Rend. Sem. Mat. Univ. Padova, Vol. 136 (2016), 265–275
DOI 10.4171/RSMUP/136-17

Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group (addendum)

Benjamin Cahen (∗)

Abstract – We extend the results of [13] to the holomorphic representations of the non-scalar type of the multi-dimensional Jacobi group.

Mathematics Subject Classification (2010). 22E27, 32M10, 46E22.

Keywords. Berezin quantization, Berezin transform; quasi-Hermitian Lie group, multi-dimensional Jacobi group, unitary representation, holomorphic representation of the non-scalar type, Stratonovich–Weyl correspondence.

1. Introduction

We use the notation of [13], Section 2. In particular, we denote by G the multi-dimensional Jacobi group and by K the subgroup of G consisting of all elements of the form $\big((0,0), c, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)\big)$ where $c \in \mathbb{R}$ and $P \in U(n)$.

Recall that the unitary representations of G considered in [13] are holomorphically induced from a unitary character of K. Here we consider, more generally, the unitary representations of G which are holomorphically induced from a unitary representation ρ of K, see [18], p. 515, and we extend the results of [13] to these representations. The main tool is then the generalized Berezin calculus for a reproducing kernel Hilbert space of vector-valued holomorphic functions, see [2], [17] and [12]. Most of the proofs are similar to those of [13] so we just sketch them briefly.

(∗) *Indirizzo dell'A.*: Université de Lorraine, Site de Metz, UFR-MIM, Département de mathématiques, Bâtiment A, Ile du Saulcy, CS 50128, F-57045, Metz cedex 01, France
E-mail: benjamin.cahen@univ-lorraine.fr

2. Representations

Let $\gamma \in \mathbb{R}$ and let ρ_0 be a unitary irreducible representation of $U(n)$ on a (finite-dimensional) complex vector space $\mathcal{V}$. Let ρ be the representation of K on $\mathcal{V}$ defined by

$$\rho\big((0,0), c, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)\big) = e^{i\gamma c}\rho_0(P).$$

We also denote by ρ_0 and ρ the corresponding representations of $\mathrm{GL}_n(\mathbb{C})$ and K^c.

Let $M_n(\mathbb{C}) = \mathfrak{n}^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{n}^-$ be the usual triangular decomposition of $M_n(\mathbb{C})$. Then ρ_0 is associated with a dominant integral weight of the form

$$\Lambda_{m_1, m_2, \ldots, m_n}\colon \mathrm{Diag}(a_1, a_2, \ldots, a_n) \longrightarrow \sum_{i=1}^{n} m_i a_i$$

where $m_1 \geq m_2 \geq \cdots \geq m_n$ and $m_i \in \mathbb{Z}$ [15], p. 274. Let $m = \sum_{i=1}^{n} m_i$. Then we have $\rho_0(zI_n) = z^m I_{\mathcal{V}}$ for each $z \in \mathbb{C}^\times$.

Recall that for each $y \in \mathbb{C}^n$ and $Y \in M_n(\mathbb{C})$ such that $Y^t = Y$, we denote

$$a(y, Y) := \big((y, 0), 0, \left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right)\big) \in \mathfrak{p}^+$$

and

$$\mathcal{D} := \{a(y, Y) \in \mathfrak{p}^+ \colon I_n - Y\bar{Y} > 0\} \cong \mathbb{C}^n \times \mathcal{B}$$

where $\mathcal{B} := \{Y \in M_n(\mathbb{C})\colon Y^t = Y,\ I_n - Y\bar{Y} > 0\}$.

Now we will apply the general considerations of [18] and [12] to the particular case of the multi-dimensional Jacobi group. Following [18], p. 497, we set $K(Z, W) := \rho(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J(g, Z) := \rho(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$ and we introduce the Hilbert space $\mathcal{H}$ of all holomorphic functions on $\mathcal{D}$ with values in $\mathcal{V}$ such that

$$\|f\|^2_{\mathcal{H}} := \int_{\mathcal{D}} \langle K(Z, Z)^{-1} f(Z), f(Z)\rangle_{\mathcal{V}}\, d\mu(Z) < +\infty$$

where μ denotes the G-invariant measure on $\mathcal{D}$ defined in [13], Section 2.

Let π be the unitary representation of G on $\mathcal{H}$ defined by

$$(\pi(g)f)(Z) = J(g^{-1}, Z)^{-1} f(g^{-1} \cdot Z).$$

In [12], we verified that π is obtained by holomorphic induction from ρ, that is, π corresponds to the natural action of G on the square-integrable holomorphic sections of the Hilbert G-bundle $G \times_\rho \mathcal{V}$ over $G/K \cong \mathcal{D}$. Moreover, π is irreducible since ρ is irreducible, see [18], p. 515.

The evaluation maps $K_Z\colon \mathcal{H} \to \mathcal{V}, f \to f(Z)$ are continuous [18], p. 539. The vector coherent states of $\mathcal{H}$ are the maps $E_Z = K_Z^*\colon \mathcal{V} \to \mathcal{H}$ defined by $\langle f(Z), v\rangle_{\mathcal{V}} = \langle f, E_Z v\rangle_{\mathcal{H}}$ for $f \in \mathcal{H}$ and $v \in \mathcal{V}$.

We have the following result, see [18], p. 540.

Proposition 2.1. (1) *There exists a constant $c_\rho > 0$ such that*

$$E_Z^* E_W = c_\rho K(Z, W)$$

for each Z, $W \in \mathcal{D}$.

(2) *For $g \in G$ and $Z \in \mathcal{D}$,*

$$E_{g\cdot Z} = \pi(g) E_Z J(g, Z)^*.$$

Now we give explicit formulas for K and J and we compute c_ρ (see Lemma 3.1 in [13]).

Proposition 2.2. (1) *Let $Z = a(y, Y) \in \mathcal{D}$ and $W = a(v, V) \in \mathcal{D}$. We set*

$$E(y, v, Y, V) := 2y^t(I_n - \bar{V}Y)^{-1}\bar{v} + y^t(I_n - \bar{V}Y)^{-1}\bar{V}y + \bar{v}^t Y(I_n - \bar{V}Y)^{-1}\bar{v}.$$

Then,

$$K(Z, W) = \exp\Big(\frac{\gamma}{4}E(y, v, Y, V)\Big)\rho_0(I_n - Y\bar{V}).$$

(2) $\mathcal{H} \neq (0)$ *if and only if $\gamma > 0$ and $m + n + 1/2 < 0$. In this case,*

$$c_\rho = \mathrm{Dim}(\mathcal{V})\Big(\frac{\gamma}{2\pi}\Big)^n J_n(-m - n - 3/2)^{-1}.$$

(3) *For each $g = \big((z_0, \bar{z}_0), c_0, \big(\begin{smallmatrix} P & Q \\ \bar{Q} & \bar{P}\end{smallmatrix}\big)\big) \in G$ and each $Z = a(y, Y) \in \mathcal{D}$, we have*

$$J(g, Z) = e^{i\gamma c_0}\exp\Big(\frac{\gamma}{4}(z_0^t\bar{z}_0 + 2\bar{z}_0{}^t P y + y^t P^t \bar{Q} y - (\bar{z}_0 + \bar{Q}y)^t(PY + Q)(\bar{Q}y + \bar{P})^{-1}(\bar{z}_0 + \bar{Q}y)\Big) \rho_0((\bar{Q}Y + \bar{P})^t)^{-1}.$$

Proof. (1) and (3) are simple calculations. To prove (2), we use the formula

$$(I_n - \bar{Y}Y)^{-1} = \mathrm{Det}(I_n - \bar{Y}Y)^{-1} C(I_n - \bar{Y}Y)^t$$

for each $Y \in \mathcal{B}$, where $C(A)$ denotes the cofactor matrix of a matrix A. From this formula, we deduce

$$\operatorname{Tr} \rho_0(I_n - \bar{Y}Y)^{-1} = \operatorname{Det}(I_n - \bar{Y}Y)^{-m} \operatorname{Tr} \rho_0(C(I_n - \bar{Y}Y)^t)$$

and then we can prove (2) by following the same lines as in the proof of Proposition 3.2 of [13], using Theorem XII.5.6 of [18]. □

Proposition 3.4 of [13] can be generalized as follows.

PROPOSITION 2.3. *For $X \in \mathfrak{g}^c$, $f \in \mathcal{H}$ and $Z \in \mathcal{D}$,*

$$d\pi(X)f(Z) = d\rho(p_{\mathfrak{k}^c}(e^{-\operatorname{ad} Z} X))\, f(Z) - (df)_Z(p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z} X)).$$

In particular,

(1) *if $X \in \mathfrak{p}^+$, then $d\pi(X)f(Z) = -(df)_Z(X)$;*
(2) *if $X \in \mathfrak{k}^c$, then $d\pi(X)f(Z) = d\rho(X)f(Z) + (df)_Z([Z,X])$;*
(3) *if $X \in \mathfrak{p}^-$, then*

$$\begin{aligned} d\pi(X)f(Z) = {} & (d\rho \circ p_{\mathfrak{k}^c})\Big(-[Z,X] + \frac{1}{2}[Z,[Z,X]]\Big) f(Z) \\ & - (df_Z \circ p_{\mathfrak{p}^+})\Big(-[Z,X] + \frac{1}{2}[Z,[Z,X]]\Big). \end{aligned}$$

Let (E_k) be a basis of $\mathfrak{p}^+$. Then, for each $f \in \mathcal{H}$ and each k, we denote

$$(\partial_k f)(Z) = \frac{d}{dt} f(Z + tE_k)|_{t=0}.$$

From the preceding proposition we deduce the following result.

PROPOSITION 2.4. *For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, $d\pi(X_1X_2\cdots X_q)$ is a sum of terms of the form $P(Z)\partial_{k_1}\partial_{k_2}\cdots\partial_{k_r}$ where $r \leq q$ and $P(Z)$ is a polynomial of degree $\leq 2q$.*

3. Berezin calculus

First we introduce the Berezin quantization map associated with ρ_0, see [3], [4], [1], [5], and [19].

Let $\tilde{\varphi}_0$ be the linear form on $M_n(\mathbb{C})$ defined by $\tilde{\varphi}_0 = -i\Lambda_{m_1,m_2,\ldots,m_n}$ on $\mathfrak{h}_0$ and $\tilde{\varphi}_0 = 0$ on $\mathfrak{n}^{\pm}$. We denote by φ_0 the restriction of $\tilde{\varphi}_0$ to $u(n)$. Then the orbit $o(\varphi_0)$ of φ_0 under the coadjoint action of $U(n)$ is then said to be associated with ρ_0.

Note that the stabilizer of φ_0 for the coadjoint action of $U(n)$ contains the torus

$$H_0 := \{\mathrm{Diag}(ia_1, ia_2, \ldots, ia_n) : a_j \in \mathbb{R}\}.$$

We say that such an element φ_0 is *regular* if the stabilizer of φ_0 is equal to H_0, see [5]. Then we can verify that φ_0 is regular if and only if one has $m_1 > m_2 > \cdots > m_n$. In the rest of the paper, we assume that φ_0 is regular.

Note also that a complex structure on $o(\varphi_0)$ is then defined by the diffeomorphism $o(\varphi_0) \simeq U(n)/H_0 \simeq \mathrm{GL}_n(\mathbb{C})/H_0^c N^-$ where N^- is the analytic subgroup of $\mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{n}^-$.

Without loss of generality, we can assume that $\mathcal{V}$ is the space of holomorphic functions on (a dense open set of) $o(\varphi_0)$ as in [5]. For $\varphi \in o(\varphi_0)$ there exists a unique function $e_\varphi \in \mathcal{V}$ (called a coherent state) such that $a(\varphi) = \langle a, e_\varphi \rangle_{\mathcal{V}}$ for each $a \in \mathcal{V}$. The Berezin calculus on $o(\varphi_0)$ associates with each operator B on $\mathcal{V}$ the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi) = \frac{\langle Be_\varphi, e_\varphi \rangle_{\mathcal{V}}}{\langle e_\varphi, e_\varphi \rangle_{\mathcal{V}}}$$

which is called the symbol of B. Then we have the following proposition, see [14], [1] and [5].

PROPOSITION 3.1. (1) *The map $B \to s(B)$ is injective.*

(2) *For each operator B on $\mathcal{V}$, we have $s(B^*) = \overline{s(B)}$.*

(3) *For each $\varphi \in o(\varphi_0)$, $k \in U(n)$ and $B \in \mathrm{End}(\mathcal{V})$, we have*

$$s(B)(\mathrm{Ad}^*(k)\varphi) = s(\rho_0(k)^{-1} B \rho_0(k))(\varphi).$$

(4) *For each $A \in u(n)$ and $\varphi \in o(\varphi_0)$, we have $s(d\rho_0(A))(\varphi) = i\langle \varphi, A \rangle$.*

We also need the following result, see [10] and [12].

PROPOSITION 3.2. *Let $Z \in \mathcal{D}$. There exists a unique element k_Z in K^c such that $k_Z^* = k_Z$ and $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Each $g \in G$ such that $g \cdot 0 = Z$ is then of the form $g = \exp(-Z^*)\,\zeta(\exp Z^* \exp Z) k_Z^{-1} h$ where $h \in K$. Consequently, the map $Z \to g_Z := \exp(-Z^*)\,\zeta(\exp Z^* \exp Z) k_Z^{-1}$ is a section for the action of G on $\mathcal{D}$.*

More explicitly, for each $Z = a(y, Y) \in \mathcal{D}$, we have

$$k_Z = \left((0,0), -\frac{i}{8} E(y, y, Y, Y), \begin{pmatrix} (I_n - Y\overline{Y})^{1/2} & 0 \\ 0 & (I_n - \overline{Y}Y)^{-1/2} \end{pmatrix} \right)$$

and

$$g_Z = \Big((-\bar{w}, -w), \frac{i}{8}(y^t(I_n - \bar{Y}Y)^{-1}\bar{Y}y - \bar{y}^t Y(I_n - \bar{Y}Y)^{-1}\bar{y}), M(Y)\Big)$$

where $w := -(I_n - \bar{Y}Y)^{-1}(\bar{y} + \bar{Y}y)$ *and*

$$M(Y) := \begin{pmatrix} (I_n - Y\bar{Y})^{-1/2} & Y(I_n - \bar{Y}Y)^{-1/2} \\ \bar{Y}(I_n - Y\bar{Y})^{-1/2} & (I_n - \bar{Y}Y)^{-1/2} \end{pmatrix}.$$

Now, following [17], [2], [12], we define the pre-symbol $S_0(A)$ of an operator A by

$$S_0(A)(Z) = c_\rho^{-1}\, \rho(k_Z^{-1}) E_Z^* A E_Z \rho(k_Z^{-1})^*$$

and the Berezin symbol $S(A)$ of A is defined as the complex-valued function on $\mathcal{D} \times o(\varphi_0)$ given by

$$S(A)(Z, \varphi) = s(S_0(A)(Z))(\varphi).$$

For each $g \in G$ and $Z \in \mathcal{D}$, let $k(g, Z) := g_Z^{-1} g^{-1} g_{g\cdot Z} \in K$. Then we can write

$$k(g, Z) = \left((0,0), c(g,Z), \begin{pmatrix} P(g,Z) & 0 \\ 0 & \overline{P(g,Z)} \end{pmatrix}\right),$$

where $c(g, Z) \in \mathbb{R}$ and $P(g, Z) \in U(n)$.

We have the following properties of S, see [12].

Proposition 3.3. (1) *Each operator A is determined by $S(A)$.*

(2) *For each operator A, we have $S(A^*) = \overline{S(A)}$.*

(3) *We have $S(I_{\mathcal{H}}) = 1$.*

(4) *For each operator A, $g \in G$, $Z \in \mathcal{D}$ and $\varphi \in o(\varphi_0)$, we have*

$$S(A)(g \cdot Z, \varphi) = S(\pi(g)^{-1} A \pi(g))(Z, \mathrm{Ad}^*(P(g,Z))\varphi).$$

Now, we give some formulas for the Berezin pre-symbol of $\pi(g)$ for $g \in G$ and for the Berezin symbol of $d\pi(X)$ for $X \in \mathfrak{g}^c$. For $\varphi \in u(n)^*$, we denote by φ^s the linear form on $\mathfrak{s}$ defined by

$$\left\langle \varphi^s, \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \right\rangle = \langle \varphi, P \rangle$$

and by φ^e the linear form on $\mathfrak{g}$ defined by

$$\left\langle \varphi^e, \left((z, \bar{z}), c, \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}\right) \right\rangle = \langle \varphi, P \rangle + \gamma c.$$

We also denote by φ^s and φ^e the extensions of φ^s and φ^e to $\mathfrak{s}^c$ and $\mathfrak{g}^c$.

Proposition 3.4 ([12]). (1) *For $g \in G$ and $Z \in \mathcal{D}$, we have*

$$S_0(\pi(g))(Z) = \rho\left(k_Z^{-1}\kappa(\exp Z^* g^{-1} \exp Z)^{-1}(k_Z^{-1})^*\right).$$

(2) *For each $X \in \mathfrak{g}$, $Z \in \mathcal{D}$ and $\varphi \in o(\varphi_0)$, we have*

$$S(d\pi(X))(Z, \varphi) = i\langle \mathrm{Ad}^*(g_Z)\varphi^e, X\rangle.$$

Recall that $\xi_0 \in \mathfrak{g}^*$ is said to be *regular* if the stabilizer $G(\xi_0)$ of ξ_0 for the coadjoint action is connected and if the Hermitian form $(Z, W) \to \langle \xi_0, [Z, W^*]\rangle$ is not isotropic [12].

Lemma 3.5. *The linear form φ_0^e is regular if and only if we have $m_j > 0$ for each j or $m_j < 0$ for each j.*

Proof. On the one hand, by using the formula for the coadjoint action of G given in [13], Section 2, we can verify that $G(\varphi_0^e)$ consists of all matrices of the form $\left((0,0), c, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)\right)$ where $c \in \mathbb{R}$ and $P \in U(n)$ is such that $\mathrm{Ad}^*(P)\varphi_0 = \varphi_0$. Since φ_0 is assumed to be regular as an element of $u(n)^*$, we get $P \in H_0$. Hence $G(\varphi_0^e) \cong \mathbb{R} \times H_0$ is connected.

On the other hand, for each $Z = a(y, Y) \in \mathcal{D}$, we have

$$\begin{aligned}\langle \varphi_0^e, [Z, Z^*]\rangle &= -\langle \varphi_0, Y\bar{Y}\rangle - \frac{i}{2}\gamma|y|^2 \\ &= -\sum_j m_j|Y_j|^2 - \frac{i}{2}\gamma|y|^2\end{aligned}$$

where $Y_1, Y_2, \ldots, Y_n$ denote the columns of Y. The result hence follows. □

Let us assume that φ_0^e is regular and denote by $\mathcal{O}(\varphi_0^e)$ the orbit of φ_0^e for the coadjoint action of G. Then we have the following proposition, see [12].

Proposition 3.6. *The map $\Psi\colon \mathcal{D} \times o(\varphi_0) \to \mathfrak{g}^*$ defined by*

$$\Psi(Z, \varphi) = \mathrm{Ad}^*(g_Z)\varphi^e$$

is a diffeomorphism form $\mathcal{D} \times o(\varphi_0)$ onto $\mathcal{O}(\varphi_0^e)$ such that

$$\Psi(g \cdot Z, \varphi) = \mathrm{Ad}^*(g)\,\Psi(Z, \mathrm{Ad}^*(P(g, Z))(\varphi))$$

for each $g \in G$, $Z \in \mathcal{D}$ and $\varphi \in o(\varphi_0)$.

More precisely, with the notation of [13], *we have*

$$\Psi(Z, \varphi) = \left(\gamma(\bar{w}, w), \gamma, \mathrm{Ad}^*(M(Y))\varphi^s - \frac{\gamma}{2}(\bar{w}, w) \times (\bar{w}, w)\right)$$

where $Z = a(y, Y) \in \mathcal{D}$ and $w := -(I_n - \bar{Y}Y)^{-1}(\bar{y} + \bar{Y}y)$.

4. Berezin transform and Stratonovich–Weyl correspondence

Here we introduce the Berezin transform associated with S and the corresponding Stratonovich–Weyl correspondence, following [12].

We fix a K-invariant measure ν on $o(\varphi_0)$ normalized as in [5], Section 2. Then the measure $\tilde{\mu} := \mu \otimes \nu$ on $\mathcal{D} \times o(\varphi_0)$ is invariant under the action of G on $\mathcal{D} \times o(\varphi_0)$ given by $g \cdot (Z, \varphi) := (g \cdot Z, \mathrm{Ad}^*(P(g, Z))^{-1}\varphi)$ and the measure $\mu_{\mathcal{O}(\varphi_0^e)} := (\Psi^{-1})^*(\tilde{\mu})$ is a G-invariant measure on $\mathcal{O}(\varphi_0^e)$.

We denote by $\mathcal{L}_2(\mathcal{H})$ the space of Hilbert-Schmidt operators on $\mathcal{H}$ endowed with the Hilbert-Schmidt norm. We also endow $\mathrm{End}(\mathcal{V})$ with the Hilbert-Schmidt norm. We denote by $L^2(\mathcal{D} \times o(\varphi_0))$ (respectively $L^2(\mathcal{D})$, $L^2(o(\varphi_0))$, $L^2(\mathcal{O}(\varphi_0^e))$) the space of functions on $\mathcal{D} \times o(\varphi_0)$ (respectively $\mathcal{D}$, $o(\varphi_0)$, $\mathcal{O}(\varphi_0^e)$) which are square-integrable with respect to the measure $\tilde{\mu}$ (respectively μ, ν, $\mu_{\mathcal{O}(\varphi_0^e)}$). Then we have the following result, see for instance [6].

Proposition 4.1. *The Berezin transform $b := ss^*$ is given by*

$$b(a)(\psi) = \int_{o(\varphi_0)} a(\varphi) \frac{|\langle e_\psi, e_\varphi \rangle_{\mathcal{V}}|^2}{\langle e_\varphi, e_\varphi \rangle_{\mathcal{V}} \langle e_\psi, e_\psi \rangle_{\mathcal{V}}} \, d\nu(\varphi)$$

for each $a \in L^2(o(\varphi_0))$

Similarly, we have the following proposition.

Proposition 4.2. *The Berezin transform $B := SS^*$ is a bounded operator of $L^2(\mathcal{D} \times o(\varphi_0))$ and that, for each $f \in L^2(\mathcal{D} \times o(\varphi_0))$, we have the following integral formula*

$$B(f)(Z, \psi) = \int_{\mathcal{D} \times o(\varphi_0)} k(Z, W, \psi, \varphi) \, f(W, \varphi) \, d\mu(W) d\nu(\varphi)$$

where

$$k(Z, W, \psi, \varphi) := \frac{|\langle \rho(\kappa(g_Z^{-1} g_W))^{-1} e_\psi, e_\varphi \rangle_{\mathcal{V}}|^2}{\langle e_\varphi, e_\varphi \rangle_{\mathcal{V}} \langle e_\psi, e_\psi \rangle_{\mathcal{V}}}.$$

Consider the left-regular representation τ of G on $L^2(\mathcal{D} \times o(\varphi_0))$ defined by

$$(\tau(g)(f))(Z, \varphi) = f(g^{-1} \cdot (Z, \varphi)).$$

Clearly, τ is unitary. Moreover, since S is G-equivariant, we immediately verify that for each $f \in L^2(\mathcal{D} \times o(\varphi_0))$ and each $g \in G$, we have $B(\tau(g) f) = \tau(g)(B(f))$.

Now, we introduce the polar decomposition of $S\colon \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D}\times o(\varphi_0))$. We can write $S = (SS^*)^{1/2}W = B^{1/2}W$ where $W := B^{-1/2}S$ is a unitary operator from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathcal{D} \times o(\varphi_0))$. Then we have the following proposition, see [12]. The main point is that W is G-equivariant since S (hence B) is G-equivariant.

PROPOSITION 4.3. (1) *$W\colon \mathcal{L}_2(\mathcal{H}) \to L^2(\mathcal{D} \times o(\varphi_0))$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi, \mathcal{D} \times o(\varphi_0))$.*

(2) *The map $\mathcal{W}$ from $\mathcal{L}_2(\mathcal{H})$ to $L^2(\mathcal{O}(\varphi_0^e))$ defined by $\mathcal{W}(f) = W(f \circ \Psi)$ is a Stratonovich–Weyl correspondence for the triple $(G, \pi, \mathcal{O}(\varphi_0^e))$.*

5. Extension of the Berezin transform

Here we generalize Proposition 5.2 of [13], that is, we extend B to a class of functions which contains $S(d\pi(X))$ for $X \in \mathfrak{g}^c$, in particular in order to define $W(d\pi(X))$.

For $Z, W \in \mathcal{D}$, we set $l_Z(W) := \log \eta(\exp Z^* \exp W) \in \mathfrak{p}^-$. We need the following lemma which is the direct generalization of Lemma 5.1 of [13].

LEMMA 5.1. (1) *For each $Z, W \in \mathcal{D}$, $V \in \mathfrak{p}^+$ and $v \in \mathcal{V}$,*

$$\frac{d}{dt}(E_Z v)(W + tV)\big|_{t=0} = -c_\rho (d\rho \circ p_{\mathfrak{k}^c})\Big([l_Z(W), V] + \frac{1}{2}[l_Z(W), [l_Z(W), V]]\Big)(E_Z v)(W).$$

(2) *For $Z, W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$,*

$$\frac{d}{dt} l_Z(W + tV)\big|_{t=0} = p_{\mathfrak{p}^-}\Big([l_Z(W), V] + \frac{1}{2}[l_Z(W), [l_Z(W), V]]\Big).$$

(3) *The function $(\partial_{k_1}\partial_{k_2}\cdots\partial_{k_q} E_Z v)(W)$ is of the form $Q(l_Z(W))(E_Z v)(W)$ where Q is a polynomial of degree $\leq 2q$ with values in* $\operatorname{End}(\mathcal{V})$.

(4) *For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the function $S_0(d\pi((X_1 X_2 \cdots X_q))$ is a sum of terms of the form $\rho(k_Z)^{-1} P(Z) Q(l_Z(Z)) \rho(k_Z)$ where P and Q are polynomials of degree $\leq 2q$ with values in* $\operatorname{End}(\mathcal{V})$.

By combining the arguments of the proof of Proposition 6.5 in [8] with those of the proof of Proposition 5.2 in [13], we then obtain the following result. Recall that $m := \sum_i m_i$.

PROPOSITION 5.2. *If $q < \frac{1}{4}(-m-2n)$ then for each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the Berezin transform of $S(d\pi(X_1X_2\cdots X_q))$ is well-defined.*

We have then generalized Proposition 5.2 of [13]. However, it seems difficult to obtain here an explicit expression for $W(d\pi(X))$, $X \in \mathfrak{g}$, as in [13], Section 6.

REFERENCES

[1] D. ARNAL – M. CAHEN – S. GUTT, *Representations of compact Lie groups and quantization by deformation,* Acad. Roy. Belg. Bull. Cl. Sci. (5) **74** (1988), no. 4-5, pp. 123–141.

[2] S. T. ALI – M. ENGLIS, *Berezin–Toeplitz quantization over matrix domains,* preprint 2006. arXiv:math-ph/0602015

[3] F. A. BEREZIN, *Quantization,* Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), pp. 1116–1175. (In Russian.) English translation, Math. USSR-Izv. **8** (1974), no. 5, pp. 1109–1165.

[4] F. A. BEREZIN, *Quantization in complex symmetric domains,* Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 363–402, 472. (In Russian.) English translation, Math. USSR-Izv. **9** (1975), no. 2, pp. 341–379.

[5] B. CAHEN, *Berezin quantization on generalized flag manifolds*, Math. Scand. **105** (2009), pp. 66–84.

[6] B. CAHEN, *Stratonovich–Weyl correspondence for compact semisimple Lie groups*, Rend. Circ. Mat. Palermo **59** (2010), pp. 331–354.

[7] B. CAHEN, *Stratonovich–Weyl correspondence for discrete series representations*, Arch. Math. (Brno) **47** (2011), pp. 41–58.

[8] B. CAHEN, *Berezin transform for non-scalar holomorphic discrete series,* Comment. Math. Univ. Carolin. **53** (2012), no. 1, pp. 1–17.

[9] B. CAHEN, *Berezin quantization and holomorphic representations,* Rend. Semin. Mat. Univ. Padova **129** (2013), pp. 277–297.

[10] B. CAHEN, *Global parametrization of scalar holomorphic coadjoint orbits of a quasi-Hermitian Lie group,* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **52** (2013), no. 1, pp. 35–48.

[11] B. CAHEN, *Stratonovich–Weyl correspondence for the Jacobi group,* Commun. Math. **22** (2014), no. 1, pp. 31–48.

[12] B. CAHEN, *Stratonovich–Weyl correspondence via Berezin quantization,* Rend. Istit. Mat. Univ. Trieste **46** (2014), pp. 157–180.

[13] B. CAHEN, *Berezin transform and Stratonovich–Weyl correspondence for the multi-dimensional Jacobi group,* Rend. Sem. Mat. Univ. Padova **136** (2016), pp. 69–93.

[14] M. Cahen – S. Gutt – J. Rawnsley, *Quantization of Kähler manifolds.* I. *Geometric interpretation of Berezin's quantization,* J. Geom. Phys. **7** (1990), pp. 45–62.

[15] R. Goodman – N. R. Wallach, *Symmetry, representations and invariants,* Graduate Texts in Mathematics, 255, Springer, Dordrecht, 2009.

[16] A. A. Kirillov, *Lectures on the orbit method,* Graduate Studies in Mathematics, 64, American Mathematical Society, Providence, R.I., 2004.

[17] S. Kilic, *The Berezin symbol and multipliers on functional Hilbert spaces,* Proc. Amer. Math. Soc. **123** (1995), no. 12, pp. 3687–3691.

[18] K-H. Neeb, *Holomorphy and convexity in Lie theory,* de Gruyter Expositions in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2000.

[19] N. J. Wildberger, *On the Fourier transform of a compact semisimple Lie group,* J. Austral. Math. Soc. Ser. A **56** (1994), no. 1, pp. 64–116.

Manoscritto pervenuto in redazione l'8 agosto 2015.