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# Cyclic non-$S$-permutable subgroups and non-normal maximal subgroups

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Abstract – A finite group $G$ is said to be a $T$-group (resp. $PT$-group, $PST$-group) if normality (resp. permutability, $S$-permutability) is a transitive relation. Ballester-Bolinches et al. gave some new characterizations of the soluble $T$-, $PT$- and $PST$-groups. A finite group $G$ is called a $T_c$-group (resp. $PT_c$-group, $PST_c$-group) if each cyclic subnormal subgroup is normal (resp. permutable, $S$-permutable) in $G$. The present work defines the $NNM_c$-, $PNM_c$-, and $SNM_c$-groups and presents new characterizations of the wider classes of soluble $T_c$-, $PT_c$-, and $PST_c$-groups.

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## 1. Introduction

In the present work, all groups are finite. Recall that a subgroup $H$ of a group $G$ is said to be $S$-permutable (or $S$-quasinormal) if $HP = PH$ for all Sylow subgroups $P$ of $G$. Kegel proved that every $S$-permutable subgroup is subnormal. A group $G$ is a $PST$-group if $S$-permutability is a transitive relation (i.e., if $H$ and $K$ are subgroups of $G$ such that $H$ is $S$-permutable in $K$ and $K$ is $S$-permutable in $G$, then $H$ is $S$-permutable in $G$). It follows from Kegel's result that $PST$-groups are exactly those groups in which every subnormal subgroup is $S$-permutable.

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Similarly, groups in which permutability (normality) is transitive relation are called $PT$-groups ($T$-groups) and can be identified with groups in which subnormal subgroups are always permutable (normal). Recall that a group $G$ is a $PST_c$-group if each cyclic subnormal subgroup is $S$-permutable in $G$. The classes of $PT_c$-groups and $T_c$-groups similarly defined as groups in which cyclic subnormal subgroups are permutable or normal, respectively. Kaplan [8] characterized soluble $T$-groups by means of their maximal subgroups and some classes of pre-Frattini subgroups. He proved a necessary and sufficient condition for a group $G$ to be a soluble $T$-group as follows: $G$ is a soluble $T$-group if and only if every non-normal subgroup of every subgroup $H$ of $G$ is contained in a non-normal maximal subgroup of $H$.

Ballester-Bolinches *et al.* [3] extended the results from Kaplan [8] and presented new characterizations for soluble $PT$- and $PST$-groups. The starting point of their results was the following: let $H$ be a proper permutable (resp. $S$-permutable) subgroup of a soluble group $G$. Using Kegel's result, $H$ is subnormal in $G$ and so $H$ is contained in a maximal subgroup of $G$ that is normal in $G$. Following Ballester-Bolinches *et al.* [3] a group $G$ is said to be a $PNM$-group (resp. $SNM$-group) if every non-permutable (resp. non-$S$-permutable) subgroup of $G$ is contained in a non-normal maximal subgroup of $G$. Many interesting results can be obtained using these concepts. For example, they proved that a group $G$ is a soluble $PT$-group (resp. $PST$-group) if and only if every subgroup of $G$ is a $PNM$-group (resp. $SNM$-group). They also showed that if $G$ is an $SNM$-group, then the nilpotent residual $G^{\mathfrak{N}}$ is supersoluble if and only if $G$ is supersoluble. Consequently, if $G$ is a group whose non-nilpotent subgroups are $SNM$-groups, then $G$ is supersoluble.

Now, we define that a group $G$ is a $PNM_c$-groups (resp. $SNM_c$-groups) if every cyclic non-permutable (resp. non-$S$-permutable) subgroup is contained in a non-normal maximal subgroup. The aim of this paper is to present new characterizations of the wider classes of soluble $T_c$-, $PT_c$-, and $PST_c$-groups. We begin with the following definition.

Definition 1.1. A group $G$ is called an $NNM_c$-group (resp. $PNM_c$-group, $SNM_c$-group) if every cyclic non-normal (resp. non-permutable, non-$S$-permutable) subgroup of $G$ is contained in a non-normal maximal subgroup of $G$.

## 2. Preliminaries

We first collect results from Ballester-Bolinches *et al.* [3], as the starting point of our results.

THEOREM 2.1. *A group $G$ is a soluble $PST$-group if and only if every subgroup of $G$ is an $SNM$-group.*

LEMMA 2.2. *Every subgroup of a group $G$ is a $PNM$-group if and only if every subgroup of $G$ is an $SNM$-group and all Sylow subgroups of $G$ are Iwasawa groups.*

It can be concluded by applying Theorem 2.1 and Lemma 2.2 that:

COROLLARY 2.3. *A group $G$ is a soluble $PT$-group if and only if every subgroup of $G$ is a $PNM$-group.*

Every subgroup of a group $G$ is an $NNM$-group if and only if every subgroup of $G$ is an $SNM$-group and all Sylow subgroups are Dedekind; thus, it can be concluded:

COROLLARY 2.4. *A group $G$ is a soluble $T$-group if and only if every subgroup of $G$ is an $NNM$-group.*

THEOREM 2.5. *If $G$ is an $SNM$-group, then the nilpotent residual $G^{\mathfrak{N}}$ is supersoluble if and only if $G$ is supersoluble.*

For the sake of easy reference, theorems from Robinson [9] have been provided. These results provide detailed information on the structure of a soluble $PST_c$-group.

THEOREM 2.6. *Let $G$ be a soluble $PST_c$-group with $F = \mathrm{Fit}(G)$ and $L = \gamma_\infty(G)$. Then the following hold*:

1) *$L$ is an abelian group of odd order*;
2) *$p'$-elements of $G$ induce power automorphisms in $L_p$ for all primes $p$*;
3) $F = C_G(L)$;
4) *$G$ splits conjugately over $L$*;
5) $F = \bar{Z}(G) \times L$;
6) $\pi(L) \cap \pi(F/L) = \emptyset$;
7) *$G$ is supersoluble.*

*Where $\gamma_\infty(G)$ is the hypercommutator subgroup or the limit of the lower central series,* $\mathrm{Fit}(G)$ *is the Fitting subgroup, and $\pi(G)$ is the set of prime divisors of the group order.*

The class of soluble $PST_c$-groups is neither subgroup nor quotient closed, which is in contrast to the behavior of soluble $PST$-groups. Robinson [9] proved:

THEOREM 2.7. *If every subgroup of a group $G$ is a $PST_c$-group, then $G$ is a soluble $PST$-group.*

THEOREM 2.8. *Let $G$ be a soluble group. If every quotient of $G$ is a $PST_c$-group, then $G$ is a $PST$-group.*

## 3. Main Results

THEOREM 3.1. (1) *Let every non-normal maximal subgroup $M$ of a group $G$ does not have a non-cyclic supplement in $G$. If every subgroup of $G$ is an $SNM_c$-group, then $G$ is a soluble $PST_c$-group.*

(2) *If every subgroup of $G$ is a $PST_c$-group, then every subgroup of $G$ is an $SNM_c$-group.*

PROOF. (1) Assume that the theorem is not true and let $G$ be a counterexample of minimal order. Then every proper subgroup of $G$ is a soluble $PST_c$-group. Using Theorem 2.6(7), every proper subgroup of $G$ is supersoluble and so $G$ is soluble.

On the other hand, there exists a cyclic subnormal subgroup $H$ of $G$ which is not $S$-permutable. Let $M$ be a maximal normal subgroup of $G$ containing $H$. There exists a non-normal maximal subgroup $L$ of $G$ containing $H$, since $G$ is an $SNM_c$-group. It is clear that $G = ML$. Since $H$ is not $S$-permutable in $G$, it follows that there exists a Sylow $p$-subgroup $P$ of $G$ such that $P$ does not permute with $H$. The choice of the minimality of $G$ implies that $H$ is $S$-permutable in $M$ and $L$. Using Corollary 1.3.3 of [1], there exist Sylow $p$-subgroups $M_0$ of $M$ and $L_0$ of $L$ where $P_0 = M_0L_0$ is a Sylow $p$-subgroup of $G$. Let $g \in G$ such that $P^g = P_0$. Hence $H$ permutes with both $M_0$ and $L_0$ and so $H$ permutes with $P_0$. Let $N$ be a minimal normal subgroup of $G$ contained in $M$. Since the factor group $G/N$ satisfies the hypothesis and $|G/N| < |G|$, then $HN$ permutes with $P$. If $(HN)P$ is a proper subgroup of $G$, then $H$ will permute with $P$. This is a contradiction. Therefore, $G = P(HN)$ and $g = xy$ such that $x \in P$ and $y \in HN$. Using Lemma 14.3.A of [5], $H$ is a normal subgroup of $HN$. Since $HP^g = P^gH$, it follows that $H^{y^{-1}} = H$ permutes with $P$, which is contrary to the assumption.

(2) It is clear. □

LEMMA 3.2. *Every subgroup of a group $G$ is a $PNM_c$-group if and only if every subgroup of $G$ is an $SNM_c$-group and all Sylow subgroups of $G$ are Iwasawa groups.*

PROOF. Assume that every subgroup of $G$ is a $PNM_c$-group. It is clear that every subgroup of $G$ is also an $SNM_c$-group. Moreover, every Sylow subgroup $P$ of $G$ is a nilpotent $PNM_c$-group. Let $H$ be a subgroup of $P$ such that $H$ is not permutable in $P$. If $H$ is cyclic, then there exists a non-normal maximal subgroup $M_1$ of $P$ such that $H \subseteq M_1$, which is a contradiction. If $H$ is non-cyclic, then $H = M\langle x\rangle$ where $M$ is a maximal subgroup of $H$ of prime index and $x \in H - M$. Either $M$ or $\langle x\rangle$ will not permute in $P$. If $\langle x\rangle$ does not permute, then there exists a non-normal maximal subgroup $M_2$ of $P$ such that $\langle x\rangle \subseteq M_2$, which is a contradiction. If $M$ does not permute in $P$, by the same argument, we have a contradiction. Hence $H$ must be permutable in $P$. This means that $P$ is an Iwasawa group.

Conversely, assume that every subgroup of $G$ is an $SNM_c$-group and all Sylow subgroups of $G$ are Iwasawa groups. Let $K$ be a cyclic $S$-permutable subgroup of a subgroup $H$ of $G$. Because all Sylow subgroups of $H$ are also Iwasawa groups, we can apply Theorem 2.1.10 of [2] to conclude that $K$ is permutable in $H$. Hence $H$ is a $PNM_c$-group. Consequently every subgroup of $G$ is a $PNM_c$-group. □

COROLLARY 3.3. (1) *Let every non-normal maximal subgroup $M$ of a group $G$ does not have a non-cyclic supplement in $G$. If every subgroup of $G$ is a $PNM_c$-group, then $G$ is a soluble $PT_c$-group.*

(2) *If every subgroup of $G$ is a soluble $PT_c$-group, then every subgroup of $G$ is a $PNM_c$-group.*

PROOF. (1) If every subgroup of $G$ is a $PNM_c$-group, then every subgroup of $G$ is an $SNM_c$-group according to Lemma 3.2 and so $G$ is a soluble $PST_c$-group. This implies that every cyclic subnormal subgroup $H$ of $G$ is $S$-permutable in $G$. Applying Theorem 2.1.10 of [2], we see that $H$ is permutable in $G$, since all Sylow subgroups of $G$ are Iwasawa groups. Thus $G$ is a soluble $PT_c$-group.

(2) It is clear. □

LEMMA 3.4. *Every subgroup of a group $G$ is an $NNM_c$-group if and only if every subgroup of $G$ is an $SNM_c$-group and all Sylow subgroups of $G$ are Dedekind groups.*

Proof. Let every subgroup of $G$ be an $NNM_c$-group. It is clear that $G$ is an $SNM_c$-group. Let $H$ be a non-normal subgroup of $P$ where $P \in \mathrm{Syl}(G)$. If $H$ is cyclic, then there exists a non-normal maximal subgroup $M_1$ of $P$ such that $H \subseteq M_1$, which is a contradiction. If $H$ is non-cyclic, then $H = M\langle x\rangle$ where $M$ is a maximal subgroup of $H$ of prime index and $x \in H - M$. Either $M$ or $\langle x\rangle$ is not normal in $P$, since $H$ is not normal in $P$. If $\langle x\rangle$ is not normal in $P$, then there exists a non-normal maximal subgroup $M_2$ of $P$ such that $\langle x\rangle \subseteq M_2$ , which is a contradiction. If $M$ is not normal in $P$, we have a similar contradiction. Thus $P$ is a Dedekind group.

Conversely, let every subgroup of $G$ be an $SNM_c$-group and every Sylow subgroup of $G$ be a Dedekind group. Let $K$ be an $S$-permutable subgroup of $H$ such that $H \leq G$. Applying Theorem 2.1.10 of [2], we see that $K$ is normal in $H$, since all Sylow subgroups of $H$ are also Dedekind groups. Hence $H$ is an $NNM_c$-group. The above argument implies that every subgroup of $G$ is an $NNM_c$-group. □

Corollary 3.5. (1) *Let every non-normal maximal subgroup $M$ of a group $G$ does not have a non-cyclic supplement in $G$. If every subgroup of $G$ is an $NNM_c$-group, then $G$ is a soluble $T_c$-group.*

(2) *If every subgroup of $G$ is a soluble $T_c$-group, then every subgroup is an $NNM_c$-group.*

Proof. (1) If every subgroup of $G$ is an $NNM_c$-group, then every subgroup of $G$ is an $SNM_c$-group and all Sylow subgroups of $G$ are Dedekind groups. Thus $G$ is a soluble $PT_c$-group. This implies that every cyclic subnormal subgroup $H$ of $G$ is permutable in $G$. Applying Theorem 2.1.10 of [2], we see that $H$ is normal in $G$, since all Sylow subgroups of $G$ are Dedekind groups. Thus $G$ is a soluble $T_c$-group.

(2) It is clear. □

Theorem 3.6. *Let $G$ and each quotient group of $G/N$ be an $SNM_c$-group. Then $G^{\mathfrak{N}}$ is supersoluble if and only if $G$ is supersoluble.*

Proof. The sufficiency of the condition is evident; we need only prove the necessity of the condition. We use induction on the order of $G$. Let $N$ be a minimal normal subgroup of $G$. Then $G^{\mathfrak{N}}N/N$ is the nilpotent residual of $G/N$ according to Proposition 2.2.8 (1) of [4]. Moreover, $G^{\mathfrak{N}}N/N$ is supersoluble and according to the hypothesis, $G/N$ is an $SNM_c$-group. By induction, $G/N$ is supersoluble. Since the class of all supersoluble groups is a saturated formation, we can suppose that $G$ has an unique minimal normal subgroup $N$ and $\Phi(G) = 1$. This means that $N = C_G(N)$ in addition $G = MN$, $M \cap N = 1$ and $\mathrm{Core}_G(M) = 1$. Let $p$ be the prime dividing $|N|$. Then $N$ has the structure of a semisimple $KG^{\mathfrak{N}}$-module where $K$ is the field of $p$ elements. Therefore, $N$ is a direct product of the minimal normal subgroups of $G^{\mathfrak{N}}$. Let $A$ be a minimal normal subgroup of $G^{\mathfrak{N}}$ contained in $N$. Then $A$ has order $p$ because $G^{\mathfrak{N}}$ is supersoluble. If $AM^{\mathfrak{N}} = \langle a \rangle M^{\mathfrak{N}}$ is not $S$-permutable in $G$, then there exists a non-normal maximal subgroup $L$ of $G$ containing $AM^{\mathfrak{N}}$. Since $A \leq L \cap N$, it follows that $N$ is contained in $L$. In particular, $G^{\mathfrak{N}}$ is contained in $L$ and $L$ is normal in $G$. This contradiction shows that $AM^{\mathfrak{N}}$ is $S$-permutable in $G$. It implies that $AM^{\mathfrak{N}}$ is subnormal in $G$ and so $N$ normalizes $AM^{\mathfrak{N}}$ according to Lemma 14.3.A of [5]. It follows that $[M^{\mathfrak{N}}, N] \leq AM^{\mathfrak{N}} \cap N = A$, which holds for every minimal normal subgroup of $G^{\mathfrak{N}}$ contained in $N$.

If $A = N$, then $N$ is of prime order and $G$ is supersoluble. Hence $N$ is a direct product of at least two minimal normal subgroups of $G^{\mathfrak{N}}$. In this case, $M^{\mathfrak{N}}$ centralizes $N$ and $M^{\mathfrak{N}} = 1$. Therefore, every subgroup of $N$ is $S$-permutable in $G$. According to Lemma 2.1.3 of [2], it follows that $N$ is of prime order. Hence $G$ is supersoluble. This establishes the theorem. □

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