Rend. Sem. Mat. Univ. Padova, Vol. 136 (2016), 257–264
DOI 10.4171/RSMUP/136-16

On the existence of Gorenstein projective precovers

J. Asadollahi (*) – T. Dehghanpour (**) – R. Hafezi (***)

Abstract – We provide a simple proof for a recent result of Bravo, Gillespie and Hovey, showing that over a left coherent ring for which the projective dimension of flat right modules is finite, the class of Gorenstein projective right modules is precovering. As an immediate application, we provide a description for the Gorenstein defect category over such rings.

Mathematics Subject Classification (2010). 11M32; 14F20, 19F27.

Keywords. Precovering class, Gorenstein projective, Gorenstein flat.

1. Introduction

Let R be ring. 'Module' will always mean 'right module'. Mod-R, resp. mod-R, denotes the category of all, resp. all finitely presented, right R-modules and $\mathcal{P}$rj-R denotes the category of projective R-modules.

An acyclic complex $\mathbf{X}$ of projective R-modules is called totally acyclic if for every $P \in \mathcal{P}\text{rj-}R$ the induced complex $\mathrm{Hom}_R(\mathbf{X}, P)$ is acyclic. An R-module G is called Gorenstein projective if it is the zeroth syzygy of a totally acyclic complex of projectives. The class of all Gorenstein projective modules will be denoted by $\mathcal{GP}\text{-}R$.

(*) *Indirizzo dell'A.*: Department of Mathematics, University of Isfahan, P.O. Box: 81746-73441, Isfahan, Iran; and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
E-mail: asadollahi@sci.ui.ac.ir; asadollahi@ipm.ir

(**) *Indirizzo dell'A.*: Department of Mathematics, University of Isfahan, P.O. Box: 81746-73441, Isfahan, Iran
E-mail: t_dehghanpoor@sci.ui.ac.ir

(***) *Indirizzo dell'A.*: School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran
E-mail: hafezi@ipm.ir

Let $\mathcal{X}$ be a class of R-modules and M be an R-module. A morphism $\varphi\colon X \to M$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-precover of M if every morphism from an object $\mathcal{X}$ to M factors through φ. $\mathcal{X}$ is called precovering if every module has an $\mathcal{X}$-precover.

One of the open problems in homological algebra is whether the class of Gorenstein projective modules is precovering in Mod-R. Holm [9, Theorem 2.10] showed that over a Gorenstein ring R, $\mathcal{GP}$-R is precovering in Mod-R. Beligiannis [2] proved that $\mathcal{GP}$-Λ is a precovering subclass of Mod-Λ, provided Λ is an artin algebra. If furthermore Λ is virtually Gorenstein, then $\mathcal{G}p$-$\Lambda := \mathcal{GP}$-$\Lambda \cap$ mod-Λ is a precovering subclass of mod-Λ, see [3, Proposition 4.7]. The most recent result in this direction appeared in [6] showing that over a left coherent ring R in which flat right modules have finite projective dimension, the class $\mathcal{GP}$-R is precovering in Mod-R. In fact, they showed that over such rings the cotorsion pair $(\mathcal{GP}\text{-}R, \mathcal{GP}\text{-}R^{\perp})$ is cogenerated by a set and hence is complete. Therefore has enough projectives, that means the class $\mathcal{GP}$-R is precovering in Mod-R, see [6, Proposition 8.10]. This result generalizes Corollary 2.13 of [10] which proves similar result for noetherian rings admitting a dualising complex and also Theorem A.1 of [12] which proves this fact for noetherian rings of finite Krull dimension. The proof presented in [6] is based on cotorsion theory techniques. In this paper, we provide a proof for this fact using more elementary approach. As an application, we provide a description for Gorenstein defect categories over such rings, that extends Theorem 6.7 of [11].

2. Facts

For the convenience of the reader we collect all the facts used throughout the paper in this short section.

2.1. An R-module C is called cotorsion if $\mathrm{Ext}^1_R(F, C) = 0$, for all flat modules F. All modules have cotorsion envelopes [5]. So every R-module M has a (co)resolution by cotorsion modules

$$0 \longrightarrow M \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow C^3 \longrightarrow \cdots.$$

Note that in this resolution all the terms, but probably, the first two are cotorsion and flat. In case M itself is flat, then C^0 also is flat and cotorsion [14, Theorem 3.4.2]. If M admits such resolution but of finite length, we say that the cotorsion dimension of M is finite.

2.2. An acyclic complex $\mathbf{F}$ of flat R-modules is called $\mathbf{F}$-totally acyclic if it remains acyclic after applying the functor $- \otimes_R E$, for every injective left R-module E. An R-module G is called Gorenstein flat if it is a syzygy of an $\mathbf{F}$-totally acyclic complex.

2.3. One may use the class of Gorenstein projective modules to assign an invariant to a module, known as the Gorenstein projective dimension of module. For every R-module M, the Gorenstein projective dimension of M, denoted $\mathrm{Gpd}_R M$, is defined to be the minimum number n such that there exists an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where G_i is Gorenstein projective, for all $i \geq 0$. If no such number n exists, we define $\mathrm{Gpd}_R M := \infty$.

2.4. Let $\mathbb{C}(R)$ denote the category of complexes over R. Recall that a complex $P \in \mathbb{C}(R)$ is a projective object if P is acyclic with projective syzygies. By [13] $\mathbb{C}(R)$ is an abelian category having enough projective objects. Moreover, by [1, 2.10] there is a functor $k^i \colon \mathbb{K}(\mathbb{C}(R)) \to \mathbb{K}(R)$, called evaluation functor, that assigns to any (bi)complex X in $\mathbb{K}(\mathbb{C}(R))$ the complex $X^i \in \mathbb{K}(R)$ sitting on the i-th row of X, where for a class $\mathcal{X}$ of modules, $\mathbb{K}(\mathcal{X})$ denotes the homotopy category of complexes over $\mathcal{X}$.

2.5. Let R be a ring with the property that all flat R-modules have finite projective dimension. This assumption, implies that there is an integer N so that the projective dimension of each flat right R-module F is less than or equal to N, see [10, Remark 2.2]. The reader may see comments in Subsection 5.1 of [8] on the ubiquity of such rings.

3. Main result

Let us begin with the following proposition. We would like to thank Edgar Enochs for providing us with an elementary proof of the part (1) ⇒ (2). See also proof of Proposition 3.4 of [9], where similar argument is used for this part.

3.1 Proposition. *Let R be a left coherent ring in which all flat (right) modules have finite projective dimension. Let $\mathbf{P}$ be a complex of projective right R-modules. Then the following are equivalent*:

(1) $\mathrm{Hom}_R(\mathbf{P}, Q)$ *is acyclic for every projective R-module Q*;

(2) $\mathbf{P} \otimes_R E$ *is acyclic for every injective left R-module E.*

Proof. (1) $\Longrightarrow$ (2). Let $\mathbf{P}$ be a complex of projective R-modules such that $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic for every projective R-module Q. Let L be a module of finite projective dimension. We use induction on the projective dimension of L to show that $\mathrm{Hom}_R(\mathbf{P}, L)$ is also acyclic. Assume inductively that $\mathrm{pd} L = n$ and the result holds for all modules of projective dimension less than n. Consider the short exact sequence $0 \to L' \to Q \to L \to 0$, where $\mathrm{pd}L' = n-1$ and Q is a projective module and apply the functor $\mathrm{Hom}_R(\mathbf{P}, -)$ to get the short exact sequence $0 \to \mathrm{Hom}_R(\mathbf{P}, L') \to \mathrm{Hom}_R(\mathbf{P}, Q) \to \mathrm{Hom}_R(\mathbf{P}, L) \to 0$ of complexes. By induction assumption the complexes $\mathrm{Hom}_R(\mathbf{P}, L')$ and $\mathrm{Hom}_R(\mathbf{P}, Q)$ are acyclic, so we get the result. Now let E be an injective left R-module. Apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to $\mathbf{P} \otimes_R E$ and use adjoint duality to get the isomorphic complex $\mathrm{Hom}_R(\mathbf{P}, \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}))$. Clearly $\mathbf{P} \otimes_R E$ is acyclic if and only if this later complex is acyclic. Since R is left coherent, $\mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z})$ is a flat right R-module. Hence, by assumption, it has finite projective dimension. So, by the first line of the proof, the complex $\mathrm{Hom}_R(\mathbf{P}, \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}))$ is acyclic. This completes the proof of this part.

(2) $\Longrightarrow$ (1). Let $\mathbf{P}$ be a complex of projective R-modules such that $\mathbf{P} \otimes_R E$ is acyclic for every injective left R-module E. Since all flat R-modules have bounded projective dimension, every module has finite cotorsion dimension, see [14, Remark 3.4.9]. In particular, every flat (projective) module has finite cotorsion flat dimension. So, by induction, we deduce that to complete the proof it is enough to show that $\mathrm{Hom}_R(\mathbf{P}, C)$ is acyclic for every cotorsion flat module C. By [14, Proposition 2.3.5], every cotorsion flat module C is a direct summand of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$. So we need to show that the complex $\mathrm{Hom}_R(\mathbf{P}, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}))$ is acyclic. By adjoint duality, this complex is acyclic if and only the complex $\mathbf{P} \otimes_R \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is acyclic. Since $\mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is an injective left R-module, it follows from our assumption that the complex $\mathbf{P} \otimes_R \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$ is acyclic. So the proof is complete. □

3.2 Remark. (1) The part (2) $\Longrightarrow$ (1) of the above proposition has been proved by Murfet and Salarian when R is a commutative noetherian ring. They also proved the converse by adding the extra assumption that R is of finite Krull dimension [12, Lemma 4.20]. Note that the techniques used in [12] to prove this result also works to prove the extended version over coherent rings. Our proof here is more elementary, just using usual homological algebra.

(2) There is also a version of the above proposition for the coherent rings. In fact, Theorem 6.7 of [6] implies that if R is a left coherent ring, then a complex $\mathbf{P}$ of projectives is acyclic with respect to the functor $- \otimes_R E$, for every absolutely pure left R-module E if and only if it is exact with respect to the functor $\mathrm{Hom}_R(-, F)$, for every flat right R-module F.

As an immediate consequence of the above proposition, we have the following corollary. See also [7, Proposition 3.7].

3.3 Corollary. *Let R be a left coherent ring in which all flat (right) modules have finite projective dimension. Then every Gorenstein projective module is Gorenstein flat.*

3.4 Proposition. *Let R be a left coherent ring in which all flat (right) modules have finite projective dimension. Then all Gorenstein flat modules have finite Gorenstein projective dimension.*

Proof. Let N be the upper bound on the projective dimension of flats. Let G be a Gorenstein flat R-module. There exists an $\mathbf{F}$-totally acyclic complex $\mathbf{F}$ of flat modules such that G is its zeroth syzygy. Consider the following soft truncation of a projective resolution of $\mathbf{F}$ in $\mathbb{C}(R)$

$$\subset_N \mathbf{P}_\bullet \colon 0 \longrightarrow \mathbf{K} \longrightarrow \mathbf{P}_{N-1} \longrightarrow \mathbf{P}_{N-2} \longrightarrow \cdots \longrightarrow \mathbf{P}_1 \longrightarrow \mathbf{P}_0 \longrightarrow \mathbf{F} \longrightarrow 0.$$

Hence the complex is acyclic in $\mathbb{C}(R)$, $\mathbf{P}_i$ is a projective complex for every $0 \leq i \leq N-1$, and $\mathbf{K} = \operatorname{Ker}(\mathbf{P}_{N-1} \longrightarrow \mathbf{P}_{N-2})$. For every $j \in \mathbb{Z}$, apply the evaluation functor k^j on the above resolution, to obtain its jth row, which is an exact sequence

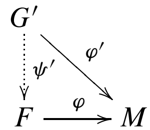
$$\begin{aligned} k^j(\subset_N \mathbf{P}_\bullet)\colon 0 \longrightarrow K^j \longrightarrow P^j_{N-1} \longrightarrow P^j_{N-2} \longrightarrow \cdots \\ \longrightarrow P^j_1 \longrightarrow P^j_0 \longrightarrow F^j \longrightarrow 0, \end{aligned}$$

where F^j is flat and each P^j_i is a projective R-module, for $0 \leq i \leq N-1$. Since projective dimension of flats is less than or equal to N, we deduce that K^j, for all $j \in \mathbb{Z}$, is projective. Therefore $\mathbf{K}$ is an acyclic complex of projectives. On the other hand, since $\mathbf{F}$ and also every complex $\mathbf{P}_i$, for $0 \leq i \leq N-1$, are acyclic with respect to the functor $- \otimes_R E$, for every injective left R-module E, we deduce easily that the complex $\mathbf{K} \otimes_R E$ is acyclic, for every injective left R-module E. Proposition 3.1 now apply to deduce that the complex $\operatorname{Hom}_R(\mathbf{K}, Q)$ is acyclic for every projective R-module Q. Hence the syzygies of $\mathbf{K}$ are Gorenstein projective. This in turn implies that the Gorenstein projective dimension of G is finite. So the proof is complete. □

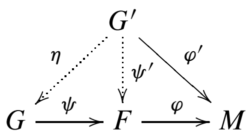
It has recently been proved that all modules have a Gorenstein flat precover [15]. We shall use this fact in the proof of the following theorem.

3.5 Theorem. *Let R be a left coherent ring in which all flat (right) modules have finite projective dimension. Then every R-module has a Gorenstein projective precover.*

Proof. Let M be an arbitrary R-module. By [15], it has a Gorenstein flat precover $F \xrightarrow{\varphi} M$. By Proposition 3.4, F has finite Gorenstein projective dimension and hence by [9, Theorem 2.10], has a Gorenstein projective precover $G \xrightarrow{\psi} F$. Using definition of Gorenstein flat precover, in view of Corollary 3.3, we have that for every Gorenstein projective module G', every morphism $\varphi' \colon G' \to M$ can be factored through φ. So there is a morphism $\psi' \colon G' \to F$ such that the following diagram is commutative:



Since $G \xrightarrow{\psi} F$ is a Gorenstein projective precover of F, there is a morphism $\eta \colon G' \to G$ such that $\psi\eta = \psi'$. Now in view of the following diagram



we deduce that $G \xrightarrow{\varphi\psi} M$ is a Gorenstein projective precover of M. Hence the proof is complete. □

3.6 Remark. Let $\mathcal{X}$ be a class of R-modules. An $\mathcal{X}$-precover $\varphi \colon X \to M$ of M is called special if $\mathrm{Ker}(\varphi) \in \mathcal{X}^{\perp}$, where orthogonal is taken with respect to Ext^1. $\mathcal{X}$ is called special precovering if every module has an special $\mathcal{X}$-precover. Of course, special precovers are much more useful.

We know from [6] that $\mathcal{GP}$-R is an special precovering class. The proof presented here does not show this. In fact, by the proof of [15, Theorem A] used in the proof of above theorem, we do not have control on the kernel of a Gorenstein flat precover of a module.

An application. Let $\mathcal{A}$ be an abelian category with enough projective object. The Gorenstein defect category $\mathbb{D}^{\mathrm{b}}_{\mathrm{defect}}(\mathcal{A})$ of $\mathcal{A}$ is defined in [4] as the Verdier quotient

$$\mathbb{D}^{\mathrm{b}}_{\mathrm{defect}}(\mathcal{A}) := \mathbb{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathcal{A})/\mathrm{Im}F,$$

where $\mathbb{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathcal{A})$ denotes the singularity category of $\mathcal{A}$ and F is a functor defined from $\underline{\mathcal{GP}\text{-}(\mathcal{A})}$ to $\mathbb{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathcal{A})$, sending a Gorenstein projective object G to the stalk complex G at degree zero.

Several descriptions for the Gorenstein defect categories are presented in Section 6 of [11]. In particular, by [11, Theorem 6.7], if $\mathcal{GP}$-$(\mathcal{A})$ is a precovering class in $\mathcal{A}$, then

$$\mathbb{D}^{\mathrm{b}}_{\mathrm{defect}}(\mathcal{A}) \simeq \mathbb{K}^{-,\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}\mathcal{A})/\mathbb{K}^{\mathrm{b}}(\mathcal{GP}\text{-}\mathcal{A}),$$

where $\mathbb{K}^{-,\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}\mathcal{A})$ denotes the full subcategory of $\mathbb{K}^{-}(\mathcal{GP}\text{-}\mathcal{A})$ consisting of all bounded above complexes $\mathbf{X}$ such that for every $G \in \mathcal{GP}$-$(\mathcal{A})$, $\mathcal{A}(G, \mathbf{X})$ is acyclic, in small degrees.

Using Theorem 3.5, we have the following result.

3.7 Proposition. *Let R be a left coherent ring in which all flat (right) modules have finite projective dimension. Then there is a triangle equivalence*

$$\mathbb{D}^{\mathrm{b}}_{\mathrm{defect}}(\mathrm{Mod}\text{-}R) \simeq \mathbb{K}^{-,\mathcal{GP}\mathrm{b}}(\mathcal{GP}\text{-}R)/\mathbb{K}^{\mathrm{b}}(\mathcal{GP}\text{-}R).$$

Proof. By Proposition 3.5, $\mathcal{GP}$-R is precovering in Mod-R. Hence similar argument as in the proof of Theorem 6.7 (i) of [11] works to prove the result. □

Acknowledgements. The research of the J. Asadollahi was in part supported by a grant from IPM (No: 92130216). The author also likes to thank the Center of Excellence for Mathematics (University of Isfahan).

References

[1] J. Asadollahi – R. Hafezi – Sh. Salarian, *Homotopy category of projective complexes and complexes of Gorenstein projective modules,* J. Algebra **399** (2014), pp. 423–444.

[2] A. Beligiannis, *Cohen–Macaulay modules, (co)torsion paris and virtually Gorenstein algebras,* J. Algebra **288** (2005), pp. 137–211.

[3] A. Beligiannis, *On algebras of finite Cohen–Macaulay type*, Adv. Math. **226** (2011), pp. 1973–2019.

[4] P. A. Bergh – D. A. Jorgensen – S. Oppermann, *The Gorenstein defect category,* Q. J. Math. **66** (2015), no. 2, pp. 459–471.

[5] L. Bican – R. El Bashir – E. Enochs, *All modules have flat covers*, Bull. London Math Soc. **33** (2001), pp. 385–390.

[6] D. Bravo – J. Gillespie – M. Hovey, *The stable module category of a general ring,* Preprint 2014. arXiv:1405.5768v1 [math.RA]

[7] L. W. Christense – A. Frankild – H. Holm, *On Gorenstein projective, injective and flat dimensions – a functorial description with applications,* J. Algebra **302** (2006), pp. 231–279.

[8] J. Gillespie, *Gorenstein complexes and recollements from cotorsion pairs,* Adv. Math. **291** (2016), 859–911.

[9] H. Holm, *Gorenstein homological dimensions,* J. Pure Appl. Algebra **189** (2004), no. 1-3, pp. 167–193.

[10] P. Jørgensen, *The homotopy category of complexes of projective modules,* Adv. Math **193** (2005), no. 1, pp. 223–232.

[11] F. Kong – P. Zhang, *From CM-finite to CM-free,* J. Pure Appl. Algebra **220** (2016), no. 2, 782–801.

[12] D. Murfet – Sh. Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), pp. 1096–1133.

[13] J. R. García Rozas, *Covers and envelops in the category of complexes of modules*, CRS Press, Boca Raton, FL, 1999.

[14] J. Xu, *Flat covers of modules,* Lecture notes in Mathematics, 1634. Springer, Berlin, 1996.

[15] G. Yang – L. Liang, *All modules have Gorenstein flat precover,* Comm. Alg. **42** (2014), pp. 3078–3085.

Manoscritto pervenuto in redazione il 4 maggio 2015.