in the picture below.

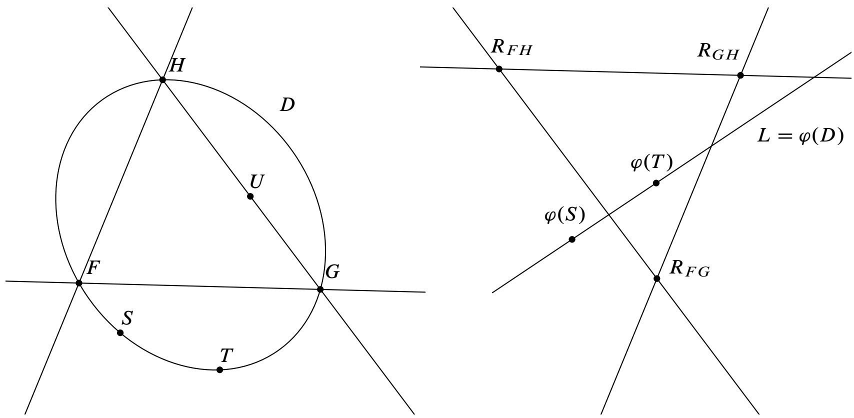

Figure 2. Subcase 1.a

If the line GH is also an ordinary line for $\mathcal{P}$, then all points in $\mathcal{P}$ are on the conic D and we are done. Otherwise, there is a point $U \in \mathcal{P}$ on the line GH as in the Figure 2.

There are at least two points S, T in $\mathcal{P}'$. These points lie then on D. If the points S, T, U are not collinear, then there is a single conic C determined by the points S, T, U, F and G and these are the only points in $\mathcal{P}$ on C. So we are done.

Note that C is the union of lines if points S, T, U are collinear and it is smooth otherwise.

Subcase 1.b. The line L goes through one of the points R_{FG}, R_{FH} or R_{GH}. Note that L cannot pass through any pair of these points, because the lines joining R_{FG}, R_{FH} and R_{GH} are not contained in the image of the Cremona transformation φ.

We assume first that L goes through R_{GH}. Then by Proposition 2.1 c) the preimage D of L is a line passing through the point F as indicated in the Figure below.

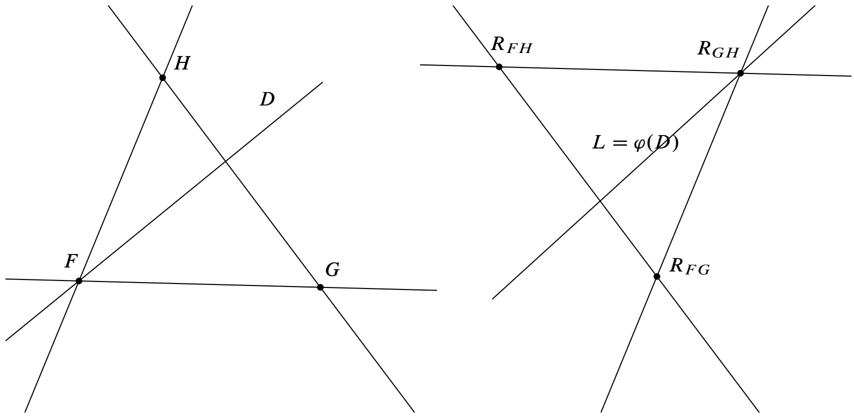

Figure 3. Subcase 1.b

In this situation $\mathcal{P}$ is contained in the union of the line D and the line GH.

Subcase 1.c. Now suppose that L goes through the point R_{FG} (the case R_{FH} is analogous). Then its preimage is a line D passing through H. If GH is an ordinary line for $\mathcal{P}$, then $\mathcal{P}$ is contained in the union of D and the line FG. If GH contains exactly 3 points from $\mathcal{P}$, then we are done by Lemma 3.2. In the remaining case GH contains at least two points U, V from $\mathcal{P}$ distinct from the points G and H. Also on D there are at least two points S, T from $\mathcal{P}$ distinct from the point H. Then the conic C through F, U, V, S and T has these 5 points in common with $\mathcal{P}$ and it is determined by these points.

Case 2. We assume now that not all points in $\mathcal{R}$ are collinear. Hence there exists an ordinary line L for $\mathcal{R}$. Let $\varphi(S)$ and $\varphi(T)$ be the points in $\mathcal{R}$ determining L.

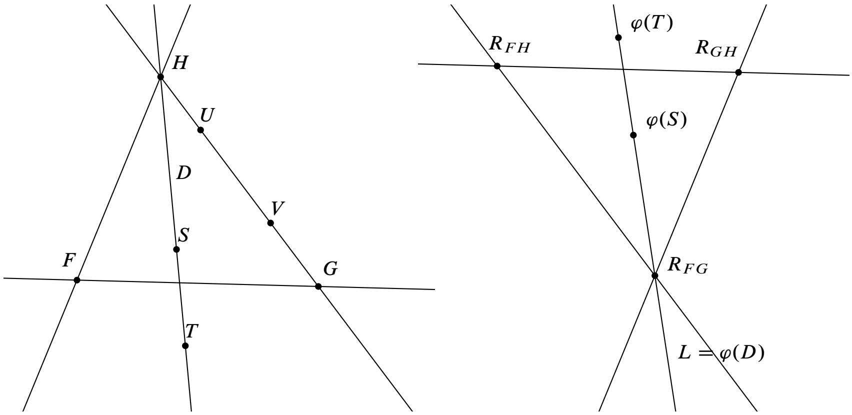

Figure 4. Subcase 1.c

Subcase 2.a. We assume that L does not pass through any of the points R_{FG}, R_{FH} and R_{GH}. This is the easiest case, since then Proposition 2.1 c) implies that the preimage of L under φ is an irreducible conic C passing through points S, T, F, G and H and there are no more points from $\mathcal{P}$ on C.

Subcase 2.b. Now we assume that L goes through the point R_{FG} (the case when L goes through R_{FH} is analogous). Then its preimage D is a line passing through the point H. There are exactly 3 points from $\mathcal{P}$ on D, namely: S, T and H. Note that the intersection point of D with the line FG does not belong to $\mathcal{P}$ (since FG is an ordinary line). Thus we are done by Lemma 3.2.

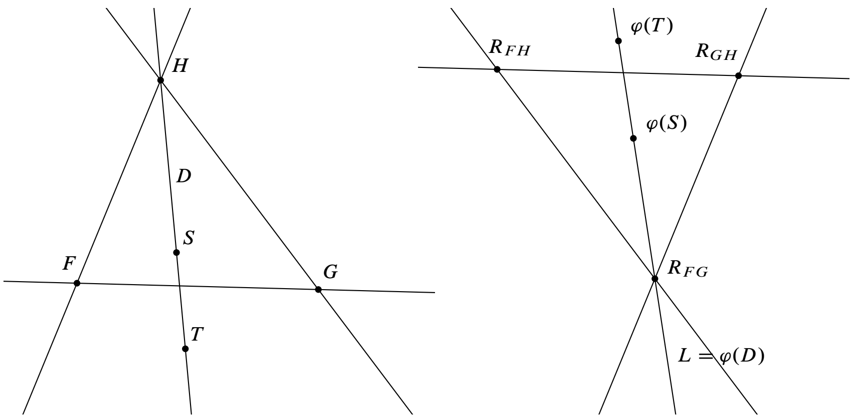

Figure 5. Subcase 2.b

Subcase 2.c. Thus we are left with the situation that *all* ordinary lines for $\mathcal{R}$ go through the point R_{GH}. Note that this point is not contained in the set $\mathcal{R}$. Let t be the number of points in $\mathcal{R}$. Then by Theorem 1.3 there are at least $\frac{3}{7}t$ ordinary lines for $\mathcal{R}$. Each of these lines contains 2 points from $\mathcal{R}$, so that there are altogether at least $\frac{6}{7}t$ points from $\mathcal{R}$ on the union of these lines. Call this set $\mathcal{W}$. Now, we consider the set $\mathcal{R}' = \mathcal{R} \cup \{R_{GH}\}$. All ordinary lines for $\mathcal{R}'$ must be of the form: a line joining R_{GH} with a point in $\mathcal{R} \setminus \mathcal{W}$. This implies that there are at most $\frac{1}{7}t$ such lines, which contradicts Theorem 1.3 for the set $\mathcal{R}'$ consisting of $t + 1$ points. Hence this subcase is not possible and the whole proof is finished.

4. Examples and further questions

We begin with an example of a set of points $\mathcal{P}$ such that *every* ordinary conic for $\mathcal{P}$ is singular. This shows that one cannot hope for Theorem 1.4 to hold assuming C smooth.

Example 4.1 (only singular ordinary conics). Let C be a smooth conic and let S be a point not on C. Let L_1, L_2, L_3 be three mutually distinct lines through S intersecting C in pairs of points P_1, P_2, Q_1, Q_2 and R_1, R_2 as indicated in the figure below.

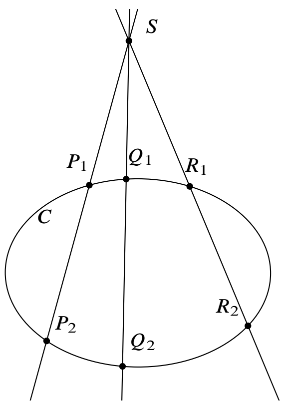

Figure 6

Then all ordinary conics for $\mathcal{P} = \{S, P_1, P_2, Q_1, Q_2, R_1, R_2\}$ split into a pair of lines through S. Indeed, an ordinary conic C must pass through the point S and, since it contains altogether 5 points from $\mathcal{P}$, it must also pass through at least one pair of points on a line L through S. But then C and L have at least 3 points in common. By Bezout's Theorem, L must be then a component of C, hence C is a singular conic.

On the other hand it might easily happen that *all* ordinary conics for some set of points are *smooth*.

Example 4.2 (only smooth ordinary conics). Let $\mathcal{P}$ be a set of points in *general* position in the plane. Then there is an ordinary conic through any 5 points from $\mathcal{P}$ and all these conics are smooth (this is more or less the definition of the "general position").

The Sylvester–Gallai Theorem fails in the finite characteristic. This is also the case for Theorem 1.4. The simplest counterexample is the following.

Example 4.3 (failure of the theorem in finite characteristic). Let $\mathbb{F}$ denote the field with 5 elements. Then $\mathbb{P}^2(\mathbb{F})$ consists of 31 points. We consider the set $\mathcal{P}$ consisting of all points in $\mathbb{P}^2(\mathbb{F})$. Then any conic C containing 5 points from $\mathcal{P}$ must contain at least one more point. Indeed, if C is non-singular (and has ≥ 5 points in $\mathbb{P}^2(\mathbb{F})$), then it consists of exactly 6 points. If it is singular, then it splits into two lines, each of them through 6 points, so that there are altogether 11 points from $\mathcal{P}$ on C.

The Sylvester–Gallai Theorem 1.1 fails also over complex numbers. The simplest example is provided by the Hesse configuration, see [1] for details.

We have expected that there exists also a complex counterexample to Theorem 1.4. However there are strong indications that this might not be the case. Of course, our proof of Theorem 1.4 presented here, relies strongly on Theorem 1.1, so that it cannot be used in the complex case. It would be very interesting to know an answer to the following question.

Problem 4.4. Decide if Theorem 1.4 is valid or not for points in the complex projective plane.

Once the problem is settled for curves of degree 1 and 2, it is natural to wonder what the situation is for curves of higher degree. Thus we repeat here the question which concludes article [13].

Problem 4.5 (curves of higher degree). Let $\mathcal{P}$ be a finite set of points in the projective plane and let d be a positive integer. Does then one of the following hold:

a) either $\mathcal{P}$ is contained in a curve of degree d

b) or there exists a curve C passing through exactly $\frac{(d+1)(d+2)}{2} - 1$ points in $\mathcal{P}$ and determined by these points?

This is not completely obvious if Problem 4.5 is the right generalization of Theorems 1.1 and 1.4. For example, for $d = 3$ one might wonder instead if either $\mathcal{P}$ is contained in a single cubic curve *singular* in a point from $\mathcal{P}$ or there exists such a curve determined by $\mathcal{P}$.

Remark 4.6 (importance of the determined condition). Note that any line is determined by 2 distinct points, so that it is not necessary to emphasize this condition in case b) of Theorem 1.1. This is no more the case for conics. In fact, it is very easy to show, that if not all points in a finite set $\mathcal{P}$ are contained in a conic, then there exists a conic through exactly 5 points in $\mathcal{P}$. So that the critical point of Theorem 1.4 is that there exist five points in $\mathcal{P}$ which *determine a single* conic.

Strangely enough the claim in the preceding Remark seems not easy to prove for curves of higher degree. So the following question can be viewed as the first step towards understanding Problem 4.5.

Problem 4.7. Let $\mathcal{P}$ be a finite set of points in the projective plane not contained in a curve of degree d. Is there a curve of degree d passing through exactly $\frac{(d+1)(d+2)}{2} - 1$ points in $\mathcal{P}$?

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