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On the existence of maximal $\mathcal{S}$-closed submodules

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Abstract – The goal of this paper is to characterize the right non-singular rings R for which every non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule. Several examples and related results are given.

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1. Introduction

The submodule $Z(M) = \{x \in M \mid xI = 0$ for an essential right ideal I of $R\}$ of a right module M over a ring R is the *singular submodule* of M. The module M is *non-singular* if $Z(M) = 0$; it is *singular* if $Z(M) = M$. The ring R is *right non-singular* if $Z(R_R) = 0$. Every right non-singular ring has a right self-injective regular maximal right ring of quotients Q^r ([5] and[7]). Whenever U is an essential submodule of M, then M/U is singular, and the converse holds if M is non-singular. A submodule U of a module M is $\mathcal{S}$-*closed* if M/U is non-singular. The $\mathcal{S}$-*closure* of a submodule U of a non-singular module M is the submodule U^* of M containing U such that $U^*/U = Z(M/U)$.

A right R-module M has *finite Goldie-dimension* if every direct sum of non-zero submodules of M is finite. The ring Q^r is semi-simple Artinian if and only if R_R has finite Goldie-dimension [7]. A ring R is a *right Goldie-ring* if it satisfies the ascending chain condition for right annihilators and has finite right Goldie-dimension. Since all right annihilators in a right non-singular ring R are $\mathcal{S}$-closed, R is a right Goldie-ring if and only if its right Goldie-dimension is finite [5].

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A non-zero module is *uniform* if all its non-zero submodules are essential. Clearly, a proper $\mathcal{S}$-closed submodule U of a non-singular module M is a maximal $\mathcal{S}$-closed submodule if and only if M/U is uniform. When working with torsion-free modules M over an integral domain, one often considers submodules of M having co-rank 1, i.e. maximal $\mathcal{S}$-closed submodules of M (e.g. [4] and [6]). [1, Proposition 2.3] shows that every non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule if R is a right non-singular right Goldie-ring. Naturally, the question arises if a right non-singular ring R has to be a right Goldie-ring if all non-singular right R-modules contain maximal $\mathcal{S}$-closed submodules.

This paper shows that this is not the case, and presents various characterizations of right non-singular rings having this property in Section 2. Theorem 2.1 also demonstrates that many of the basic properties usually associated with torsion-free modules over integral domains remain valid for non-singular right R-modules exactly if R is a right non-singular ring for which all non-singular right R-modules contain maximal $\mathcal{S}$-closed submodules. Reduced rings satisfying the conditions of Theorem 2.1 are discussed in detail in Section 3. Finally, Section 4 investigates which additional conditions a ring R needs to satisfy to ensure that it is a right Goldie-ring provided all non-singular modules contain maximal $\mathcal{S}$-closed submodules.

2. Maximal $\mathcal{S}$-closed submodules

Let e be an idempotent of a ring R. It is *primitive* if it cannot be written as the sum of two non-zero orthogonal idempotents. Therefore, e is primitive if and only if eR is indecomposable. More generally, a direct summand U of a module M is indecomposable if and only if it is of the form $U = e(M)$ for some primitive idempotent e of the endomorphism ring $\mathrm{End}_R(M)$ of M. On the other hand, an R-module M is *superdecomposable* if it has no non-zero indecomposable direct summands, i.e. its endomorphism ring contains no primitive idempotents.

If S is a subset of a right R-module M, then $\mathrm{ann}_R(S) = \{r \in R \mid Sr = 0\}$ is *the annihilator of* S. In the case $S \subseteq R$, we use the symbols $\mathrm{ann}^\ell_R(S)$ and $\mathrm{ann}^r_R(S)$ to distinguish between left and right annihilators. Finally, $\dim_R M$ denotes the Goldie-dimension of a finite dimensional module M.

Theorem 2.1. *The following conditions are equivalent for a right non-singular ring* R:

a) *if* e *is a non-zero idempotent of* Q^r, *then there are orthogonal idempotents* e_1 *and* e_2 *of* Q^r *such that* $e_1 \neq 0$ *is primitive and* $e = e_1 + e_2$;

b) Q^r *does not have any superdecomposable direct summands*;

c) *every proper $\mathcal{S}$-closed submodule of a non-singular right R-module M is contained in an $\mathcal{S}$-closed submodule V of M such that M/V is uniform* $(0 < \dim_R M/V < \infty)$;

d) *every non-zero non-singular right R-module M contains an $\mathcal{S}$-closed submodule V of M such that M/V is uniform* $(0 < \dim_R M/V < \infty)$;

e) $\bigcap \{U \mid U$ *is a maximal $\mathcal{S}$-closed submodule of* $M\} = 0$ *for all non-singular right R-modules M*;

f) *every non-zero non-singular right R-module M contains a non-zero uniform (finite dimensional) submodule which can be chosen to be $\mathcal{S}$-closed*;

g) *every non-zero non-singular right R-module M contains an essential submodule which is the direct sum of uniform (finite dimensional) modules.*

Proof. Let M be a non-singular right R-module. Clearly, if M contains a submodule U with $0 < \dim_R U < \infty$, then U contains a non-zero uniform submodule and an essential submodule which is a finite direct sum of uniform submodules. Thus, uniformity and finite dimensionality are equivalent in conditions f) and g). Moreover, if U is a proper $\mathcal{S}$-closed submodule of M such that M/U has finite Goldie-dimension, then U is contained in a maximal $\mathcal{S}$-closed submodule. To see this, observe that M/U contains an $\mathcal{S}$-closed submodule of the form V/U for some submodule $U \subseteq V \subseteq M$ such that $[M/U]/[V/U] \cong M/V$ is uniform. Therefore, uniformity and finite dimensionality are equivalent in conditions c) and d) too.

a) $\Longrightarrow$ *c*). Let U be a proper $\mathcal{S}$-closed submodule of a non-singular right R-module M. Since M/U is non-singular, U cannot be essential in M. Choose $0 \neq x \in M$ such that $U \cap xR = 0$. Because M is non-singular, we can find a non-zero right ideal I of R such that $\mathrm{ann}_R(x) \cap I = 0$. Then $xI \cong I$. If V is a submodule of M which contains U and is maximal with respect to the property that $V \cap xI = 0$, then V is $\mathcal{S}$-closed and $V \oplus xI$ is essential in M. Hence, $M/(V \oplus xI) \cong [M/V]/[(V \oplus xI)/V]$ is singular. Since M/V is non-singular, $(V \oplus xI)/V \cong I$ is an essential submodule of M/V. Therefore, M/V is isomorphic to a submodule of Q^r_R. Once we have shown that every non-zero submodule A of Q^r_R contains a maximal $\mathcal{S}$-closed submodule, then we can select an $\mathcal{S}$-closed submodule W of M containing V such that $[M/V]/[W/V] \cong M/W$ is uniform.

Let $0 \neq A \subseteq Q^r_R$, and consider $0 \neq a \in A$. Since Q^r is a regular ring, we can find an idempotent e of Q^r such that $aQ^r = eQ^r$. By a), there are orthogonal idempotents e_1 and e_2 such that $e_1 \neq 0$ is primitive and $e = e_1 + e_2$. We can find an essential right ideal L of R such that $0 \neq e_1 L \subseteq aR \subseteq A$ since aR is essential in $aQ^r = e_1Q^r \oplus e_2Q^r$. But $e_1L \cap (e_2Q^r \cap A) = 0$ yields that $B = e_2Q^r \cap A$ is a proper submodule of A. It is $\mathcal{S}$-closed in A since

$$A/B = A/(e_2Q^r \cap A) \cong (A + e_2Q^r)/e_2Q^r \subseteq e_1Q^r$$

is non-singular.

It remains to show that e_1Q^r is uniform. If this is not the case, then we can find $0 \neq x, y \in e_1Q^r$ with $xR \cap yR = 0$. If $z \in xQ^r \cap yQ^r$, then there is an essential right ideal K of R such that $zK \subseteq xR \cap yR = 0$ since xR and yR are essential in xQ^r and yQ^r respectively. Since Q^r is non-singular, $z = 0$. Because $xQ^r \oplus yQ^r$ is a finitely generated right ideal of the regular ring Q^r, there is an idempotent $f \in Q^r$ such that $fQ^r = xQ^r \oplus yQ^r \subseteq e_1Q^r$. Then $e_1Q^r = xQ^r \oplus yQ^r \oplus [(1-f)Q^r \cap e_1Q^r]$ which contradicts the fact that e_1Q^r is indecomposable. Therefore, e_1Q^r is a uniform R-module, and the same holds for A/B.

Since $c) \implies d)$ is obvious, we consider $d) \implies g)$. Let M be a non-zero non-singular R-module. By d), there is a proper $\mathcal{S}$-closed submodule V of M such that M/V is uniform. As before, we can find $0 \neq x \in M$ with $xR \cap V = 0$. Then, xR is uniform since it is isomorphic to a submodule of M/V, and the same holds for its $\mathcal{S}$-closure in M.

Let $\{U_i \mid i \in I\}$ be the collection of uniform submodules of M. Since the set $\{J \subseteq I \mid \Sigma_J U_j \text{ is direct}\}$ is inductive, it contains a maximal element J_0. If $W = \bigoplus_{J_0} U_j$ is not essential in M, then there is $0 \neq y \in M$ such that $yR \cap W = 0$. By the results of the last paragraph, xR contains a uniform submodule U_i for some $i \notin J_0$, which contradicts the maximality of J_0.

Since $g) \implies f)$ is obvious, we continue by showing $f) \implies e)$. Suppose that

$$V = \bigcap \{U \mid U \text{ is a maximal } \mathcal{S}\text{-closed submodule of } M\} \neq 0$$

for some non-singular module M. By f), V contains a non-zero uniform submodule X. If W is a submodule of M which is maximal with respect to the property $X \cap W = 0$, then W is $\mathcal{S}$-closed in M and $X \oplus W$ is essential in M. Thus, the module $M/(X \oplus W) \cong [M/W]/[X \oplus W)/W]$ is singular. Since M/W is non-singular, $(X \oplus W)/W \cong X$ is a uniform essential submodule of M/W. Consequently, M/W is uniform too. Therefore, $V \subseteq W$ and $0 \neq X = X \cap W = 0$, a contradiction.

$e) \implies b)$. Let e be a non-zero idempotent of Q^r. By e), the non-zero module eQ^r has to contain a proper $\mathcal{S}$-closed submodule U such that eQ^r/U is uniform since

$$\bigcap \{U \mid U \text{ is a maximal } \mathcal{S}\text{-closed submodule of } eQ^r\} = 0.$$

We pick $0 \neq x \in eQ^r$ such that $U \cap xR = 0$ which is possible since U cannot be essential in eQ^r. Because Q^r is regular, $xQ^r = e_1Q^r$ for some non-zero idempotent $e_1 \in Q^r$. Then $e_1Q^r \cap U = xQ^r \cap U = 0$ since R_R is essential in Q^r_R. Because $e_1Q^r \cong [e_1Q^r \oplus U]/U \subseteq eQ^r/U$ is uniform as an R-module, e_1Q^r is indecomposable as a Q^r-module. Since $eQ^r = e_1Q^r \oplus [(1-e_1)Q^r \cap eQ^r]$, the module eQ^r cannot be superdecomposable.

b) $\Longrightarrow$ *a*). Let e be a non-zero idempotent of Q^r which is not primitive. Since eQ^r is not superdecomposable, we can write $eQ^r = U \oplus V$ where U is a non-zero indecomposable Q^r-module. If $e = u + v$ for some $u \in U$ and $v \in V$, then u and v are orthogonal idempotents of Q^r with $U = uQ^r$ and $V = vQ^r$. To see this, observe that $u = eu = u^2 - vu$ yields $u - u^2 = vu \in U \cap V = 0$. Hence, $u = u^2$ and $vu = 0$. By symmetry, $v = v^2$ and $uv = 0$. Thus, $U = uQ^r$ and $V = vQ^r$. Since U is indecomposable, u has to be primitive using the same arguments. □

Condition e) shows that if R satisfies the conditions of the last theorem then every non-singular right R-module M contains many maximal $\mathcal{S}$-closed submodules unless M is uniform.

A non-singular ring R is a right Goldie-ring exactly if direct sums of non-singular injectives are injective [7]. Similarly, we can describe the right non-singular rings R satisfying the conditions of Theorem 2.1 in terms of the non-singular injective right R-modules.

Corollary 2.2. *The following conditions are equivalent for a right non-singular ring R:*

a) *every non-zero non-singular right R-module M contains a maximal $\mathcal{S}$-closed submodule*;

b) *every non-zero non-singular injective right R-module has a non-zero indecomposable direct summand*;

c) *every non-singular injective right R-module E is a direct summand of $\Pi_I E_i$ where each E_i is isomorphic to an indecomposable direct summand of Q^r_R.*

Proof. *a*) $\Longrightarrow$ *c*). Let E be a non-zero non-singular injective right R-module. By Theorem 2.1, E has an essential submodule of the form $\bigoplus_I U_i$ such that each U_i is uniform. Since E is injective, it contains an injective hull E_i of U_i which is indecomposable since U_i is uniform. Therefore, $\Pi_I E_i$ contains an injective hull of

$\bigoplus_I U_i$. Since E is an essential extension of this submodule, there is a monomorphism $E \to \Pi_I E_i$ which splits since E is injective. It remains to show that each E_i is isomorphic to a submodule of Q^r. Since U_i is non-singular, there is a non-zero right ideal J_i of R which is isomorphic to a submodule of U_i arguing as in the proof of $a) \implies c)$ of Theorem 2.1. Since U_i is uniform, E_i is an injective hull of J_i too, and hence is isomorphic to a direct summand of Q^r_R.

$c) \implies b)$. Let E be a non-zero non-singular injective right R-module. By c), E is isomorphic to a direct summand of $\Pi_I E_i$ where each E_i is an indecomposable injective right R-module. We can find a projection map $\pi_j\colon \Pi_I E_i \to E_j$ such that $\pi_j(E) \neq 0$. Since E_j is uniform, $U = U \cap \ker\ \pi_j$ is a maximal $\mathcal{S}$-closed submodule of E. Then E contains an injective hull E' of U. Since E'/U is singular, and U is $\mathcal{S}$-closed, $U = E'$ and $E = U \oplus E_1$. Because E/U is uniform, E_1 is indecomposable.

$b) \implies a)$. By b), Q^r_R has an indecomposable direct summand E_1. By Theorem 2.1, a) holds. □

Corollary 2.2 not necessarily guarantees that every non-singular injective module with an essential submodule, which is a direct sum of uniform submodules, is itself a direct sum of uniform modules unless R is a right Goldie-ring as Example 3.3 shows.

We continue with a first example of a right non-singular ring of infinite Goldie-dimension such that all non-singular right R-modules contain maximal $\mathcal{S}$-closed submodules.

Example 2.3. The endomorphism ring R of an infinite dimensional vector-space V is a right selfinjective regular ring, and therefore is its own maximal right ring of quotients. If e is any non-zero idempotent of R, then $V = e(V) \oplus (1-e)(V)$ and $e(V) = U \oplus W$ where $dim\ U = 1$. Arguing as in the proof of $b) \implies a)$ of Theorem 2.1, we can find two orthogonal idempotents e_U and e_W of R such that $e = e_U + e_W$ and $e_U(V) = U$. Clearly, e_U is primitive.

Further examples of rings satifying the conditions of Theorem 2.1 can be constructed using the next result:

Corollary 2.4. *Let R be a right non-singular ring such that every non-singular R-module contains a maximal $\mathcal{S}$-closed submodule. Every ring S which is Morita-equivalent to R has this property too.*

Proof. Let $\mathcal{F}: \mathcal{M}_R \to \mathcal{M}_S$ be a category equivalence with inverse $\mathcal{G}$. If $E \neq 0$ is a non-singular injective S-module, then $\mathcal{G}(E)$ is a non-zero non-singular injective R-module since both non-singularity and injectivity are Morita-equivalent. By Theorem 2.1, $\mathcal{G}(E)$ has a non-zero indecomposable summand E_1. Then $\mathcal{F}(E_1)$ is a non-zero indecomposable summand of $\mathcal{F}\mathcal{G}(E) \cong E$. Another application of Theorem 2.1 yields that every non-zero non-singular S-module contains a maximal $\mathcal{S}$-closed submodule. □

A right non-singular ring R is a *right Utumi-ring* if every $\mathcal{S}$-closed right ideal is the right annihilator of some subset of R. The right and left Utumi-rings are precisely the right and left non-singular rings for which $Q^\ell = Q^r$. Since Condition a) of Theorem 2.1 is right-left symmetric, we obtain

Corollary 2.5. *Let R be a right and left Utumi-ring. If every non-zero non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule, then the same holds for all non-zero non-singular left R-modules.*

On the other hand, the conditions in Theorem 2.1 need not be right-left symmetric if R is a not a right and left Utumi-ring. For instance, there exist rings R without zero divisors which have infinite right Goldie-dimension, but have left Goldie-dimension 1, see [2]. Clearly, no non-zero submodule of R_R can be uniform.

Corollary 2.6. *Let R be a ring without zero-divisors. Every non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule if and only if Q^r is a division algebra.*

Proof. By [2], R_R has either infinite Goldie-dimension or $\dim_R R_R = 1$. □

3. Reduced rings

A ring R is *reduced* if it contains no nilpotent elements. Therefore, the ring in Example 2.3 is not reduced. The idempotents of a reduced ring are central [7].

Theorem 3.1. *The following conditions are equivalent for a right non-singular ring R:*

a) Q^r is a reduced ring, and every non-singular R-module contains a maximal $\mathcal{S}$-closed submodule;

b) $Q^r \cong \Pi_I D_i$ where each D_i is a division algebra.

Proof. $a) \implies b)$. By Theorem 2.1, Q^r_R has an essential submodule U of the form $\bigoplus_I D_i$ where each D_i is uniform. Since Q^r_R is injective, it contains a copy of the injective hull, $E(D_i)$, of D_i, which is an essential extension of D_i. Therefore, we may assume that each D_i is injective, and hence is a direct summand of Q^r_R. If $x \in D_i$ and $q \in Q^r$, then there exits an essential right ideal I of R such that $qI \subseteq R$. Then $(xq)I = x(qI) \subseteq D_i$. Since D_i is $\mathcal{S}$-closed in Q^r, we obtain $xq \in D_i$. Therefore, $D_i Q^r = D_i$ for all $i \in I$. Because of $Q^r \cong \mathrm{End}_{Q^r}(Q^r) = \mathrm{End}_R(Q^r)$, there exit idempotents $e_i \in Q^r$ such that $D_i = e_i Q^r$ for all $i \in I$. Since Q^r is reduced, each e_i is central. Therefore, $e_i e_j \in e_i Q^r \cap e_j Q^r = 0$ for $i \neq j$ so that $\{e_i \mid i \in I\}$ is a family of orthogonal idempotents.

Observe that $e_i Q^r e_i = e_i Q^r$ since e_i is central. Therefore, each D_i is a subring of Q^r with identity e_i. To show that D_i is a division algebra, let $0 \neq x \in D_i$. Then $xQ^r \subseteq D_i$, and there is an idempotent $e \in Q^r$ with $xQ^r = eQ^r$ because Q^r is regular. Since e is central, we have

$$D_i = e_i Q^r = e_i e Q^r \oplus e_i(1-e)Q^r = eQ^r \oplus (e_i - e)Q^r = xQ^r \oplus (e_i - e)Q^r.$$

However, D_i is uniform, so that $D_i = xQ^r = (e_i x)Q^r = x(e_i Q^r) = xD_i$. Thus, there is $y \in D_i$ with $xy = e_i$, and D_i is a division algebra.

Let S be the ring $\Pi_I D_i$, and define an R-module homomorphism $\alpha\colon Q^r \to S$ by $\alpha(q) = (e_i q)_{i \in I}$ for $q \in Q^r$. Observe that α is a ring homomorphism too because e_i is central for all $i \in I$. Since the idempotents $\{e_i \mid i \in I\}$ are orthogonal, $\alpha(e_i)$ has e_i as an entry in the i^{th}-coordinate, while all its other coordinates are 0. Therefore, $\alpha(U) = \bigoplus_I D_i \subseteq \Pi_I D_i$. For $0 \neq q \in Q^r$, choose $r \in R$ with $0 \neq qr \in U$ which is possible because U is an essential submodule of Q^r_R. There are $i_1, \ldots, i_m \in I$ and $q_1, \ldots q_m \in Q^r$ such that $qr = e_{i_1} q_1 + \cdots + e_{i_m} q_m$ and $e_{i_j} q_j \neq 0$ for all $j = 1, \ldots, m$. Since $\alpha(qr)$ has $e_{i_j} q_j \neq 0$ as its ${i_j}^{\text{th}}$-coordinate for $j = 1, \ldots, m$, we obtain $\alpha(q)r = \alpha(qr) \neq 0$; and α is a monomorphism.

Consequently, the subring $\alpha(Q^r)$ of S is isomorphic to Q^r and has the properties $\alpha(1_{Q^r}) = (e_i)_{i \in I} = 1_S$ and $\bigoplus_I D_i \subseteq \alpha(Q^r)$. In particular, $\alpha(Q^r)$ is right self-injective. Therefore, $\alpha(Q^r)$ is a direct summand of S when the latter is viewed as an $\alpha(Q^r)$-module. On the other hand, for every non-zero $a \in \Pi_I D_i$, there is an idempotent $b \in \bigoplus_I D_i \subseteq \alpha(Q^r)$ such that $0 \neq ab \in \bigoplus_I D_i \subseteq \alpha(Q^r)$. Therefore, $\alpha(Q^r)$ also is an essential $\alpha(Q^r)$-submodule of S when the latter is viewed as a right module over $\alpha(Q^r)$. However, this is only possible if $\alpha(Q^r) = S$.

$b) \implies a)$. The ring $\Pi_I D_i$ clearly satisfies the idempotent condition of Part a) of Theorem 2.1, and has no nilpotent elements. Hence, a) holds. $\square$

A commutative ring is non-singular if and only if it is semi-prime [2] in which case $Q = Q^r$ is commutative too.

Corollary 3.2. *The following conditions are equivalent for a semi-prime commutative ring R:*

a) *every non-singular R-module contains a maximal $\mathcal{S}$-closed submodule*;

b) $Q \cong \Pi_I F_i$ *where each F_i is a field.*

Non-commutative examples of rings with infinite Goldie-dimension satisfying the conditions of Theorem 2.1 can be constructed by combining Theorem 3.1 or Corollary 3.2 with Corollary 2.4.

We now give an example that demonstrates that the idempotents of Q^r and not those of R play the decise role in our discussion.

Example 3.3. Let I be an index-set and R be an integral domain with field of quotients F. For each $i \in I$, choose a non-zero proper ideal J_i of R. Inside $Q = F^I$, consider the subrings $S_1 = R1_Q + R^{(I)}$ and $S_2 = R1_Q + \bigoplus_I J_i$ which are both semi-prime, and have Q as its maximal ring of quotients. Therefore, both S_1 and S_2 satisfy the conditions of Theorem 2.1. The ring S_1 has the same primitive idempotents as Q, while S_2 has only the trivial idempotents 0 and 1_R.

However, there exist reduced rings R which do not satisfy the conditions of Theorem 2.1.

Example 3.4. Let F be a field, and consider the rings $Q = F^I$ as in Example 3.3 and $\bar{Q} = Q/F^{(I)}$. For $x = (x_i) \in Q$, let $\operatorname{spt}(x) = \{i \in I \mid x_i \neq 0\}$. Every non-empty subset J of I induces an idempotent e_J by setting $e_J(i) = 1$ if $i \in J$ and $e_J(i) = 0$ otherwise. For every non-zero $x \in Q$, there is a unique $x' \in Q$ with $\operatorname{spt}(x') = \operatorname{spt}(x)$ and $xx' = e_{\operatorname{spt}(x)}$. To simplify our notation, we write $\bar{x}$ for the coset $x + F^{(I)}$. Since Q is regular, the same holds for $\bar{Q}$. Clearly, any non-zero idempotent e of $\bar{Q}$ is of the form $\overline{e_J}$ for some infinite subset J of I. Partitioning J into two disjoint infinite subsets J_1 and J_2 yields two non-zero orthogonal idempotents $f_i = \overline{e_{J_i}}$ of $\bar{Q}$ with $e = f_1 + f_2$. Therefore, e is not primitive.

To see that $\bar{Q}$ does not satisfy the conditions of Theorem 2.1, consider a proper $\mathcal{S}$-closed ideal I of $\bar{Q}$. Since I is not essential, there is $x \in \bar{Q}$ such that $x\bar{Q} \cap I = 0$. Because $\bar{Q}$ is regular, we can find an idempotent $e \in \bar{Q}$ with $x\bar{Q} = e\bar{Q}$. By the last paragraph, $e\bar{Q} = f_1\bar{Q} \oplus f_2\bar{Q}$ for non-zero orthogonal idempotents f_1 and f_2. Thus, $\bar{Q}/I$ has Goldie-dimension at least 2, and I cannot be maximal $\mathcal{S}$-closed in $\bar{Q}$. Thus, $\bar{Q}$ does not contain any maximal $\mathcal{S}$-closed submodules.

Replacing F by an integral domain R which is not a field yields an example of a ring without primitive idempotents which is not regular.

To illustrate the significance of the idempotent condition in Theorem 2.1 further, we consider the following result.

PROPOSITION 3.5. *Let R be a right non-singular ring such that every non-zero non-singular right R-module contains a maximal $\mathcal{S}$-closed submodule. A non-singular right R-module M with $\dim_R M > 1$ is injective if all its maximal $\mathcal{S}$-closed submodules are injective. However, this may fail if R does not satisfy the conditions of Theorem 2.1.*

PROOF. Select a maximal $\mathcal{S}$-closed submodule U of M. Because U is injective, $M = U \oplus V$ for some uniform submodule V of M. However, V is contained in a maximal $\mathcal{S}$-closed submodule W of M, which also is injective. Hence, W contains an injective hull E of V. Since V is $\mathcal{S}$-closed in M, we have $V = E$, and M is injective.

To see that this result may fail if R does not satisfy the conditions of Theorem 2.1, observe that the ring $\bar{Q}$ in Example 3.4 is not semi-simple Artinian since it does not contain any primitive idempotents. Therefore, it contains a proper essential right ideal I. If J is any proper $\mathcal{S}$-closed submodule of I, then there is an element $x \in I$ with $J \cap x\bar{Q} = 0$. However, we can argue as in Example 3.4 to show that $x\bar{Q}$ has Goldie-dimension at least 2 because $\bar{Q}$ is regular. Thus, I does not contain any maximal $\mathcal{S}$-closed submodules. Therefore, I is not an injective $\bar{Q}$-module although all its maximal $\mathcal{S}$-closed submodules (there are none!) are injective. □

4. Right Goldie-rings

In this section, we investigate which additional conditions a ring R with the property that all non-singular right R-modules contain maximal $\mathcal{S}$-closed submodules needs to satisfy to ensure that R is a right Goldie-ring. We want to remind the reader that a ring satisfies the ascending chain condition (ACC) for annihilators on one side if and only if it satisfies the descending chain condition (DCC) for annihilators on the other side [2].

Our first result illustrates how [1, Proposition 2.3] fits into the framework of Theorem 2.1.

COROLLARY 4.1. *If R is right non-singular right Goldie-ring, then every non-zero non-singular module contains a maximal $\mathcal{S}$-closed submodule.*

Proof. By [7], Q^r is semi-simple Artinian. Therefore, every non-zero idempotent of Q^r is a sum of primitive idempotents. $\square$

The next result shows that a ring R satisfying any of the conditions of Theorem 2.1 is a right Goldie-ring if Q^r does not have too many primitive idempotents. However, Example 3.4 shows that a ring may not contain any primitive idempotents although it contains infinite families of orthogonal idempotents.

Theorem 4.2. *A right non-singular ring R is a right Goldie-ring if it satisfies any one of the following conditions*:

a) *Q^r has no infinite family of orthogonal primitive idempotents, and every non-zero non-singular right R-module M contains a maximal $\mathcal{S}$-closed submodule*;

b) *R is a right Utumi-ring which satisfies the ACC or DCC for right annihilators, and every non-zero non-singular right R-module M contains a maximal $\mathcal{S}$-closed submodule*;

c) *${}_RR$ is essential in ${}_RQ^r$, and R satisfies the ACC or DCC for right annihilators.*

Proof. a) By Theorem 2.1, $Q^r = e_1Q^r \oplus (1-e_1)Q^r$ for some primitive idempotent e_1 of Q^r. Applying Theorem 2.1 again yields $1-e_1$ is either primitive or $1-e_1 = e_2 + f$ where e_2 is primitive and f an idempotent of Q^r with $fe_2 = e_2f = 0$. Repeating the process, we eventually obtain $1 = e_1 + \cdots + e_n$ for orthogonal primitive idempotents $e_1, \ldots, e_n$ of Q^r since Q^r does not contain any infinite family of orthogonal primitive idempotents. Because $E_i = e_iQ^r$ is an indecomposable injective, it is uniform arguing as in the proof of Theorem 2.1. Thus, Q^r and R have Goldie-dimension n.

b) Since every right R-module contains a maximal $\mathcal{S}$-closed submodule, we can find a descending chain $R = I_0 \supsetneqq I_1 \supsetneqq \cdots \supsetneqq I_n \supsetneqq \cdots$ of $\mathcal{S}$-closed right ideals such that I_n/I_{n+1} is uniform. By Theorem 2.1, we also obtain an ascending chain $0 = J_0 \subsetneqq J_1 \subsetneqq \cdots \subsetneqq J_n \subsetneqq \cdots$ of $\mathcal{S}$-closed right ideals such that J_{n+1}/J_n is uniform. Since R is right Utumi, I_n and J_n are right annihilators for all $n < \omega$. Because R satisfies the ACC or the DCC for right annihilators, at least one of these two chains has to terminate, i.e. either $I_m = 0$ or $J_m = R$ for some $m < \omega$. In either case, R has finite right Goldie-dimension.

c) For $q \in Q^r$, choose an essential right ideal I of R such that $qI \subseteq R$. Then $\mathrm{ann}^\ell_R(qI) = \mathrm{ann}^\ell_{Q^r}(qQ^r) \cap R$ since if $x \in R$ satisfies $xqI = 0$, then $xq = 0$ because Q^r_R is non-singular. Thus, every left ideal of R of the form $\mathrm{ann}^\ell_{Q^r}(qQ^r) \cap R$ is the left annihilator of some subset of R.

Let e and f be non-zero orthogonal idempotents of Q^r, and select an essential right ideal I of R such that $eI, fI \subseteq R$. Since $(e+f)Q^r = eQ^r \oplus fQ^r$, we have

$$\mathrm{ann}^\ell_R((e+f)I) = \mathrm{ann}^\ell_{Q^r}((e+f)Q^r) \cap R \subseteq \mathrm{ann}^\ell_{Q^r}(eQ^r) \cap R = \mathrm{ann}^\ell_R(eI).$$

Suppose $\mathrm{ann}^\ell_R((e+f)I) = \mathrm{ann}^\ell_R(eI)$, and observe $0 \neq Rf \subseteq \mathrm{ann}_{Q^r}(eQ^r)$. Since ${}_RR$ is essential in ${}_RQ^r$, we can find a non-zero $r \in R$ such that $0 \neq rf \in R$. Then

$$rf \in \mathrm{ann}_{Q^r}(eQ^r) \cap R = \mathrm{ann}^\ell_R(eI) = \mathrm{ann}^\ell_R((e+f)I)$$

yields $0 = rf(e+f) = rf^2 = rf$, a contradiction; and $\mathrm{ann}^\ell_R((e+f)I) \subsetneq \mathrm{ann}^\ell_R(eI)$.

If Q^r does not contain any infinite family of orthogonal idempotents, then there are primitive idempotents $e_1, \ldots, e_n$ of Q^r such that $1 = e_1 + \cdots + e_n$. Because e_iQ^r is an indecomposable injective module, it has Goldie-dimension 1 for $i = 1, \ldots, n$. Then $Q^r_R = e_1Q^r \oplus \cdots \oplus e_nQ^r$ yields $\dim_R R = n < \infty$.

Therefore, suppose that Q^r contains an infinite family of orthogonal idempotents $\{e_n \mid n < \omega\}$. If R satisfies the ACC for right annihilators, then setting $f_n = e_1 + \cdots + e_n$ induces a strictly descending chain $\{\mathrm{ann}^\ell_{Q^r}(f_nQ^r) \cap R \mid n < \omega\}$ of left annihilators of R. This is not possible since R also satisfies the DCC for left annihilators.

On the other hand, if R satisfies the DCC for right annihilators, then we consider the idempotents $f_n = 1 - (e_1 + \cdots + e_n)$ instead. We obtain the strictly ascending chain $\{\mathrm{ann}^\ell_{Q^r}(f_nQ^r) \cap R \mid n < \omega\}$ of left annihilators of R since $f_{n+1} + e_n = f_n$ for all $n < \omega$, which contradicts the fact that R also satisfies the DCC for right annihilators. □

The endomorphism ring R of an infinite dimensional vector-space is a regular right self-injective ring which is not left self-injective. Thus, R is its own maximal right ring of quotients, but cannot be its left ring of quotients. Therefore, the fact that ${}_RR$ is essential in ${}_RQ^r$ does not imply that R is right and left Utumi in general. The examples in Sections 2 and 3 also show that the chain conditions on the right annihilators cannot be omitted from Parts b) and c) of the last theorem.

Rings similar to those discussed in the third part of the last theorem were investigated by Faith in [3, Theorem 2.14A and Theorem 2.14B]. However, our result does not assume a priori that R is right and left Utumi in contrast to [3].

A right Goldie-ring R such that ${}_RR$ is essential in ${}_RQ^r$ has to be right and left Utumi since the fact that Q^r is semi-simple Artininan yields that Q^r is also the maximal left ring of quotients of R. Combining this observation with Theorem 4.2 yields the equivalence of a) and b) in

Corollary 4.3. *The following conditions are equivalent for a right non-singular ring R:*

a) *R is a right and left Utumi right and left Goldie-ring*;

b) *R satisfies the ACC or DCC for right annihilators and has the property that ${}_R R$ is essential in ${}_R Q^r$*;

c) *R is a right and left Utumi-ring such that every subset X of R for which* $\mathrm{ann}^r_R(X) = 0$ *contains a finite subset X' with* $\mathrm{ann}^r_R(X') = 0$ [3].

Proof. The equivalence of a) and c) is a direct consequence of [3, Theorem 2.14A and Theorem 2.14B]. □

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