Rend. Sem. Mat. Univ. Padova, Vol. 136 (2016), 225–255
DOI 10.4171/RSMUP/136-15

Hard Lefschetz theorem in p-adic cohomology

Daniel Caro (∗)

Abstract – In this paper, we give a p-adic analogue of the hard Lefschetz Theorem.

Mathematics Subject Classification (2010). 14F30.

Keywords. Hard Lefschetz theorem, p-adic cohomology.

Introduction

Let $\mathbb{F}_q$ be a finite field of characteristic p. In the context of Grothendieck's l-adic étale cohomology, with $l \neq p$, in using Weil conjecture and a generalization of the theorem of Hadamard and de La Vallée Poussin, Deligne proved the so called hard Lefschetz theorem for the constant coefficient in the case of a smooth and projective variety of pure dimension over $\mathbb{F}_q$. Later, using Gabber's purity theorem and its consequences (e.g. the semi-simplicity of a pure perverse sheaf) this has been extended to the relative case and for pure perverse sheaves in [5] (see also [11, IV.4.1] for the essentially same proof but with a different presentation). More precisely, let $f: X \to Y$ be a projective morphism defined over $\mathbb{F}_q$ and η be the Chern class of the relative line bundle in $H^2(X, \overline{\mathbb{Q}}_l(1))$. Let $\mathcal{E}$ be a pure perverse sheaf on X. Then for every positive integer r, the homomorphisms induced by η^r between the perverse cohomology groups

$$^pH^{-r}\mathbb{R}f_*(\mathcal{E}) \longrightarrow {}^pH^{-r}\mathbb{R}f_*(\mathcal{E})(r)$$

are isomorphisms. The main purpose of this paper is to check a p-adic analogue of this hard Lefschetz Theorem as follows: we replace "perverse sheaf" by "arithmetic left (by default) $\mathcal{D}$-module endowed with a Frobenius structure" and we use

(∗) *Indirizzo dell'A.*: Laboratoire de Mathématiques Nicolas Oresme, Université de Caen Campus 2, 14032 Caen Cedex, France
E-mail: daniel.caro@unicaen.fr

the push forward as defined in Berthelot's theory of arithmetic $\mathcal{D}$-modules. In particular, when $Y = \operatorname{Spec} k$, X is smooth and projective of pure dimension d, E is a pure overconvergent F-isocrystal, we get the isomorphisms

$$H_{\mathrm{rig}}^{d-r}(E) \xrightarrow{\sim} H_{\mathrm{rig}}^{d+r}(E)(r)$$

(see the Corollary 2.8 of the paper). Finally, we recall that in the context of crystalline cohomology, which is the first attempt to get a nice p-adic cohomology over varieties of characteristic p, Berthelot checked a weak Lefschetz theorem (see [6]).

To check this p-adic analogue, we have followed the proof in the l-adic context written in [11, IV.4.1] (just because the author prefers the exposition). As a p-adic analogue of the original proof, two main ingredients of our proof are the semi-simplicity of a pure arithmetic $\mathcal{D}$-module (see [3, 4.3.1]) and the construction and the properties of the trace map given in [1, 1.5]. Then, this paper can be considered as a natural application of these works. We follow here their terminology and notation.

Let us describe the contents of the paper. In the first chapter, we study the properties of the Serre subcategory consisting of relative constant objects. In the second chapter, we introduce the p-adic analogue of the Brylinsky–Radon transform and use its properties to prove the hard Lefschetz Theorem. We have tried to write the proofs only when the p-adic analogues were not straightforward. Finally, in the last chapter, for the sake of completeness, we check the inversion formula satisfied by Radon transform.

Acknowledgment

The author would like to thank Tomoyuki Abe for many explanations concerning the trace maps and their properties and Weizhe Zheng for his interest and questions concerning a p-adic analogue of the hard Lefschetz Theorem.

Notation. In this paper, we fix a positive integer s, a complete discrete valuation ring $\mathcal{V}$ of mixed characteristic $(0, p)$. Its residue field is denoted by k, and assume it to be perfect. We also suppose that there exists a lifting $\sigma \colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ of the s-th Frobenius automorphism of k. We put $q := p^s$, $K := \mathrm{Frac}(\mathcal{V})$. We fix an isomorphism $\iota \colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$.

In this context, a realizable variety X will mean a k-variety such that there exists an immersion of the form $X \hookrightarrow \mathcal{P}$ into a smooth proper p-adic formal scheme over $\mathcal{V}$. We denote by $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ the derived category of overholonomic complexes of (left by default) $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-modules (for the notion of overholonomicity, see [7]). We denote the derived category of overholonomic complexes of arithmetic $\mathcal{D}$-modules on X by $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$. We recall that this is by definition the full subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$ of complexes $\mathcal{E}$ such that $\mathbb{R}\Gamma^{\dagger}_{X}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$. Up to a canonical equivalence of categories, this does not depend on the choice of such an immersion $X \hookrightarrow \mathcal{P}$. From [3, 1.2], we have a canonical t-structure on $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$, whose heart is denoted by $\mathrm{Ovhol}(X/K)$. We recall that, if $\mathfrak{U}$ is an open set of $\mathcal{P}$ containing X and such that X is closed in $\mathfrak{U}$, then a complex $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ belong to $\mathrm{Ovhol}(X/K)$ if and only if $\mathcal{E}|\mathfrak{U}$ is isomorphic to an overholonomic $\mathcal{D}^{\dagger}_{\mathfrak{U},\mathbb{Q}}$-module. Beware that an objet of $\mathrm{Ovhol}(X/K)$ is not necessary an overholonomic $\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}}$-module but is in general a object of $D^{\mathrm{b}}_{\mathrm{ovhol}}(\mathcal{D}^{\dagger}_{\mathcal{P},\mathbb{Q}})$. Following [3, 1.2], the i-th cohomological space of an object $\mathcal{E}$ of $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ will be denoted by $\mathcal{H}^{i}_{t}(\mathcal{E}) \in \mathrm{Ovhol}(X/K)$. We will keep the notation concerning cohomological operators as defined in [3, 1.1].

We will also use the categories defined in [2, 1.5]: let $\mathrm{Hol}_{F}(X/K)'$ be the full subcategory of $\mathrm{Ovhol}(X/K)$ whose objects can be endowed with some s'-th Frobenius structure for some integer s' which is a multiple of s, and let $\mathrm{Hol}_{F}(X/K)$ be the thick abelian subcategory generated by $\mathrm{Hol}_{F}(X/K)'$ in $\mathrm{Ovhol}(X/K)$. We denote by $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ the triangulated full subcategory of $D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K)$ such that the cohomologies are in $\mathrm{Hol}_{F}(X/K)$. For any integer n, we define the twist of Tate over $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ as follows: the twist (n) is the identity (and then the forgetful functor $F\text{-}D^{\mathrm{b}}_{\mathrm{hol},F}(X/K) \to D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ commutes with the twist of Tate). For simplicity and if there is no risk of confusion with the notion of holonomicity of Berthelot, we will write $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ instead of $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$ and $\mathrm{Hol}(X/K)$ instead of $\mathrm{Hol}_{F}(X/K)$. With this notation, we get $F\text{-}D^{\mathrm{b}}_{\mathrm{ovhol}}(X/K) = F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. Be careful that this notation is a bit misleading since in general we do not know even with Frobenius structures if the notion of holonomicity of Berthelot and the notion of overholonomicity coincide (but this is not misleading with Frobenius structure in the case where we can embed the variety into of smooth projective formal schemes over $\mathcal{V}$; see [10]).

1. Constant objects with respect to smooth $\mathbb{P}^d$-fibration morphisms

1.1. Let $g\colon U \to T$ be a morphism of realizable varieties. Let $\mathcal{F}, \mathcal{G} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(T/K)$. We have the morphisms

(1.1.1)
$$\epsilon_g\colon g^!(\mathcal{F}) \otimes g^+(\mathcal{G}) \underset{\mathrm{adj}}{\longrightarrow} g^! g_!(g^!(\mathcal{F}) \otimes g^+(\mathcal{G})) \underset{\mathrm{proj}}{\overset{\sim}{\longrightarrow}} g^!(g_! g^!(\mathcal{F}) \otimes \mathcal{G}) \underset{\mathrm{adj}}{\longrightarrow} g^!(\mathcal{F} \otimes \mathcal{G})$$

where proj (resp. adj) means the projection isomorphism constructed in [3, A.6] (resp. the adjunction isomorphism corresponding to the adjoint functors $(g_!, g^!)$). Since the projection isomorphisms and adjunction isomorphisms are transitive, then so is for ϵ_g i.e., for any $h\colon V \to T$ morphism of realizable varieties, the diagram

$$(1.1.2)\qquad \begin{tikzcd} h^! g^!(\mathcal{F}) \otimes h^+ g^+(\mathcal{G}) \arrow[r, "\epsilon_h"] \arrow[d, "\sim"] & h^!(g^!(\mathcal{F}) \otimes g^+(\mathcal{G})) \arrow[r, "h^!\epsilon_g"] & h^! g^!(\mathcal{F} \otimes \mathcal{G}) \arrow[d, "\sim"] \\ (g \circ h)^!(\mathcal{F}) \otimes (g \circ h)^+(\mathcal{G}) \arrow[rr, "\epsilon_{g\circ h}"] & & (g \circ h)^!(\mathcal{F} \otimes \mathcal{G}) \end{tikzcd}$$

is commutative.

1.2 (Poincaré duality). Let $f\colon X \to S$ be a smooth equidimensional morphism of relative dimension d of realizable varieties. T. Abe has checked (see [1, 1.5.13]) that the morphism

$$(1.2.1)\qquad \theta_f\colon f^+[d] \longrightarrow f^![-d](-d),$$

which is induced by adjunction from the trace map Tr: $f_! f^+[2d](d) \to \mathrm{Id}$, is an isomorphism of t-exact functors (when f is moreover proper, this trace map can be compared with that defined by Virrion in [13]). This isomorphism satisfies several compatibility properties (see [1, 1.5]), e.g. it is transitive.

1.3. We keep the notation of 1.2. Let $\mathcal{F}, \mathcal{G} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(S/K)$. Diagram 1 is commutative. Indeed, the pentagon is commutative from [1, 1.5.1.Var5]. The other parts of the diagram are commutative by definition and functoriality. Hence we get the canonical commutative square:

$$(1.3.1)\qquad \begin{tikzcd} f^+(\mathcal{F} \otimes \mathcal{G})[2d](d) \arrow[r, "\sim"] \arrow[d, "\theta_f"', "\sim"] & f^+(\mathcal{F}[2d](d)) \otimes f^+(\mathcal{G}) \arrow[d, "\theta_f \otimes \mathrm{Id}"', "\sim"] \\ f^!(\mathcal{F} \otimes \mathcal{G}) & f^!(\mathcal{F}) \otimes f^+(\mathcal{G}). \arrow[l, "\epsilon_f"] \end{tikzcd}$$

This implies that the bottom morphism of (1.3.1) is also an isomorphism.

$$
\begin{tikzcd}
\boxed{f^{+}(\mathcal{F}\otimes\mathcal{G})[2d](d)} \arrow[r, "\sim"] \arrow[d, "\mathrm{adj}"] & \boxed{f^{+}(\mathcal{F}[2d](d))\otimes f^{+}(\mathcal{G})} \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{adj}"] & f^{!}f_{!}f^{+}(\mathcal{F}[2d](d))\otimes f^{+}(\mathcal{G}) \arrow[r, "\mathrm{Tr}\otimes\mathrm{Id}"] \arrow[d, "\mathrm{adj}"] & \boxed{f^{!}(\mathcal{F})\otimes f^{+}(\mathcal{G})} \arrow[ddd, "\epsilon_f"] \\
f^{!}f_{!}f^{+}(\mathcal{F}\otimes\mathcal{G})[2d](d) \arrow[r, "\sim"] \arrow[dd, "\mathrm{Tr}"] & f^{!}f_{!}(f^{+}(\mathcal{F}[2d](d))\otimes f^{+}(\mathcal{G})) \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{proj}"', "\sim"] & f^{!}f_{!}(f^{!}f_{!}f^{+}(\mathcal{F}[2d](d))\otimes f^{+}(\mathcal{G})) \arrow[d, "\mathrm{proj}"', "\sim"] & \\
& f^{!}(f_{!}f^{+}(\mathcal{F}[2d](d))\otimes\mathcal{G}) \arrow[r, "\mathrm{adj}"] \arrow[dr, equal] & f^{!}(f_{!}f^{!}f_{!}f^{+}(\mathcal{F}[2d](d))\otimes\mathcal{G}) \arrow[d, "\mathrm{adj}"] & \\
\boxed{f^{!}(\mathcal{F}\otimes\mathcal{G})} & & f^{!}(f_{!}f^{+}(\mathcal{F}[2d](d))\otimes\mathcal{G}) \arrow[ll, "\mathrm{Tr}\otimes\mathrm{Id}"] \arrow[r, "\mathrm{Tr}\otimes\mathrm{Id}"] & f^{!}(\mathcal{F}\otimes\mathcal{G})
\end{tikzcd}
$$

Diagram 1

1.4 Definition. Let $f\colon X \to S$ be an equidimensional smooth morphism of relative dimension d of realizable varieties.

(1) The objects of the essential image of the functor

$$f^+\colon (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(S/K) \longrightarrow (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$$

are called *constant* (*with respect to* f).

(2) The objects of the essential image of the functor

$$f^+[d]\colon (F\text{-})\,\mathrm{Hol}(S/K) \longrightarrow (F\text{-})\,\mathrm{Hol}(X/K)$$

are called *constant* (*with respect to* f). We denote by $f^+[d](F\text{-})\,\mathrm{Hol}(S/K)$ its essential image.

1.5. Let X be a realizable k-variety and $p_X\colon X \to \operatorname{Spec} k$ be the structural morphism. We denote by $K_X := p_X^+(K)$ the constant coefficient of X. The complex K_X is the p-adic analogue of the constant sheaf $\mathbb{Q}_l$ over X. Let $\mathcal{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. We notice that $K_X \otimes \mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

1.6 Proposition. *Let $u\colon Y \hookrightarrow X$ be a closed immersion of pure codimension r in X of smooth realizable k-varieties. Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$.*

(1) *There exists a natural functorial morphism of $(F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(Y/K)$ of the form*

$$\theta_u\colon u^+(\mathcal{E}) \longrightarrow u^!(\mathcal{E})[2r](r). \tag{1.6.1}$$

(2) *If (locally on X) the complex $\mathcal{E}$ is constant with respect to a smooth equidimensional morphism $f\colon X \to S$ of realizable varieties such that $f\circ u$ is also smooth, then θ_u is an isomorphism.*

Proof. This can be checked as in [11, II.11.2]: with the notation and hypothesis of the second part, putting $g := f \circ u$ and $d_g := \dim Y - \dim S$, for any $\mathcal{K} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(S/K)$, by using the isomorphism (1.2.1), we get the isomorphism

$$\theta_u\colon u^+(f^+\mathcal{K}) \underset{\theta_g}{\xrightarrow{\sim}} u^!(f^!\mathcal{K})[-2d_g](d_g) \xrightarrow[u^!(\theta_f^{-1})]{\sim} u^!(f^+\mathcal{K})[2r](r). \tag{1.6.2}$$

In particular, we get

$$\theta_u\colon u^+(K_X) \xrightarrow{\sim} u^!(K_X)[2r](r).$$

We remark that

$$\theta_u\colon u^+(K_X) \xrightarrow{\sim} u^!(K_X)[2r](r)$$

does not depend on the choice of f which can be for instance the structural morphism of X (indeed, since Y is smooth, this is up to a shift a morphism of overconvergent isocrystal on Y and then we can suppose S smooth ; then this is a consequence of the transitivity of the isomorphisms of the form (1.2.1)).

$$
\begin{tikzcd}
u^{+} f^{+}(\mathcal{K}' \otimes \mathcal{K}) \arrow[rr, "\sim"] \arrow[d, "\theta_g"', "\sim"] \arrow[dd, bend right=60, "\theta_u"'] & & u^{+} f^{+}(\mathcal{K}') \otimes u^{+} f^{+}(\mathcal{K}) \arrow[d, "\theta_g \otimes \mathrm{Id}"', "\sim"] \arrow[dd, bend left=60, "\theta_u \otimes \mathrm{Id}"] \\
u^{!} f^{!}(\mathcal{K}' \otimes \mathcal{K})[-2d_g](d_g) & u^{!}(f^{!}(\mathcal{K}') \otimes f^{+}(\mathcal{K}))[-2d_g](d_g) \arrow[l, "\sim", "\epsilon_f"'] & u^{!} f^{!}(\mathcal{K}') \otimes u^{+} f^{+}(\mathcal{K})[-2d_g](d_g) \arrow[l, "\sim", "\epsilon_u"'] \arrow[ll, bend right=10, "\epsilon_g", "\sim"'] \\
u^{!} f^{+}(\mathcal{K}' \otimes \mathcal{K})[-2r](r) \arrow[r, "\sim"] \arrow[u, "\theta_f", "\sim"'] & u^{!}(f^{+}(\mathcal{K}') \otimes f^{+}(\mathcal{K}))[-2r](r) \arrow[u, "\theta_f \otimes \mathrm{Id}", "\sim"'] & u^{!} f^{+}(\mathcal{K}') \otimes u^{+} f^{+}(\mathcal{K})[-2r](r) \arrow[l, "\sim", "\epsilon_u"'] \arrow[u, "\theta_f \otimes \mathrm{Id}", "\sim"']
\end{tikzcd}
$$

Diagram 2

For any $\mathcal{K}, \mathcal{K}' \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(S/K)$, using (1.1.2) and (1.3.1), we check the commutativity of Diagram 2, where ϵ_f, ϵ_g (and then ϵ_u) are some isomorphisms because of the commutativity of (1.3.1).

More generally (in the context of the first part of the proposition), we define $\theta_u : u^+(\mathcal{E}) \to u^!(\mathcal{E})[2r](r)$ so that the diagram

(1.6.3)
$$\begin{tikzcd}
u^+(\mathcal{E}) \arrow[r, "\sim"] \arrow[d, dotted, "\theta_u"] & u^+(K_X \otimes \mathcal{E}) \arrow[r, "\sim"] & u^+(K_X) \otimes u^+(\mathcal{E}) \arrow[d, "\theta_u \otimes \mathrm{Id}"', "\sim"] \\
u^!(\mathcal{E})[2r](r) & u^!(K_X \otimes \mathcal{E})[2r](r) \arrow[l, "\sim"'] & u^!(K_X) \otimes u^+(\mathcal{E})[2r](r), \arrow[l, "\epsilon_u"']
\end{tikzcd}$$

where

$$\theta_u : u^+(K_X) \xrightarrow{\sim} u^!(K_X)[2r](r)$$

is defined in (1.6.2) with f equal to the structural morphism of X, is commutative.

We go back to the second part of the proposition, i.e. suppose now that $\mathcal{E} = f^+(\mathcal{K})$. By using the commutativity of Diagram 2 applied to the case $\mathcal{K}' := K_S$, the isomorphism $\theta_u : u^+(\mathcal{E}) \xrightarrow{\sim} u^!(\mathcal{E})[2r](r)$ defined in (1.6.2) is equal to that defined in (1.6.3). Hence, θ_u is indeed an isomorphism in this case. □

1.7. With the notation of (1.6.1), we get the morphism

$$u_!(\theta_u) : u_! u^+(\mathcal{E}) \longrightarrow u_! u^!(\mathcal{E})[2r](r).$$

By adjunction, we have

$$u_! u^!(\mathcal{E}) \longrightarrow \mathcal{E},$$

which gives by composition

$$\phi_u := \mathrm{adj} \circ u_!(\theta_u) : u_! u^+(\mathcal{E}) \longrightarrow \mathcal{E}[2r](r).$$

The goal of this paragraph is to check that the diagram below

(1.7.1)
$$\begin{tikzcd}
u_! u^+(\mathcal{E}) \arrow[r, "\mathrm{proj}", "\sim"'] \arrow[d, "\phi_u"] & u_! u^+(K_X) \otimes \mathcal{E} \arrow[d, "\phi_u \otimes \mathrm{Id}"] \\
\mathcal{E}[2r](r) \arrow[r, "\sim"] & K_X \otimes \mathcal{E}[2r](r).
\end{tikzcd}$$

is commutative. It is sufficient to check the commutativity of Diagram 3.

$$
\begin{tikzcd}
u_!u^+(\mathcal{E}) \arrow[r, "\sim"] \arrow[d, "u_!(\theta_u)"] \arrow[dd, bend right=60, "\phi_u"'] & u_!u^+(K_X \otimes \mathcal{E}) \arrow[r, "\sim"] \arrow[d, "u_!(\theta_u)"] & u_!(u^+(K_X) \otimes u^+(\mathcal{E})) \arrow[r, "\mathrm{proj}", "\sim"'] \arrow[d, "u_!(\theta_u \otimes \mathrm{Id})"] & u_!u^+(K_X) \otimes \mathcal{E} \arrow[d, "u_!(\theta_u)\otimes \mathrm{Id}"'] \arrow[dd, bend left=60, "\phi_u \otimes \mathrm{Id}"] \\
u_!u^!(\mathcal{E})[2r](r) \arrow[r, "\sim"] \arrow[d, "\mathrm{adj}"] & u_!u^!(K_X \otimes \mathcal{E})[2r](r) \arrow[d, "\mathrm{adj}"] & u_!(u^!(K_X) \otimes u^+(\mathcal{E}))[2r](r) \arrow[l, "\epsilon_u"'] \arrow[r, "\mathrm{proj}", "\sim"'] & u_!u^!(K_X) \otimes \mathcal{E}[2r](r) \arrow[d, "\mathrm{adj}"'] \\
\mathcal{E}[2r](r) \arrow[r, "\sim"] & K_X \otimes \mathcal{E}[2r](r) \arrow[rr, equal] & & K_X \otimes \mathcal{E}[2r](r)
\end{tikzcd}
$$

Diagram 3

From (1.6.3), the middle upper square is commutative. The commutativity of the other squares is checked by functoriality. It remains to check the commutativity of the rectangle, which comes from the commutativity of Diagram 4.

We end this paragraph with a remark: from the commutativity of (1.7.1), we can construct the morphism

$$\phi_u : u_! u^+(\mathcal{E}) \longrightarrow \mathcal{E}[2r](r)$$

and then by adjunction

$$\theta_u : u^+(\mathcal{E}) \longrightarrow u^!(\mathcal{E})[2r](r)$$

from

$$\phi_u : u_! u^+(K_X) \xrightarrow{\sim} K_X[2r](r).$$

1.8. Let $u: Z \hookrightarrow X$ be a closed immersion of pure codimension r in X of smooth realizable k-varieties. Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. From Proposition 1.6, we have the morphism $\theta_u : u^+\mathcal{E} \to u^!\mathcal{E}[2r](r)$. The composition of the following three morphisms:

$$\eta_{u,\mathcal{E}} : \mathcal{E} \xrightarrow{\mathrm{adj}} u_+ u^+ \mathcal{E} \xrightarrow{\sim} u_! u^+ \mathcal{E} \xrightarrow{u_!(\theta_u)} u_! u^! \mathcal{E}[2r](r) \xrightarrow{\mathrm{adj}} \mathcal{E}[2r](r) \tag{1.8.1}$$

is an element of $\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(\mathcal{E}, \mathcal{E}[2r])$ (resp. $\mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(\mathcal{E}, \mathcal{E}[2r](r))$). By using the commutativity of (1.7.1), we check that

$$\eta_{u,\mathcal{E}} = \eta_{u,K_X} \otimes \mathrm{Id}_{\mathcal{E}}\,. \tag{1.8.2}$$

1.9 Remark. We keep the notation of 1.8. Following the notation of [1, 3.1.1 and 3.1.6], we put

$$H^{2r}_Z(X)(r) := \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(K_X, u_+ u^! K_X[2r](r))$$

and

$$H^{2r}(X)(r) := \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(K_X, K_X[2r](r)).$$

From [1, 3.1.6], the composition

$$u_+(\theta_u) \circ \mathrm{adj}: K_X \longrightarrow u_+ u^! K_X[2r](r)$$

is called the cycle class of Y and is denoted by $\mathrm{cl}_X(Z) \in H^{2r}_Z(X)(r)$. Since $u_!$ is a left adjoint functor of $u^!$ and since $u_+ \xrightarrow{\sim} u_!$, we get a canonical homomorphism $H^{2r}_Z(X)(r) \to H^{2r}(X)(r)$ which sends $\mathrm{cl}_X(Z)$ to η_{u,K_X}, see (1.8.1).

$$\begin{tikzcd}
u_!u^!(K_X \otimes \mathcal{E})[2r](r) \arrow[d, equal] & & u_!(u^!(K_X) \otimes u^+(\mathcal{E}))[2r](r) \arrow[ll, "\epsilon_u"'] \arrow[d, "\mathrm{adj}"] \arrow[dd, bend left=60, equal] \\
u_!u^!(K_X \otimes \mathcal{E})[2r](r) \arrow[d, "\mathrm{adj}"] & u_!u^!(u_!u^!(K_X) \otimes \mathcal{E})[2r](r) \arrow[l, "\mathrm{adj}"'] \arrow[d, "\mathrm{adj}"] & u_!u^!u_!(u^!(K_X) \otimes u^+(\mathcal{E}))[2r](r) \arrow[l, "\mathrm{proj}", "\sim"'] \arrow[d, "\mathrm{adj}"] \\
K_X \otimes \mathcal{E}[2r](r) & u_!u^!(K_X) \otimes u^+(\mathcal{E}))[2r](r) \arrow[l, "\mathrm{adj}"'] & u_!(u^!(K_X) \otimes u^+(\mathcal{E}))[2r](r) \arrow[l, "\mathrm{proj}", "\sim"']
\end{tikzcd}$$

Diagram 4

In order to check Theorem 1.15 below we will need the following lemmas:

1.10 Lemma. *Let*

$$\begin{tikzcd} Z \arrow[r, hook, "u"] & X \\ Z' \arrow[u, "g"] \arrow[r, hook, "u'"] & X' \arrow[u, "f"'] \end{tikzcd}$$

be a cartesian square so that u and u' are closed immersions of pure codimension r of smooth realizable k-varieties. Let $\mathcal{E}_X \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ and $\mathcal{E}_{X'} := f^+(\mathcal{E}_X)$. Let $\eta_{u,\mathcal{E}_X} \colon \mathcal{E}_X \to \mathcal{E}_X[2r](r)$ and $\eta_{u',\mathcal{E}_{X'}} \colon \mathcal{E}_{X'} \to \mathcal{E}_{X'}[2r](r)$ be the morphisms as defined in (1.8.1). Then we get $f^+(\eta_{u,\mathcal{E}_X}) = \eta_{u',\mathcal{E}_{X'}}$.

Proof. Thanks to equation (1.8.2), we can suppose $\mathcal{E}_X = K_X$. This comes from [1, 3.2.6] (see also Remark 1.9). □

1.11 Lemma. *Let $\pi \colon \mathbb{P}^d \to \operatorname{Spec} k$ be the canonical projection. Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d/K)$. Let H be the zero set of a section of the fundamental line bundle $\mathcal{O}_{\mathbb{P}^d}(1)$ and $u \colon H \hookrightarrow \mathbb{P}^d$ be the closed immersion. The morphism $\eta_{u,\mathcal{E}} \colon \mathcal{E} \to \mathcal{E}[2](1)$ as defined in (1.8.1) does not depend on the choice of the hyperplane H and will be denoted by $\eta_{\pi,\mathcal{E}}$.*

Proof. We can suppose $\mathcal{E} = K_X$. Let H_1, H_2 be respectively the zero set of two sections of $\mathcal{O}_{\mathbb{P}^d}(1)$. From (1.8.1), for $i = 1, 2$, the closed immersions $u_i \colon H_i \hookrightarrow \mathbb{P}^d$ induce the morphisms $\eta_i \colon K_X \to K_X[2](1)$. For $i = 1, 2$, we put

$$\psi_i \colon K \xrightarrow{\mathrm{adj}} \pi_+\pi^+K = \pi_+K_X \xrightarrow{\pi_+(\eta_i)} \pi_+K_X[2](1).$$

By adjunction, $\eta_1 = \eta_2$ if and only if $\psi_1 = \psi_2$. There exists an isomorphism $\sigma \colon \mathbb{P}^d \xrightarrow{\sim} \mathbb{P}^d$ so that $\sigma^{-1}(H_1) = H_2$. From Lemma 1.10, we get $\sigma^+(\eta_1) = \eta_2$ and then $\sigma_+(\eta_2) = \eta_1$. Since $\pi \circ \sigma = \pi$, this implies that $\psi_2 = \psi_1$. □

1.12. Let S be a realizable variety, $\pi \colon \mathbb{P}^d \to \operatorname{Spec} k$ and $\pi_S \colon \mathbb{P}^d_S \to S$, $f \colon \mathbb{P}^d_S \to \mathbb{P}^d$ be the canonical projections. Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d_S/K)$. With the notation of Lemma 1.11, we put

$$\eta_{\pi_S,\mathcal{E}} := f^+(\eta_{\pi,K_{\mathbb{P}^d}}) \otimes \mathrm{Id}_{\mathcal{E}} \colon \mathcal{E} \longrightarrow \mathcal{E}[2](1). \tag{1.12.1}$$

Let $S' \to S$ be a morphism of realizable varieties and $a \colon \mathbb{P}^d_{S'} \to \mathbb{P}^d_S$ be the induced morphism. Then, we remark that

$$a^+(\eta_{\pi_S,\mathcal{E}}) = \eta_{\pi_{S'},a^+(\mathcal{E})}. \tag{1.12.2}$$

1.13 Lemma. *Let $\pi\colon \mathbb{P}^d_S \to S$ be the canonical projection and $\iota\colon \mathbb{P}^{d'}_S \hookrightarrow \mathbb{P}^d_S$ be a closed S-immersion such that $\pi' := \pi \circ \iota$ is the canonical projection. Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d_S/K)$. We have the equality*

$$\iota^+(\eta_{\pi,\mathcal{E}}) = \eta_{\pi',\iota^+(\mathcal{E})}. \tag{1.13.1}$$

Proof. By construction, see (1.12.1), we can suppose $\mathcal{E} = K_{\mathbb{P}^d_S}$. By using (1.12.2), we reduce to treat the case $S = \operatorname{Spec} k$. Then, this comes from Lemma 1.10. □

1.14 Lemma. *Let S be a realizable variety, $q\colon X = \mathbb{A}^d_S \to S$ be the canonical projection. Let $\mathcal{E} \in \mathrm{Hol}(S/K)$.*

(1) *For any $i \neq 0$, we have $\mathcal{H}^i_t q_+ q^+(\mathcal{E}) = 0$ and $\mathcal{H}^{2d-i}_t q_! q^+(\mathcal{E}) = 0$.*

(2) *We have $\mathcal{H}^0_t q_+ q^+(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ and $\mathcal{H}^{2d}_t q_! q^+(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ in $\mathrm{Hol}(S/K)$.*

Proof. From (1.2.1), we can only consider the pushforward case. By transitivity of the pushforward, we reduce to the case where $d = 1$. The complex $q_+q^+(\mathcal{E})$ is isomorphic to the relative de Rham cohomology of $\mathbb{A}^1_S/S$ of $q^+[d](\mathcal{E}) \in \mathrm{Hol}(\mathbb{A}^1_S/K)$. Then, this is an easy computation. □

1.15 Theorem. *Let $\pi\colon \mathbb{P}^n_S \to S$ be the canonical projection, $\iota\colon X \hookrightarrow \mathbb{P}^n_S$ be a closed immersion such that, for any closed point s of S, $f^{-1}(s) \xrightarrow{\sim} \mathbb{P}^d_{k(s)}$ where $k(s)$ is the residue field of s and $f := \pi \circ \iota$ (we might call such a morphism f a $\mathbb{P}^d$-fibration morphism). Let $\mathcal{E} \in (F\text{-})D^{\mathrm{b}}_{\mathrm{hol}}(S/K)$. With the notation of (1.12.1), we put*

$$\eta = \iota^+\eta_{\pi,\pi^+(\mathcal{E})[-2](-1)}\colon f^+(\mathcal{E})[-2](-1) \longrightarrow f^+(\mathcal{E}).$$

By composition, for any integer $i \geq 0$, we get

$$\eta^i\colon f^+(\mathcal{E})[-2i](-i) \longrightarrow f^+(\mathcal{E}).$$

By adjunction, this is equivalent to have a morphism of the form

$$\mathcal{E}[-2i](-i) \longrightarrow f_+ \circ f^+(\mathcal{E}),$$

which by abuse of notation will still be denoted by η^i. The following map

$$\textstyle\bigoplus_{i=0}^d \eta^i\colon \bigoplus_{i=0}^d \mathcal{E}[-2i](-i) \longrightarrow f_+ \circ f^+(\mathcal{E}) \tag{1.15.1}$$

is an isomorphism.

Proof. Diagram 5, where the vertical arrows of the top are the projection isomorphisms (recall that since f is proper, we have $f_+ \xrightarrow{\sim} f_!$), is commutative (indeed, the commutativity of the square below comes from the definition (1.12.1), that of the other square is functorial and that of the rectangle is left to the reader). By using the commutativity of Diagram 5, we can suppose $\mathcal{E} = K_S$.

The fact that the morphism (1.15.1) is actually an isomorphism can be checked after pulling back by the closed immersions induced by the closed points of S. Hence, by using (1.12.2), we can suppose that $S = \operatorname{Spec} k$ and $X = \mathbb{P}^d$. From Lemma 1.13, we can suppose that $d = n$, i.e. ι is the identity and f is the canonical projection $\mathbb{P}^d \to \operatorname{Spec} k$.

We proceed by induction on $d \geq 0$. The case $d = 0$ is obvious. So, we can suppose $d \geq 1$. Let $q\colon \mathbb{A}^d \to \operatorname{Spec} k$ the projection, $H := X \setminus \mathbb{A}^d$ be the hyperplane at the infinity, $u\colon H \hookrightarrow X$ the induced closed immersion, $g := f \circ u$. We put $\tilde{\eta}^i := u^+(\eta^i)\colon g^+(K)[-2i](-i) \to g^+(K)$. Again by abuse of notation, let $\tilde{\eta}^i\colon K[-2i](-i) \to g_+g^+(K)$ be the morphism induced by adjunction. From the transitivity of the adjunction morphism, we get the commutativity of the left square:

$$\begin{tikzcd}
\eta^i\colon & K[-2i](-i) \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & f_+f^+(K)[-2i](-i) \arrow[r, "f_+(\eta^i)"] \arrow[d, "\mathrm{adj}"] & f_+f^+(K) \arrow[d, "\mathrm{adj}"] \\
\tilde{\eta}^i\colon & K[-2i](-i) \arrow[r, "\mathrm{adj}"] & f_+u_+u^+f^+(K)[-2i](-i) \arrow[r, "g_+(\tilde{\eta}^i)"] & f_+u_+u^+f^+(K).
\end{tikzcd}$$

This induces the following commutative square

$$\begin{tikzcd}
\bigoplus_{i=0}^{d-1}\eta^i\colon & \bigoplus_{i=0}^{d-1} K[-2i](-i) \arrow[r] \arrow[d, equal] & f_+f^+(K) \arrow[d, "\mathrm{adj}"] \\
\bigoplus_{i=0}^{d-1}\tilde{\eta}^i\colon & \bigoplus_{i=0}^{d-1} K[-2i](-i) \arrow[r, "\sim"] & g_+g^+(K).
\end{tikzcd} \tag{1.15.2}$$

a) From (1.13.1), we get the equality $\tilde{\eta} = \eta_{g,g^+(K)}$. By using the induction hypothesis applied to g, the arrow of the bottom of the diagram (1.15.2) is an isomorphism. We denote by $\tau_{\leq 2d-1}$ the truncation functor of the canonical t-structure of [3]. Since we have the exact triangle of localization

$$q_!q^+(K) \longrightarrow f_+f^+(K) \longrightarrow g_+g^+(K) \longrightarrow +1$$

and Lemma 1.14, then after having applied the functor $\tau_{\leq 2d-1}$ to the right morphism of (1.15.2) we get an isomorphism (for the degree $2d-1$, we use that $\mathcal{H}_t^{2d-1} g_+g^+(K) = 0$). By considering (1.15.2), this implies that the truncation $\tau_{\leq 2d-1}(\oplus_{i=0}^{d}\eta^i)$ of (1.15.1) is an isomorphism.

$$
\begin{tikzcd}
K_S \otimes \mathcal{E} \arrow[r, "\mathrm{adj} \otimes \mathrm{Id}_{\mathcal{E}}"] \arrow[dd, "\sim"] & f_+ f^+(K_S) \otimes \mathcal{E} \arrow[r, equal] & f_+(K_X) \otimes \mathcal{E} \arrow[r, "f_+(\eta) \otimes \mathrm{Id}_{\mathcal{E}}"] \arrow[d, "\mathrm{proj}"', "\sim"] & f_+(K_X[2](1)) \otimes \mathcal{E} \arrow[d, "\mathrm{proj}"', "\sim"] \\
 & & f_+(K_X \otimes f^+\mathcal{E}) \arrow[r, "f_+(\eta \otimes \mathrm{Id}_{f^+(\mathcal{E})})"] \arrow[d, "\sim"] & f_+(K_X[2](1) \otimes f^+(\mathcal{E})) \arrow[d, "\sim"] \\
\mathcal{E} \arrow[rr, "\mathrm{adj}"] & & f_+ f^+(\mathcal{E}) \arrow[r, "f_+(\eta_{f^+(\mathcal{E})})"] & f_+ f^+(\mathcal{E}[2](1))
\end{tikzcd}
$$

Diagram 5

b) Now, consider Diagram 6, where the right arrow of the bottom is an isomorphism because of Proposition 1.6.(2); the commutativity of the right square follows from the definition (1.8.1) and the notation (1.12.1). Hence, by using the induction hypothesis, we check that after having applied the functor $\tau_{\geq 2d}$ to Diagram 6, the composition of the arrows of the bottom becomes an isomorphism. A cone of the right morphism of Diagram 6 is isomorphic to $q_+ q^+(K)$. From (1.14), we get $\tau_{\geq 2d-1} q_+ q^+(K) = 0$. Hence, by applying $\tau_{\geq 2d}$ to the right morphism of Diagram 6, we get an isomorphism. This implies that $\tau_{\geq 2d}(\eta^d)$ is an isomorphism. Hence, so is $\tau_{\geq 2d}(\bigoplus_{i=0}^{d} \eta^i)$. Using the step a) of the proof, we can conclude. □

1.16 Corollary. *We keep the geometrical notation of (1.15) and we suppose f smooth. Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\leq 0}$, $\mathcal{F} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\geq 0}$, $\mathcal{G} \in D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\leq 0}$, and $\mathcal{H} \in D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\geq 0}$. Then*

$$\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(f^+(\mathcal{G}), f^+(\mathcal{H})) = \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{G}, \mathcal{H});$$

and

$$\mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(f^+(\mathcal{E}), f^+(\mathcal{F})) = \mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{E}, \mathcal{F}). \tag{1.16.1}$$

Proof. Since $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\leq 0}$ and $\mathcal{F} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\geq 0}$, then

$$\mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{E}, \mathcal{F}) = \mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{H}^0_t\mathcal{E}, \mathcal{H}^0_t\mathcal{F}). \tag{1.16.2}$$

Since f is smooth, then the functor $f_+ f^+$ preserves $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)^{\geq 0}$. Hence, by adjunction, we get

$$\begin{aligned}\mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(X/K)}(f^+(\mathcal{E}), f^+(\mathcal{F})) &= \mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{E}, f_+ f^+(\mathcal{F}))\\ &= \mathrm{Hom}_{F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(S/K)}(\mathcal{H}^0_t\mathcal{E}, \mathcal{H}^0_t f_+ f^+(\mathcal{F})).\end{aligned}$$

With

$$\mathcal{H}^0_t\mathcal{F} \xrightarrow[\text{Theorem 1.15}]{\sim} \mathcal{H}^0_t f_+ \circ f^+(\mathcal{H}^0_t\mathcal{F}) \xrightarrow{\sim} \mathcal{H}^0_t f_+ \circ f^+(\mathcal{F}),$$

then we obtain the last equality of (1.16.1). The proof without Frobenius is identical. □

1.17 Proposition. *We keep the notation and hypotheses of Corollary 1.16.*

(1) *The functor $f^+[d]\colon (F\text{-})\,\mathrm{Hol}(S/K) \to (F\text{-})\,\mathrm{Hol}(X/K)$ is t-exact and fully faithful.*

(2) *For any $\mathcal{E}, \mathcal{F} \in \mathrm{Hol}(S/K)$, the functor $f^+[d]$ induces the equality*

$$\mathrm{Ext}^1_{\mathrm{Hol}(S/K)}(\mathcal{E}, \mathcal{F}) = \mathrm{Ext}^1_{\mathrm{Hol}(X/K)}(f^+[d](\mathcal{E}), f^+[d](\mathcal{F})). \tag{1.17.1}$$

$$\begin{tikzcd}
K[-2d](-d) \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & f_+f^+(K)[-2d](-d) \arrow[r, "f_+(\eta^{d-1})"] \arrow[d, "\mathrm{adj}"] & f_+f^+(K)[-2](-1) \arrow[r, "f_+(\eta)"] \arrow[d, "\mathrm{adj}"] & f_+f^+(K) \\
K[-2d](-d) \arrow[r, "\mathrm{adj}"] \arrow[rr, bend right, "\tilde{\eta}^{d-1}"'] & f_+u_+u^+f^+(K)[-2d](-d) \arrow[r, "g_+(\tilde{\eta}^{d-1})"] & f_+u_+u^+f^+(K)[-2](-1) \arrow[r, "\sim", "g_+(\theta_u)"'] & f_+u_+u^!f^+(K) \arrow[u, "\mathrm{adj}"']
\end{tikzcd}$$

Diagram 6

Proof. Since f is smooth, the functor $f^+[d]$ is t-exact. From Corollary 1.16, we get its full faithfulness. Since the canonical morphism $\mathcal{F}[1] \to \tau_{\leq 0} f_+ f^+(\mathcal{F}[1])$ is an isomorphism (use (1.15.1)), we get the second assertion (by using similar techniques than in the proof of Corollary 1.16). $\square$

1.18 Remark. We keep the notation and hypotheses of Corollary 1.16. Let $\mathcal{E} \in \mathrm{Hol}(S/K)$. Since the pull back under Frobenius commutes with the functor $f^+[d]$, then we get a bijection between Frobenius structures on $\mathcal{E}$ and Frobenius structures on $f^+[d](\mathcal{E})$. Moreover, let $\mathcal{F}, \mathcal{G} \in F\text{-}\mathrm{Hol}(S/K)$ and $\phi: \mathcal{F} \to \mathcal{G}$ a morphism of $\mathrm{Hol}(S/K)$. Then ϕ commutes with Frobenius if and only if so is $f^+[d](\phi)$.

1.19 Proposition. *We keep the notation and hypotheses of Corollary 1.16. We suppose furthermore that the morphism f has locally a section.*

(1) *The functor $f^+[d]: (F\text{-})\,\mathrm{Hol}(S/K) \to (F\text{-})\,\mathrm{Hol}(X/K)$ sends simple objects to simple objects.*

(2) *The functor $f^+[d]: (F\text{-})\,\mathrm{Hol}(S/K) \to (F\text{-})\,\mathrm{Hol}(X/K)$ has the right adjoint functor $\mathcal{H}_t^{-d} \circ f_+: (F\text{-})\,\mathrm{Hol}(X/K) \to (F\text{-})\,\mathrm{Hol}(S/K)$ and the left adjoint functor $\mathcal{H}_t^{d} \circ f_!(d): (F\text{-})\,\mathrm{Hol}(X/K) \to (F\text{-})\,\mathrm{Hol}(S/K)$.*

Proof. Let $\mathcal{E}$ be a simple object of $(F\text{-})\,\mathrm{Hol}(S/K)$. From [3, 1.4.9.(i)] (without Frobenius structures, use the fact that $\mathrm{Isoc}^{\dagger\dagger}(S/K) \cap \mathrm{Hol}(S/K)$ is a Serre subcategory of $\mathrm{Hol}(S/K)$, where $\mathrm{Isoc}^{\dagger\dagger}(S/K)$ is constructed in [9]) there exist an open dense smooth subscheme S' of S, an irreducible object $\mathcal{E}' \in (F\text{-})\,\mathrm{Hol}(S'/K)$ which is also an object of $(F\text{-})\,\mathrm{Isoc}^{\dagger\dagger}(S'/K)$ such that $\mathcal{E} \xrightarrow{\sim} u_{!+}(\mathcal{E}')$ where $u: S' \hookrightarrow S$ is the inclusion. Put $X' := f^{-1}(S')$, $f': X' \to S'$, $v: X' \hookrightarrow X$. By adjunction, we remark that the canonical morphism $u_!(\mathcal{E}') \to u_+(\mathcal{E}')$ is the only one so that we get the identity over S'. With this remark, since

$$f^+[d] \circ u_!(\mathcal{E}') \xrightarrow{\sim} v_! \circ f'^+[d](\mathcal{E}'),$$

and

$$f^+[d] \circ u_+(\mathcal{E}') \xrightarrow{\sim} f^![-d](-d) \circ u_+(\mathcal{E}') \xrightarrow{\sim} v_+ \circ f'^![-d](-d)(\mathcal{E}') \\ \xrightarrow{\sim} v_+ \circ f'^+[d](\mathcal{E}'),$$

since the functors $f^+[d]$ and $f'^+[d]$ are t-exact, then after applying $\mathcal{H}_t^0$ to these isomorphisms we get the isomorphism

$$f^+[d] \circ u_{!+}(\mathcal{E}') \xrightarrow{\sim} v_{!+} \circ f'^+[d](\mathcal{E}').$$

Since the functor $v_{!+}$ preserves the irreducibility, then we reduce to the case where S is affine, smooth, irreducible, and where moreover $\mathcal{E} \in (F\text{-})\operatorname{Isoc}^{\dagger\dagger}(S/K)$. Hence

$$f^+[d](\mathcal{E}) \in (F\text{-})\operatorname{Isoc}^{\dagger\dagger}(X/K) \cap (F\text{-})\operatorname{Hol}(X/K).$$

Let $0 \neq \mathcal{G} \in (F\text{-})\operatorname{Hol}(X/K)$ be a subobject of $f^+[d](\mathcal{E})$. Since the category $(F\text{-})\operatorname{Isoc}^{\dagger\dagger}(X/K) \cap (F\text{-})\operatorname{Hol}(X/K)$ is a Serre subcategory of $(F\text{-})\operatorname{Hol}(X/K)$, then $\mathcal{G} \in (F\text{-})\operatorname{Isoc}^{\dagger\dagger}(X/K)$. Since the generic rank of an overconvergent isocrystal is preserved under pull-backs, since f has locally a section, since X is irreducible (because the fibers of f are irreducible) then we can conclude that $\mathcal{G} = f^+[d](\mathcal{E})$ and hence $f^+[d](\mathcal{E})$ is a simple object.

The last part comes from the left t-exactness of $f_+[-d]$ and the right t-exactness of $f_![d](d)$, from the fact that the couples $(f^+[d], f_+[-d])$ and $(f_![d](d), f^![-d](-d))$ are adjoint functors, and from the isomorphism

$$f^+[d] \xrightarrow{\sim} f^![-d](-d)$$

of (1.2.1). □

1.20 Proposition. *We keep the notation and hypotheses of Proposition 1.19. Let $\mathcal{E} \in (F\text{-})\operatorname{Hol}(X/K)$.*

(1) *The category of constant objects with respect to f is a thick subcategory of $(F\text{-})\operatorname{Hol}(X/K)$.*

(2) *The object*

$$\mathcal{H}_t^0 \circ f^+ \circ f_+(\mathcal{E}) = f^+[d] \circ (\mathcal{H}_t^{-d} f_+)(\mathcal{E})$$

is the largest constant with respect to f subobject of $\mathcal{E}$ in the category $(F\text{-})\operatorname{Hol}(X/K)$.

(3) *The object*

$$\mathcal{H}_t^{2d} \circ f^+ \circ f_!(\mathcal{E}(d)) = f^+[d] \circ (\mathcal{H}_t^{d} f_!)(\mathcal{E}(d))$$

is the largest constant with respect to f quotient object of $\mathcal{E}$ in the category $(F\text{-})\operatorname{Hol}(X/K)$.

Proof. The thickness of the category of constant objects without Frobenius structures comes from the equality (1.17.1). With Remark 1.18, we get the thickness with Frobenius structures. The rest is similar to the proof of [11, III.11.3] i.e. this comes from the general fact [11, III.11.1] and from Proposition 1.19. □

2. The Brylinski–Radon transform and the Hard Lefschetz Theorem

Let $d \geq 1$ be an integer and $\mathbb{P}^d$ be the d-dimensional projective space defined over k, let $\check{\mathbb{P}}^d$ be the dual projective space over k, which parameterizes the hyperplanes in $\mathbb{P}^d$, let H be the universal incidence relation, i.e. the closed subvariety of $\mathbb{P}^d \times \check{\mathbb{P}}^d$ so that $(x, h) \in H$ if and only if the point $x \in h$. Let Y be a realizable k-variety. We denote by

$$i \colon H \times Y \hookrightarrow \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y$$

the canonical immersion and by

$$p_1 \colon \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y \longrightarrow \mathbb{P}^d \times Y, \quad p_2 \colon \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y \longrightarrow \check{\mathbb{P}}^d \times Y,$$
$$\tilde{p}_1 \colon \check{\mathbb{P}}^d \times Y \longrightarrow Y, \qquad \tilde{p}_2 \colon \mathbb{P}^d \times Y \longrightarrow Y$$

the canonical projections, and we set $\pi_1 := p_1 \circ i$, $\pi_2 := p_2 \circ i$.

2.1 Definition. We define the Brylinski–Radon transform

$$\mathrm{Rad} \colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K) \longrightarrow F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\check{\mathbb{P}}^d \times Y/K)$$

by posing, for any $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$,

$$\mathrm{Rad}(\mathcal{E}) := \pi_{2+}\pi_1^{+}(\mathcal{E})[d-1]. \tag{2.1.1}$$

For any $n \in \mathbb{Z}$, we put $\mathrm{Rad}^n := \mathcal{H}^n_t \circ \mathrm{Rad}$.

2.2 Definition. Let U be the open complement of the closed subvariety $H \times Y$ in $\mathbb{P}^d \times \check{\mathbb{P}}^d \times Y$. Let $j \colon U \hookrightarrow \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y$ be the open immersion and $q_1 := p_1 \circ j$, $q_2 := p_2 \circ j$. We define the modified Radon transform

$$\mathrm{Rad}_! \colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K) \longrightarrow F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\check{\mathbb{P}}^d \times Y/K)$$

by posing, for any $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$,

$$\mathrm{Rad}_!(\mathcal{E}) := q_{2!} \circ q_1^{+}[d](\mathcal{E}). \tag{2.2.1}$$

For any integer $n \in \mathbb{Z}$, we put $\mathrm{Rad}^n_!(\mathcal{E}) := \mathcal{H}^n_t\, \mathrm{Rad}_!(\mathcal{E})$.

2.3. (1) Since the functor $q_1^+[d]$ is t-exact and the functor $q_{2!}$ is left t-exact (because q_2 is affine, e.g. see [3, 1.3.13] but this is obvious here since q_2 is moreover smooth) then $\mathrm{Rad}_!$ is left t-exact.

(2) The exact triangle

$$i_+ \circ i^+[-1] \longrightarrow j_! j^+ \longrightarrow \mathrm{Id} \longrightarrow i_+ \circ i^+$$

induces for any $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$ the exact triangle

$$\mathrm{Rad}(\mathcal{E}) \longrightarrow \mathrm{Rad}_!(\mathcal{E}) \longrightarrow p_{2+} p_1^+[d](\mathcal{E}) \longrightarrow \mathrm{Rad}(\mathcal{E})[1]. \tag{2.3.1}$$

2.4 Lemma. *Let* $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$.

(1) *We have the isomorphism*

$$\mathcal{E}[-2d](-d) \xrightarrow{\sim} q_{1!} q_1^+(\mathcal{E}).$$

(2) *We have the isomorphism*

$$\tilde{p}_{1+}(\mathrm{Rad}_!(\mathcal{E})) \xrightarrow{\sim} \tilde{p}_{2+}(\mathcal{E})[-d](-d).$$

(3) *If* $\mathcal{E} \in (F\text{-})D^{\geq 0}(\mathbb{P}^d \times Y/K)$*, then*

$$\tilde{p}_{1+}(\mathrm{Rad}_!(\mathcal{E})) \in (F\text{-})D^{\geq 0}(Y/K).$$

Proof. We put $\eta = \eta_{p_1, p_1^+(\mathcal{E})[-2](-1)}$ and $\tilde{\eta}^i := i^+(\eta^i)$. In order to check the first isomorphism, by 1.5 and the projection formula (see [3, A.6]) we get

$$q_{1!} q_1^+(\mathcal{E}) \xrightarrow{\sim} q_{1!}(q_1^+(K_U) \otimes q_1^+(\mathcal{E})) \xrightarrow{\sim} q_{1!} q_1^+(K_U) \otimes \mathcal{E}.$$

Hence, we can suppose $\mathcal{E} = K_{\mathbb{P}^d \times Y}$. Then, by using a base change theorem, we can suppose $Y = \operatorname{Spec} k$. We put $\mathcal{G} = K_{\mathbb{P}^d}[-d] \in F\text{-}\mathrm{Hol}(\mathbb{P}^d/K)$. Consider the diagram below

$$\begin{tikzcd}
q_{1!} q_1^+(\mathcal{G}) \arrow[r] & p_{1+} p_1^+(\mathcal{G}) \arrow[r, "\mathrm{adj}"] & p_{1+} i_+ i^+ p_1^+(\mathcal{G}) \arrow[r, "+"] & {} \\
\mathcal{G}[-2d](-d) \arrow[r] & \bigoplus_{i=0}^{d} \mathcal{G}[-2i](-i) \arrow[r] \arrow[u, "\sim"', "\oplus_{i=0}^{d} \eta^i"] & \bigoplus_{i=0}^{d-1} \mathcal{G}[-2i](-i) \arrow[r, "+"] \arrow[u, "\sim"', "\oplus_{i=0}^{d-1} \tilde{\eta}^i"] & {}
\end{tikzcd} \tag{2.4.1}$$
,

where the rows are exact triangles and where we keep the abuse of notation in Theorem 1.15, i.e. the morphism η^i (resp. $\tilde{\eta}^i$) means the morphism induced by adjunction with respect to the couple (p_1^+, p_{1+}) (resp. (q_1^+, q_{1+})) from η^i (resp. $\tilde{\eta}^i$). By transitivity of the adjunction, we get that the square of (2.4.1) is commutative. Moreover, we recall that the vertical arrows are isomorphisms thanks to (1.15.1). By applying the functor $\tau_{\geq 2d}$ to the diagram (2.4.1), we get

$$\begin{aligned} q_{1!}q_1^+(\mathcal{G}) &\xrightarrow[\text{Lemma 1.14}]{\sim} \tau_{\geq 2d} q_{1!}q_1^+(\mathcal{G}) \\ &\xrightarrow{\sim} \tau_{\geq 2d}\, p_{1+}p_1^+(\mathcal{G}) \xleftarrow{\sim} \tau_{\geq 2d} \bigoplus_{i=0}^{d} \mathcal{G}[-2i](-i) \xleftarrow{\sim} \mathcal{G}[-2d](-d), \end{aligned}$$

which finishes the proof of the first isomorphism. We get the second isomorphism from the first one by composition:

$$\begin{aligned} \tilde{p}_{1+}(\mathrm{Rad}_!(\mathcal{E})) &\xrightarrow{\sim} \tilde{p}_{1!} \circ q_{2!} \circ q_1^+[d](\mathcal{E}) \\ &\xrightarrow{\sim} \tilde{p}_{2!} \circ q_{1!} \circ q_1^+[d](\mathcal{E}) \xrightarrow{\sim} \tilde{p}_{2!}(\mathcal{E})[-d](-d). \end{aligned}$$

Finally, since $\tilde{p}_{2!}[-d]$ is left t-exact, we obtain the third property from the second one. □

2.5 Lemma. *Let $\mathcal{E} \in (F\text{-})D^{\geq 0}(\mathbb{P}^d \times Y/K)$. Then $\mathrm{Rad}_!^0(\mathcal{E})$ is left reduced with respect to $\tilde{p}_1$, i.e. does not have any nontrivial constant with respect to $\tilde{p}_1$ subobject.*

Proof. The proof is the same than [11, IV.2.7]: from Proposition 1.20 we reduce to prove that $\mathcal{H}_t^{-d}\, \tilde{p}_{1+}(\mathrm{Rad}_!^0(\mathcal{E})) = 0$. Since $\mathrm{Rad}_!(\mathcal{E}) \in (F\text{-})D^{\geq 0}(\check{\mathbb{P}}^d \times Y/K)$ (see the property 2.3.(1)), since $\mathcal{H}_t^{-d}\, \tilde{p}_{1+}$ is left t-exact, then we get the isomorphism

$$\mathcal{H}_t^{-d}\, \tilde{p}_{1+}(\mathrm{Rad}_!^0(\mathcal{E})) \xrightarrow{\sim} \mathcal{H}_t^{-d}\, \tilde{p}_{1+}(\mathrm{Rad}_!(\mathcal{E})).$$

We conclude by using 2.4.(3) □

2.6. Let $f: X \to Y$ be a projective morphism of realizable varieties. Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. A morphism $\eta: \mathcal{E} \to \mathcal{E}[2](1)$ is called a "Chern class of a relative hyperplane for the projective morphism f" if there exists a closed immersion $\iota: X \hookrightarrow \mathbb{P}^d_Y$ so that $f = \pi \circ \iota$, where $\pi: \mathbb{P}^d_Y \to Y$ is the canonical projection, and so that, with the notation 1.12,

$$\eta = \mathrm{Id}_{\mathcal{E}} \otimes \iota^+(\eta_{\pi, K_{\mathbb{P}^d_Y}}).$$

By using the projection isomorphisms (see [3, A.6]), we remark that the map $\iota_+(\eta): \iota_+(\mathcal{E}) \to \iota_+(\mathcal{E})[2](1)$ is canonically isomorphic to $\mathrm{Id}_{\iota_+(\mathcal{E})} \otimes \eta_{\pi, K_{\mathbb{P}^d_Y}}$.

2.7 Theorem (hard Lefschetz Theorem). *We suppose here that k is a finite field with p^s elements. Let $f\colon X \to Y$ be a projective morphism of realizable varieties. Let $\mathcal{E} \in F\text{-}\mathrm{Hol}(X/K)$ be an ι-pure module and $\eta\colon \mathcal{E} \to \mathcal{E}[2](1)$ be a Chern class of a relative hyperplane for f (see the definition in 2.6). For any positive integer r, we obtain by composition $\eta^r\colon \mathcal{E} \to \mathcal{E}[2r](r)$. We get the homomorphism*

$$\mathcal{H}_t^{-r} f_+(\eta^r)\colon \mathcal{H}_t^{-r} f_+(\mathcal{E}) \longrightarrow \mathcal{H}_t^{r} f_+(\mathcal{E})(r). \tag{2.7.1}$$

The homomorphism (2.7.1) is an isomorphism.

Proof. We follow the proof of the hard Lefschetz Theorem of [11, IV.4.1] which is similar to that of [5].

0. Since the assertion is local on Y, one can suppose Y affine and smooth. Using 2.6, we reduce to the case where f is the projection $\mathbb{P}^d_Y \to Y$ and $\eta = \mathrm{Id}_{\mathcal{E}} \otimes \eta_{f,K_{\mathbb{P}^d_Y}}$. Then we keep the notation of Section 2 of our paper.

1. In this step, we treat the case $r = 1$. We put

$$\mathcal{G} = p_1^+[d](\mathcal{E}) \in F\text{-}\mathrm{Hol}(\mathbb{P}^d \times \check{\mathbb{P}}^d \times Y/K).$$

Following (1.8.1), we get from the closed immersion $i\colon H \times Y \hookrightarrow \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y$ the morphism

$$\zeta := \eta_{i,\mathcal{G}}\colon \mathcal{G} \to \mathcal{G}[2](1).$$

a) Let $\operatorname{Spec} k \hookrightarrow \check{\mathbb{P}}^d$ be a rational point,

$$t\colon Y \hookrightarrow \check{\mathbb{P}}^d \times Y \quad \text{and} \quad s\colon \mathbb{P}^d \times Y \hookrightarrow \mathbb{P}^d \times \check{\mathbb{P}}^d \times Y$$

the induced closed immersions. Since $s^{-1}(H \times Y)$ is an hyperplane of $\mathbb{P}^d \times Y$, since s is a section of p_1, using Lemma 1.10 we get $s^+[-d](\zeta) = \eta$. Since

$$p_{2+} p_1^+ \xrightarrow{\sim} \tilde{p}_1^+ \tilde{p}_{2+} \quad \text{and} \quad \tilde{p}_{2+} s^+ \xrightarrow{\sim} t^+ p_{2+},$$

since the functor $s^+[-d]$ (resp. $t^+[-d]$) is acyclique for the constant objects with respect to p_1 (resp. $\tilde{p}_1$), we get that

$$t^+[-d]\mathcal{H}_t^{-1} p_{2+}(\zeta) \xrightarrow{\sim} \mathcal{H}_t^{-1} \tilde{p}_{2+} s^+[-d](\zeta) = \mathcal{H}_t^{-1} \tilde{p}_{2+}(\eta).$$

Hence, this is enough to check that $\mathcal{H}_t^{-1} p_{2+}(\zeta)\colon \mathcal{H}_t^{-1} p_{2+}(\mathcal{G}) \to \mathcal{H}_t^{1} p_{2+}(\mathcal{G})(1)$ is an isomorphism. For simplicity, we denote this morphism ζ.

b) Consider Diagram 7, where

$$\check{\mathcal{G}} = p_1^+[d](\mathbb{D}(\mathcal{E})) \in F\text{-}\mathrm{Hol}(\mathbb{P}^d \times \check{\mathbb{P}}^d \times Y/K),$$

the horizontal isomorphisms of the middle square are constructed using the commutativity of the dual functor with the functor $p_1^+[d]$ (this is the Poincare duality (1.2.1)) and with proper push-forwards, where the right square is the dual of the left square used for $\mathbb{D}(\mathcal{E})$ instead of $\mathcal{E}$, is commutative. By transitivity of the relative duality isomorphism and by definition of the adjunction morphisms, we get the commutativity of the middle square. The commutativity of the other squares of Diagram 7 are tautological (e.g. for the second square, this is the construction (1.8.1)). Moreover, the second left arrow of the bottom row is indeed an isomorphism because of Proposition 1.6.(2).

c) Since π_1 is smooth of relative dimension $d-1$ and π_2 is proper, we get $\mathbb{D}\,\mathrm{Rad}^0(\mathbb{D}(\mathcal{E})) \xrightarrow{\sim} \mathrm{Rad}^0(\mathcal{E})$. From both properties of 2.3, we get the first exact sequence

$$0 \longrightarrow \mathcal{H}_t^{d-1} p_{2+} p_1^+(\mathcal{E}) \longrightarrow \mathrm{Rad}^0(\mathcal{E}) \longrightarrow \mathrm{Rad}_!^0(\mathcal{E}); \tag{2.7.2}$$

the second one is induced by duality:

$$\mathbb{D}\,\mathrm{Rad}_!^0(\mathbb{D}\mathcal{E}) \longrightarrow \mathrm{Rad}^0(\mathcal{E}) \longrightarrow \mathbb{D}\mathcal{H}_t^{d-1} p_{2+} p_1^+(\mathbb{D}\mathcal{E}) \longrightarrow 0, \tag{2.7.3}$$

By construction, the first morphism of (2.7.2) and the last one of (2.7.3) are respectively the left vertical arrow and the right vertical arrow of Diagram 7. Using Lemma 2.5 we check that $\mathrm{Rad}_!^0(\mathcal{E})$ (resp. $\mathbb{D}\,\mathrm{Rad}_!^0(\mathbb{D}\mathcal{E})$) is left (resp. right) reduced with respect to $\tilde{p}_1$, i.e. does not have any nontrivial constant with respect to $\tilde{p}_1$ subobject (resp. quotient). This implies that $\mathcal{H}_t^{d-1} p_{2+} p_1^+(\mathcal{E})$ (resp. $\mathbb{D}\mathcal{H}_t^{d-1} p_{2+} p_1^+(\mathbb{D}\mathcal{E})$) is the maximal constant with respect to $\tilde{p}_1$ subobject (resp. quotient) of $\mathrm{Rad}^0(\mathcal{E})$.

d) Since π_1 is smooth, π_2 is proper and $\mathcal{E}$ is ι-pure then so is $\mathrm{Rad}^0(\mathcal{E})$. Hence $\mathrm{Rad}^0(\mathcal{E})$ is semi-simple in the category $\mathrm{Hol}(\check{\mathbb{P}}^d \times Y/K)$ (see [3, 4.3.1]). By considering Diagram 7 and using the step 1.c), this implies that the morphism

$$\zeta\colon \mathcal{H}_t^{-1} p_{2+}(\mathcal{G}) \longrightarrow \mathcal{H}_t^1 p_{2+}(\mathcal{G})(1)$$

is an isomorphism in $\mathrm{Hol}(\check{\mathbb{P}}^d \times Y/K)$. Since ζ is also a morphism of the category $F\text{-}\mathrm{Hol}(\check{\mathbb{P}}^d \times Y/K)$, then ζ is an isomorphism of $F\text{-}\mathrm{Hol}(\check{\mathbb{P}}^d \times Y/K)$.

$$\begin{tikzcd}
\mathcal{H}_t^{d-1} p_{2+} p_1^{+}(\mathcal{E}) \arrow[r, equal] \arrow[d] & \mathcal{H}_t^{-1} p_{2+}(\mathcal{G}) \arrow[r, "\zeta"] \arrow[d, "\mathrm{adj}"] & \mathcal{H}_t^{1} p_{2+}(\mathcal{G})(1) \arrow[r, "\sim"] & \mathbb{D}\mathcal{H}_t^{-1} p_{2+}(\check{\mathcal{G}})(1) \arrow[r, equal] & \mathbb{D}\mathcal{H}_t^{d-1} p_{2+} p_1^{+}(\mathbb{D}(\mathcal{E}))(1) \\
\mathrm{Rad}^0(\mathcal{E}) & \mathcal{H}_t^{-1} p_{2+} i_{+} i^{+}(\mathcal{G}) \arrow[l, "\sim"'] \arrow[r, "\sim", "\theta_i"'] & \mathcal{H}_t^{1} p_{2+} i_{+} i^{!}(\mathcal{G})(1) \arrow[r, "\sim"] \arrow[u, "\mathrm{adj}"] & \mathbb{D}\mathcal{H}_t^{-1} p_{2+} i_{+} i^{+}(\check{\mathcal{G}})(1) \arrow[u, "\mathbb{D}(\mathrm{adj})"] & \mathbb{D}\,\mathrm{Rad}^0(\mathbb{D}(\mathcal{E}))(1) \arrow[l, "\sim"'] \arrow[u]
\end{tikzcd}$$

Diagram 7

2. We proceed by induction on r. Suppose $r \geq 2$. We put

$$\widetilde{\mathcal{G}} := i^{+}(\mathcal{G})[-1] \xrightarrow{\sim} \pi_1^{+}[d-1](\mathcal{E}).$$

The morphism ζ induces by pull-back

$$\tilde{\zeta}^{r-1} := i^{+}(\zeta^{r-1})[-1] \colon \widetilde{\mathcal{G}} \longrightarrow \widetilde{\mathcal{G}}[2r](r).$$

Consider Diagram 8, which is commutative and where the middle arrow of the middle row is an isomorphism because of Proposition 1.6.(2). By considering the long exact sequence induced by the exact triangle (2.3.1), since $\mathrm{Rad}_!$ is left exact and $r \geq 2$, we check that the adjunction morphism

$$\mathcal{H}_t^{-r} p_{2+}(\mathcal{G}) \longrightarrow \mathcal{H}_t^{-r} p_{2+} i_{+} i^{+}(\mathcal{G})$$

is an isomorphism. This implies that the right vertical arrow of Diagram 8 is an isomorphism. To check that the arrow of the bottom of Diagram 8 is an isomorphism, it is sufficient to prove that $t^{+}[-d]\mathcal{H}_t^{-(r-1)}\pi_{2+}(\tilde{\zeta}^{r-1})$ is an isomorphism (the rational point $\operatorname{Spec} k \hookrightarrow \check{\mathbb{P}}^d$ can vary).

We put

$$\tilde{\imath} \colon s^{-1}(H \times Y) \hookrightarrow \mathbb{P}^d \times Y, \quad \tilde{s} \colon s^{-1}(H \times Y) \hookrightarrow H \times Y,$$

and

$$\tilde{\pi}_2 := \tilde{p}_2 \circ \tilde{s}.$$

From Lemma 1.10, since $s^{+}[-d](\mathcal{G}) \xrightarrow{\sim} \mathcal{E}$, we have $s^{+}[-d](\zeta) = \eta_{\tilde{\imath},\mathcal{E}}$. Hence, since $\tilde{\imath}$ is a closed immersion induced by an hyperplane of $\mathbb{P}^d \times Y$, putting

$$\widetilde{\mathcal{E}} := \tilde{\imath}^{+}[-1](\mathcal{E}) \in F\text{-}\mathrm{Hol}(s^{-1}(H)/K),$$

we remark that

$$\tilde{\eta} := \tilde{\imath}^{+}[-1] s^{+}[-d](\zeta) \colon \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{E}}[2](1)$$

is a Chern class of a relative hyperplane for the projective morphism $\tilde{\pi}_2$. Hence, since $\widetilde{\mathcal{E}}$ is pure, by using the induction hypothesis, we get that $\mathcal{H}_t^{-(r-1)}\tilde{\pi}_{2+}(\tilde{\eta}^{r-1})$ is an isomorphism. Finally, with the same arguments than in the step 1.a), we check the first isomorphism:

$$\begin{aligned} t^{+}[-d]\mathcal{H}_t^{-(r-1)}\pi_{2+}(\tilde{\zeta}^{r-1}) &\xrightarrow{\sim} \mathcal{H}_t^{-(r-1)}\tilde{\pi}_{2+}\tilde{s}^{+}[-d](\tilde{\zeta}^{r-1}) \\ &= \mathcal{H}_t^{-(r-1)}\tilde{\pi}_{2+}\tilde{s}^{+}[-d] i^{+}[-1](\zeta^{r-1}) \xrightarrow{\sim} \mathcal{H}_t^{-(r-1)}\tilde{\pi}_{2+}\tilde{\imath}^{+}[-1] s^{+}[-d](\zeta^{r-1}) \\ &\xrightarrow{\sim} \mathcal{H}_t^{-(r-1)}\tilde{\pi}_{2+}(\tilde{\eta}^{r-1}), \end{aligned}$$

which implies that $t^{+}[-d]\mathcal{H}_t^{-(r-1)}\pi_{2+}(\tilde{\zeta}^{r-1})$ is also an isomorphism. □

$$
\begin{tikzcd}
\mathcal{H}_t^{-r} p_{2+}(\mathcal{G}) \arrow[r, "\xi^{r-1}"] \arrow[d, "\mathrm{adj}"] & \mathcal{H}_t^{r-2} p_{2+}(\mathcal{G}) \arrow[r, "\xi"] \arrow[d, "\mathrm{adj}"] & \mathcal{H}_t^{r} p_{2+}(\mathcal{G})(1) \arrow[r, "\sim"] & \mathbb{D}\mathcal{H}_t^{-r} p_{2+}(\check{\mathcal{G}})(1) \\
\mathcal{H}_t^{-r} p_{2+} i_+ i^+(\mathcal{G}) \arrow[r, "\xi^{r-1}"] \arrow[d, "\sim"] & \mathcal{H}_t^{r-2} p_{2+} i_+ i^+(\mathcal{G}) \arrow[r, "\sim", "\theta_i"'] \arrow[d, "\sim"] & \mathcal{H}_t^{r} p_{2+} i_+ i^!(\mathcal{G})(1) \arrow[r, "\sim"] \arrow[u, "\mathrm{adj}"'] & \mathbb{D}\mathcal{H}_t^{-r} p_{2+} i_+ i^+(\check{\mathcal{G}})(1) \arrow[u, "\mathbb{D}(\mathrm{adj})"'] \\
\mathcal{H}_t^{-(r-1)} \pi_{2+}(\widetilde{\mathcal{G}}) \arrow[r, "\tilde{\xi}^{r-1}"] & \mathcal{H}_t^{r-1} \pi_{2+}(\widetilde{\mathcal{G}}) & &
\end{tikzcd}
$$

Diagram 8

2.8 Corollary. *We suppose here that k is a finite field with p^s elements. Let $f\colon X \to \operatorname{Spec} k$ be a smooth and projective morphism with X of pure dimension d. Let E is a pure overconvergent F-isocrystal. Then for every positive integer r, we get the isomorphisms $H^{d-r}_{\mathrm{rig}}(E) \xrightarrow{\sim} H^{d+r}_{\mathrm{rig}}(E)(r)$.*

Proof. In this context, we can use the comparison theorem between the rigid cohomology and the push forward as defined in the theory of arithmetic $\mathcal{D}$-modules, see for instance [4, 5.9]. □

2.9 Remark. (1) In the context of Corollary 2.8, we need the hypothesis that X is smooth because otherwise we do not know how to get an equivalence between the category of overconvergent F-isocrystal over X as defined by Berthelot and some coefficients defined in the framework of arithmetic $\mathcal{D}$-modules (one might hope to remove the smoothness condition: see the question [1, 1.3.11]).

(2) Thanks to a very recent preprint of Christopher Lazda (see for instance Lemma [12, 5.5]), one should get some relative version of Corollary 2.8, i.e. in the case where $f\colon X \to Y$ is a smooth and projective morphism of realizable varieties with Y no necessary equal to $\operatorname{Spec} k$.

3. The dual Brylinski–Radon and the inversion formula

We keep the notation Section 2 of our paper.

3.1 Definition. We define the dual Brylinski–Radon transform

$$\mathrm{Rad}^{\vee}\colon F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\check{\mathbb{P}}^d \times Y/K) \longrightarrow F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$$

by posing, for any $\check{\mathcal{E}} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\check{\mathbb{P}}^d \times Y/K)$,

$$\mathrm{Rad}^{\vee}(\check{\mathcal{E}}) := \pi_{1+}\pi_2^{+}(\check{\mathcal{E}})[d-1]. \tag{3.1.1}$$

3.2 Lemma. *Let $\iota\colon X := (H \times_{\check{\mathbb{P}}^d} \check{H}) \times Y \hookrightarrow \mathbb{P}^d \times \check{\mathbb{P}}^d \times \mathbb{P}^d \times Y$ be the canonical embedding, $p_{13}\colon \mathbb{P}^d \times \check{\mathbb{P}}^d \times \mathbb{P}^d \times Y \to \mathbb{P}^d \times \mathbb{P}^d \times Y$ be the projection and $\pi = p_{13} \circ \iota$. Let $\Delta\colon \mathbb{P}^d \times Y \hookrightarrow \mathbb{P}^d \times \mathbb{P}^d \times Y$ be the diagonal immersion (and the identity over Y). Let $\mathcal{F} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d \times Y/K)$. We have the isomorphism of $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d \times Y/K)$ of the form*

$$\Delta_+ \circ \Delta^+(\mathcal{F})[2-2d](1-d) \oplus \bigoplus_{i=0}^{d-2} \mathcal{F}[-2i](-i) \xrightarrow{\sim} \pi_+\pi^+(\mathcal{F}).$$

Proof. By using the projection isomorphisms (see [3, A.6]), we can suppose $\mathcal{F} = K_{\mathbb{P}^d \times \mathbb{P}^d \times Y}$. Hence, by using some base change theorems (induced by the projection $Y \to \operatorname{Spec} k$), we can suppose $Y = \operatorname{Spec} k$. We put $\widetilde{K} := K_{\mathbb{P}^d \times \mathbb{P}^d}$. Consider the following cartesian squares

(3.2.1)
$$\begin{tikzcd}
X \arrow[r, hook, "\iota"] \arrow[rr, bend left, "\pi"] & \mathbb{P}^d \times \check{\mathbb{P}}^d \times \mathbb{P}^d \arrow[r, "p_{13}"] & \mathbb{P}^d \times \mathbb{P}^d \\
\widetilde{X} \arrow[u, hook, "\widetilde{\Delta}"] \arrow[r, hook, "\tilde{\iota}"] \arrow[rr, bend right, "\tilde{\pi}"'] \arrow[ur, phantom, "\Box"] & \mathbb{P}^d \times \check{\mathbb{P}}^d \arrow[u, hook] \arrow[r, "p_1"] \arrow[ur, phantom, "\Box"] & \mathbb{P}^d . \arrow[u, hook, "\Delta"']
\end{tikzcd}$$

Put $\tilde{\pi} := p_1 \circ \tilde{\iota}$, $\eta := \iota^+ \eta_{p_{13}, K_{\mathbb{P}^d \times \check{\mathbb{P}}^d \times \mathbb{P}^d}}$ and $\tilde{\eta} := \tilde{\iota}^+ \eta_{p_1, K_{\mathbb{P}^d \times \check{\mathbb{P}}^d}}$. From (1.12.2), we check $\widetilde{\Delta}^+(\eta) = \tilde{\eta}$. By using the construction of Theorem 1.15, we get the morphism

(3.2.2)
$$\textstyle\bigoplus_{i=0}^{d-2} \eta^i \colon \bigoplus_{i=0}^{d-2} \widetilde{K}[-2i](-i) \longrightarrow \pi_+ \pi^+(\widetilde{K}).$$

Since π is outside $\Delta(\mathbb{P}^d)$ a $\mathbb{P}^{d-2}$-fibration, from Theorem 1.15, we deduce that the morphism (3.2.2) is an isomorphism outside $\Delta(\mathbb{P}^d)$. Hence, a cone of (3.2.2) is in the essential image of Δ_+. Since

$$\Delta^+ \pi_+ \pi^+(\widetilde{K}) \xrightarrow{\sim} \tilde{\pi}_+ \widetilde{\Delta}^+ \pi^+(\widetilde{K}) \xrightarrow{\sim} \tilde{\pi}_+ \tilde{\pi}^+ \Delta^+(\widetilde{K}),$$

by applying Δ^+ to the morphism 3.2.2, we get

(3.2.3)
$$\textstyle\bigoplus_{i=0}^{d-2} \tilde{\eta}^i \colon \bigoplus_{i=0}^{d-2} \Delta^+ \widetilde{K}[-2i](-i) \longrightarrow \tilde{\pi}_+ \tilde{\pi}^+ \Delta^+(\widetilde{K}).$$

Since $\tilde{\pi}$ is a $\mathbb{P}^{d-1}$-fibration, from Theorem 1.15, we remark that the cone of the morphism (3.2.3) is isomorphic to $\Delta^+(\widetilde{K})[2-2d](1-d)$. Moreover, using Theorem 1.15 again we build the morphism

$$\Delta^+ \pi_+ \pi^+(\widetilde{K}) \longrightarrow \Delta^+(\widetilde{K})[2-2d](1-d)$$

and by adjunction the second morphism of the sequence in $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d / K)$:

(3.2.4)
$$\textstyle\bigoplus_{i=0}^{d-2} \widetilde{K}[-2i](-i) \xrightarrow[(3.2.2)]{} \pi_+ \pi^+(\widetilde{K}) \to \Delta_+ \circ \Delta^+(\widetilde{K})[2-2d](1-d).$$

Since $\pi_+\pi^+(\widetilde{K})$ is ι-pure, then $\pi_+\pi^+(\widetilde{K})$ is semisimple in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d/K)$ (see [3, 4.3.6]). Then, we get the isomorphism

$$\Delta_+ \circ \Delta^+(\widetilde{K})[2-2d](1-d) \oplus \bigoplus_{i=0}^{d-2} \widetilde{K}[-2i](-i) \xrightarrow{\sim} \pi_+\pi^+(\widetilde{K})$$

in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d/K)$ which induces the morphisms of (3.2.4). For any $i = 0, \dots, d-2$, we have

$$\begin{aligned}
&\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d/K)}(\Delta_+ \circ \Delta^+(\widetilde{K})[2-2d], \widetilde{K}[-2i]) \\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d/K)}(\Delta^+(\widetilde{K})[2-2d], \Delta^!\widetilde{K}[-2i]) \\
&\quad \xrightarrow[(1.6.1)]{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d/K)}(\Delta^!(\widetilde{K})[2], \Delta^!\widetilde{K}[-2i]) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times \mathbb{P}^d/K)}(\widetilde{K}[-2i], \Delta_+ \circ \Delta^+(\widetilde{K})[2-2d]) \\
&\quad \xrightarrow{\sim} \mathrm{Hom}_{D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d/K)}(\Delta^+\widetilde{K}[-2i], \Delta^+(\widetilde{K})[2-2d]) = 0.
\end{aligned}$$

Hence, we get the compatibility with Frobenius. □

3.3 Proposition (Radon inversion formula). *Let $\mathcal{E} \in F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(\mathbb{P}^d \times Y/K)$. Then the following formula holds*

$$\mathrm{Rad}^\vee \circ \mathrm{Rad}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}(1-d) \oplus \tilde{p}_2^+[d](\phi(\mathcal{E})), \tag{3.3.1}$$

where $\phi(\mathcal{E}) := \bigoplus_{i=0}^{d-2} \tilde{p}_{2+}(\mathcal{E})[d-2-2i](-i)$.

Proof. With the notation of Lemma 3.2, let respectively

$$u, v\colon \mathbb{P}^d \times \mathbb{P}^d \times Y \longrightarrow \mathbb{P}^d \times Y$$

be the left and middle projection. Then, by using the base change theorem, more precisely, look at the cartesian square defining the fibered product

$$X = (H \times Y) \times_{\check{\mathbb{P}}^d \times Y} (\check{H} \times Y),$$

we get

$$\mathrm{Rad}^\vee \circ \mathrm{Rad}(\mathcal{E}) \xrightarrow{\sim} v_+\pi_+\pi^+u^+(\mathcal{E})[2d-2].$$

Hence we obtain:

$$\begin{aligned}
&\mathrm{Rad}^\vee \circ \mathrm{Rad}(\mathcal{E}) \\
&\quad \xrightarrow[\text{Lemma } 3.2]{\sim} v_+\Delta_+ \circ \Delta^+(u^+(\mathcal{E}))(1-d) \oplus \bigoplus_{i=0}^{d-2} v_+u^+(\mathcal{E})[2d-2i-2](-i) \\
&\quad \xrightarrow{\sim} \mathcal{E}(1-d) \oplus \bigoplus_{i=0}^{d-2} \tilde{p}_2^+[d]\tilde{p}_{2+}(\mathcal{E})[d-2i-2](-i).
\end{aligned}$$

□

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Manoscritto pervenuto in redazione il 6 marzo 2015.