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A new characterization of some families of finite simple groups

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Abstract – Let G be a finite group. A vanishing element of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G. Denote by $\mathrm{Vo}(G)$ the set of the orders of vanishing elements of G. In this paper, we prove that if G is a finite group such that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ and $|G| = |M|$, then $G \cong M$, where M is a sporadic simple group, an alternating group, a projective special linear group $L_2(p)$, where p is an odd prime or a finite simple K_n-group, where $n \in \{3, 4\}$. These results confirm the conjecture posed in [17] for the simple groups under study.

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1. Introduction

Let G be a finite group. A *vanishing element* of G is an element $g \in G$ such that $\chi(g) = 0$ for some irreducible complex character χ of G. We will denote by $\mathrm{Van}(G)$ the set of vanishing elements of G and by $\mathrm{Vo}(G)$ the set of the orders of elements in $\mathrm{Van}(G)$. According to [3] and [14], we know that the set $\mathrm{Vo}(G)$ can release some information about the structure of a finite group G. For instance, Theorem C of [15] as a strengthening of (Corollary 3, [14]) states that if p is a

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prime divisor of $|G|$ and G does not have any vanishing element of order divisible by p, then G has a normal Sylow p-subgroup. It is also shown in [36] that if G is a finite group such that $\mathrm{Vo}(G) = \mathrm{Vo}(A_5)$, then $G \cong A_5$, i.e., the alternating group A_5 is characterizable by the set of orders of vanishing elements. According to this result, one may ask the following question:

> *are all finite nonabelian simple groups characterizable by the set of orders of vanishing elements*?

The answer to this question is not affirmative in general. For example, for the simple linear group $L_3(5)$, we have $\mathrm{Vo}(L_3(5)) = \mathrm{Vo}(\mathrm{Aut}(L_3(5))$ but $L_3(5) \ncong \mathrm{Aut}(L_3(5))$ because $|L_3(5)| \neq |\mathrm{Aut}(L_3(5))|$. Therefore, in [17], the following conjecture was put forward.

Conjecture. *Let G be a finite group and let M be a finite nonabelian simple group. If $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ and $|G| = |M|$, then $G \cong M$.*

Also, in [17], an affirmative answer was given to this conjecture for the simple groups $L_2(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, A_7, $Sz(8)$ and $Sz(32)$. In this paper, we first prove that the conjecture is confirmed for all sporadic simple groups, the alternating groups and projective special linear group $L_2(p)$, where p is an odd prime. So, we have the following result.

Theorem A. *Let G be a finite group and M be a sporadic simple group, an alternating group or a projective special linear group $L_2(p)$, where p is an odd prime. If $|G| = |M|$ and $\mathrm{Vo}(G) = \mathrm{Vo}(M)$, then $G \cong M$.*

The finite simple group G is called a K_n-group if its order has exactly n distinct prime divisors, where $n \in \mathbb{N}$. The following lemma determines all K_n-groups, where $n \in \{3, 4\}$:

Lemma 1.1 ([4], [18], [30], [35]). *Let G be a finite simple K_n-group.*

(1) *If $n = 3$, then G is isomorphic to one of the following groups*:

$$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2).$$

(2) *If $n = 4$, then G is isomorphic to one of the following groups*:

(a) $A_7,\ A_8,\ A_9,\ A_{10},\ M_{11},\ M_{12},\ J_2,\ L_2(16),\ L_2(25),\ L_2(49),\ L_2(81),\ L_2(97),\ L_2(243),\ L_2(577),\ L_3(4),\ L_3(5),\ L_3(7),\ L_3(8),\ L_3(17),\ L_4(3),\ S_4(4),\ S_4(5),\ S_4(7),\ S_4(9),\ S_6(2),\ O_8^+(2),\ G_2(3),\ U_3(4),\ U_3(5),\ U_3(7),\ U_3(8),\ U_3(9),\ U_4(3),\ U_5(2),\ Sz(8),\ Sz(32),\ {}^3D_4(2),\ {}^2F_4(2)'$;

(b) $L_2(r)$, *where* r *is a prime*, $r^2-1 = 2^a.3^b.v$, $v > 3$ *is a prime*, $a, b \in \mathbb{N}$;

(c) $L_2(2^m)$, *where* m, $2^m - 1$ *and* $(2^m + 1)/3$ *are primes greater than* 3;

(d) $L_2(3^m)$, *where* m , $(3^m + 1)/4$ *and* $(3^m - 1)/2$ *are odd primes.*

As a second result of this paper, we show the validity of the conjecture for the groups listed in Lemma 1.1. In fact, we have the following result.

THEOREM B. *Let* G *be a finite group and let* M *be a simple* K_3*-group or a simple* K_4*-group. If* $|G| = |M|$ *and* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$*, then* $G \cong M$.

Throughout this paper, we use the following notation. Let G be a finite group, p be a prime number and m be a positive integer. The number of Sylow p-subgroups of G is denoted by $n_p(G)$. Also, $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G. The notation $p^m \parallel |G|$ means that p^m divides $|G|$ but p^{m+1} does not divide $|G|$. Also, by $\omega(G)$ we denote the set of orders of elements of group G. All further notation is standard and can be found in [12], for instance.

2. Preliminaries

One of the main keys for the proof of our results is a result by Dolfi, et al. in [15] on the vanishing prime graph of a finite group and its relationship with the Gruenberg–Kegel graph. For this reason, we will recall the required definitions in the following.

Given a finite set of positive integers X, the prime graph $\Pi(X)$ is defined as the simple undirected graph whose vertices are the primes p such that there exists an element of X divisible by p, and two distinct vertices p, q are adjacent if and only if there exists an element of X divisible by pq. For a finite group G, the graph $\Pi(\omega(G))$, which we denote by $GK(G)$ is also known as the Gruenberg–Kegel graph of G. Also, the prime graph $\Pi(\mathrm{Vo}(G))$, which in this paper we denote by $\Gamma(G)$, is called the vanishing prime graph of G.

We denote by $t(G)$ the number of connected components of $GK(G)$ and by $\pi_i(G)$, $i = 1, 2, \ldots, t(G)$, the ith connected component of $GK(G)$. If the order of G is even, we set $2 \in \pi_1(G)$. We also, denote by $\pi(n)$ the set of all primes dividing n, where n is a natural number. Now $|G|$ can be expressed as the product of $m_1, m_2, \ldots, m_{t(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i(G)$. We call $m_1, m_2, \ldots, m_{t(G)}$ the order components of G and we write $\mathrm{OC}(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$, the set of order components of G. A finite simple group S is said to be characterizable by its order components, if $S \cong G$ for each finite group G such that $\mathrm{OC}(G) = \mathrm{OC}(S)$.

A 2-Frobenius group is a group G that has proper normal subgroups K and L such that L is a Frobenius group with kernel K and G/K is a Frobenius group with kernel L/K. The following lemma determines the structure of the finite group with disconnected Gruenberg–Kegel graph:

Lemma 2.1 ([31]). *Let G be a finite group. If $t(G) \geq 2$, then the structure of G is as follows.*

(1) *G is either a Frobenius group or a 2-Frobenius group.*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and K/H is a nonabelian simple group.*

Lemma 2.2 ([8]). *Let G be a Frobenius group of even order with kernel F and complement H. Then*

(1) *$t(G) = 2$, $\{\pi_1(G), \pi_2(G)\} = \{\pi(|H|), \pi(|F|)\}$;*

(2) *if H is a nonsolvable group, then there exists $H_0 \leq H$ such that $H_0 = L_2(5) \times Z$, where $(2\cdot 3\cdot 5, |Z|) = 1$ and the Sylow subgroups of Z are cyclic.*

Lemma 2.3 ([5]). *If G is a 2-Frobenius group with normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, then*

(1) *$t(G) = 2$, $\pi_1(G) = \pi(|G/K|) \cup \pi(|H|)$ and $\pi_2(G) = \pi(|K/H|)$;*

(2) *G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq \mathrm{Aut}(K/H)$;*

(3) *G is solvable.*

A group G is said to be a nearly 2-Frobenius group if there exist two normal subgroups F and L of G with the following properties: $F = F_1 \times F_2$ is nilpotent, where F_1 and F_2 are normal subgroups of G, furthermore G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group.

Lemma 2.4 ([15], [16], [24]). (1) *If G is a finite nonabelian simple group, then $GK(G) = \Gamma(G)$, unless $G \cong A_7$.*

(2) *If G is a solvable Frobenius group with Frobenius kernel F and Frobenius complement H, then either $GK(G) = \Gamma(G)$ or $\Gamma(G)$ coincides with the connected component of $GK(G)$ with vertex set $\pi(|H|)$.*

(3) *If G is a solvable group, then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius group or a nearly 2-Frobenius group.*

(4) *Let G be a solvable group with a Fitting subgroup $F(G)$. If x is a non-vanishing element of G, then $xF(G)$ is a 2-element of $G/F(G)$.*

(5) *Let N be a normal subgroup of G. If $xN \in \mathrm{Van}(G/N)$, then $xN \subseteq \mathrm{Van}(G)$.*

Lemma 2.5. (1) *Let S be a simple group with disconnected Gruenberg–Kegel graph, except $U_4(2), U_5(2)$. If G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S)$, then G is neither Frobenius nor 2-Frobenius.*

(2) *Let $S \in \{U_4(2), U_5(2)\}$. If G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(S)$, then G is a 2-Frobenius group or $G \cong S$.*

Proof. (1) is Main Theorem of [28]. Also, according to [28], there are 2-Frobenius groups U and W with $\mathrm{OC}(U) = \mathrm{OC}(U_4(2))$ and $\mathrm{OC}(W) = \mathrm{OC}(U_4(2))$. If G is a finite group with $\mathrm{OC}(G) = \mathrm{OC}(U_4(2)) = \{2^6.3^4, 5\}$ and G is not a 2-Frobenius group, then by (Theorem 1, [28]) and Lemma 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and K/H is a nonabelian simple group. Since $|G| = 2^6.3^4.5$, according to [33], $K/H \cong A_5, A_6$ or $U_4(2)$. If $K/H \cong A_5, A_6$, then since $G/H \leq \mathrm{Aut}(K/H)$, we have $3 \mid |H|$. Let $H_3 \in \mathrm{Syl}_3(H)$ and $G_5 \in \mathrm{Syl}_5(G)$. Thus $|H_3| = 3^i$, where $i = 2$ or 3. Since G does not have an element of order 15, we can conclude that G_5 acts fixed point freely on H_3 and hence, $5 \mid (3^i - 1)$ $(i = 2, 3)$, a contradiction. Thus $K/H \cong U_4(2)$ which implies that $G \cong U_4(2)$, as desired. If $\mathrm{OC}(G) = \mathrm{OC}(U_5(2)) = \{2^{10}.3^5.5, 11\}$, and G is not a 2-Frobenius group, then a similar argument implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(|H|) \cup \pi(|G/K|) \subseteq \pi_1(G)$, H is nilpotent and $K/H \cong L_2(11), M_{11}, M_{12}$ or $U_5(2)$. So, it is enough to replace the roles of 5 and 11 in the previous argument to get $G \cong U_5(2)$. □

Let p be a prime number. Recall that a character χ in $\mathrm{Irr}(G)$ is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer (Theorem 8.17, [23]) if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

Lemma 2.6 (Proposition 2.1, [14]). *Let S be a nonabelian simple group and p a prime number. If S is of Lie type, or if $p \geq 5$, then there exists $\chi \in \mathrm{Irr}(S)$ of p-defect zero.*

Remark 2.7. If χ vanishes on a p-element of G, then $\chi(1)$ is divisible by p.

Proof. According to (Corollary 22.26, [25]) the proof is straightforward. □

Lemma 2.8 ([32]). *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic nonabelian simple groups and $G/K \leq \mathrm{Out}(K/H)$.*

Lemma 2.9. *Let G be a finite group of even order. Suppose that there exists $p \in \pi(|G|)$ such that p and 2 are nonadjacent in $\mathrm{GK}(G)$. If G is nonsolvable, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\mathrm{Out}(K/H)|$ and $K/H \leq G/H \leq \mathrm{Aut}(K/H)$.*

Proof. According to Theorem 3 in [11] and the proof of Lemma 1 in [32], the proof is straightforward. □

Lemma 2.10 (Theorem 1, [2]). *Let G be a finite nonsolvable simple group whose order g is divisible by $p > g^{\frac{1}{3}}$. Then G is isomorphic either to $L_2(p)$, where $p > 3$ is a prime or $L_2(p-1)$, where $p > 3$ is a Fermat prime.*

3. Main Results

The following general results play a role in the proof of Theorems A and B.

Lemma 3.1. *Let G be a finite group and let S be a finite simple group with disconnected Gruenberg–Kegel graph such that $S \ncong A_7$ and there exists $2 \leq i \leq t(S)$ such that for every $p \in \pi_i(S)$, we have $p \parallel |S|$. If $\mathrm{Vo}(G) = \mathrm{Vo}(S)$ and $|G| = |S|$, then $m_i(S) \in \mathrm{OC}(G)$. Particularly, the Gruenberg–Kegel graph of G is disconnected.*

Proof. According to Lemma 2.4(1) and the fact that $\mathrm{Vo}(G) = \mathrm{Vo}(S)$, we have $\Gamma(G) = \Gamma(S) = GK(S)$. Since $|G| = |S|$, there exists $2 \leq i \leq t(S)$ such that for every $p \in \pi_i(S)$, we have p divides $|G|$ and p^2 does not divide $|G|$. Suppose the assertion of the lemma is false. Thus there exists $q \in \pi_j(S)$, where $1 \leq j \leq t(S)$ and $i \neq j$, such that p and q are adjacent in $GK(G)$. Since $p \mid |S|$, according to Lemma 2.6 and the fact that $\mathrm{Vo}(S) = \mathrm{Vo}(G)$, we have $p \in \mathrm{Vo}(G)$. So G has an element g of order p such that $\chi(g) = 0$ for some irreducible complex character χ of G. Now, Remark 2.7 implies that p divides $\chi(1)$. Since $p \parallel |S|$ and $|S| = |G|$, χ is an irreducible character of p-defect zero of G. Thus p and q are adjacent in $\Gamma(G)$, which is a contradiction to the fact that $\Gamma(G) = \Gamma(S) = GK(S)$. □

According to the above lemma, we have the following corollary.

Corollary 3.2. *Let G be a finite group and S be a finite simple group with disconnected Gruenberg–Kegel graph except A_7. Assume that for every $p \in \pi_i(S)$, where $2 \leq i \leq t(S)$, we have $p \parallel |S|$. If $\mathrm{Vo}(G) = \mathrm{Vo}(S)$ and $|G| = |S|$, then $\mathrm{OC}(G) = \mathrm{OC}(S)$.*

Proof of Theorem A. The proof of Theorem A falls naturally into three parts.

Part 1. Let M be a sporadic simple group. Then according to [31], the Gruenberg–Kegel graph components of M are shown in Table 1 and hence, M and G satisfy the conditions of Corollary 3.2. Thus according to [6], we have $G \cong M$.

Table 1. The Gruenberg–Kegel graph components of some simple groups

M	Restriction on M	$\pi_1(M)$	m_2	m_3	m_4	m_5	m_6
A_n	$6 < n = p, p+1, p+2$ not both $n, n-2$ prime	$\frac{n!}{2p}$	p				
A_p	$6 < p$ $p, p-2$ are primes	$\frac{(p)!}{2p(p-2)}$	p	$p-2$			
M_{12}		$\{2,3,5\}$	11				
J_2		$\{2,3,5\}$	7				
Ru		$\{2,3,5,7,13\}$	29				
He		$\{2,3,5,7\}$	17				
McL		$\{2,3,5,7\}$	11				
Co_1		$\{2,3,5,7,11,13\}$	23				
Co_3		$\{2,3,5,7,11\}$	23				
Fi_{22}		$\{2,3,5,7,11\}$	13				
HN		$\{2,3,5,7,11\}$	19				
$L_2(q)$	$3 < q \equiv \varepsilon (\mathrm{mod} 4), \varepsilon = \pm 1$	$\pi(q-\varepsilon)$	$\pi(q)$	$\frac{q+\varepsilon}{2}$			
$L_2(q)$	$3 < q$, q even	$\{2\}$	$q-1$	$q+1$			
$L_3(4)$		$\{2\}$	3^2	5	7		
$L_3(q)$	$q \neq 2, 4$	$\pi(q(q^2-1))$	$\frac{q^3-1}{(q-1)(3,q-1)}$				
$L_4(3)$		$\{2,3,5\}$	13				
$S_4(q)$		$\pi(q(q^2-1))$	$\frac{q^2+1}{(2,q-1)}$				
$S_6(2)$		$\{2,3,5\}$	7				
$O_8^+(2)$		$\{2,3,5\}$	7				
$G_2(3)$		$\{2,3\}$	7	13			
$U_3(q)$		$\pi(q(q^2-1))$	$\frac{q^3+1}{(q+1)(3,q+1)}$				
$U_4(2)$		$\{2,3\}$	5				
$U_4(3)$		$\{2,3\}$	5	7			
$U_5(2)$		$\{2,3,5\}$	11				
${}^3D_4(2)$		$\{2,3,7\}$	13				
${}^2F_4'(2)$		$\{2,3,5\}$	13				
M_{11}		$\{2,3\}$	5	11			
M_{23}		$\{2,3,5,7\}$	11	23			
M_{24}		$\{2,3,5,7\}$	11	23			
J_3		$\{2,3,5\}$	17	19			
HiS		$\{2,3,5\}$	7	11			
Suz		$\{2,3,5,7\}$	11	13			
Co_2		$\{2,3,5,7\}$	11	23			
Fi_{23}		$\{2,3,5,7,11,13\}$	17	23			
F_3		$\{2,3,5{\cdot}7,13\}$	19	31			
F_2		$\{2,3,5,7,11,13,17,19,23\}$	31	47			
M_{22}		$\{2,3\}$	5	7	11		
J_1		$\{2,3,5\}$	7	11	19		
$O'N$		$\{2,3,5,7\}$	11	19	31		
LyS		$\{2,3,5,7,11\}$	31	37	67		
Fi_{24}'		$\{2,3,5,7,11,13\}$	17	23	29		
F_1		$\{2,3,5,7,11,13,17,19,23,$ $29,31,47\}$	41	59	71		
J_4		$\{2,3,5,7,11\}$	23	29	31	37	43

Part 2. Let $M = A_n$ be an alternating group. If $\mathrm{GK}(G)$ is not connected, then according to Table 1, one of the numbers n, $n-1$ or $n-2$ is prime. Thus Corollary 3.2 and [1] imply that $G \cong M$. So, to complete the proof, we should consider the case $\mathrm{GK}(G)$ is connected, i.e., $n, n-1$ and $n-2$ are not primes. We will prove the cases $n = 10$ and $n \geq 16$, separately.

- If $n = 10$, then $\mathrm{Vo}(G) = \{2, \dots, 10, 12, 15, 21\}$ and $|G| = 2^7.3^4.5^2.7$. Since 7 divides $|G|$ and 7^2 does not divide $|G|$, Remark 2.7 implies that G has an irreducible character of 7-defect zero. Thus G does not have any element of order 14. Now we claim that G is nonsolvable. If not, then G has a subgroup K of order 35. We can easily see that K is nilpotent and hence, G has an element of order 35. But this is a contradiction to the fact that G has an irreducible character of 7-defect zero and $35 \notin \mathrm{Vo}(G)$.

  Now from Lemma 2.9 we deduce that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\mathrm{Out}(K/H)|$ and $K/H \leq G/H \leq \mathrm{Aut}(K/H)$. According to $|G|$ and [33], K/H is one of the simple groups A_n, where $n \in \{5, 6, 7, 8, 9, 10\}$, $U_4(2)$, $L_3(4)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, J_2. Moreover, we know that in these cases, $\mathrm{Out}(K/H)$ is a $\{2, 3\}$-group. So we have the following three characterizable cases.

  Case 1. If 7 does not divide $|K/H|$, then K/H is one of the groups A_5, A_6 or $U_4(2)$. In this case, we can easily see that $|H| = 35k$, where 35 and k are coprimes. Let P be a Sylow 7-subgroup H, then the Frattini argument implies that $G = HN_G(P)$ and hence, $5 \mid |C_G(P)|$. Thus G has an element of order 35. But this is a contradiction to the fact that G has an irreducible character of 7-defect zero and $35 \notin \mathrm{Vo}(G)$.

  Case 2. If 7 divides $|K/H|$ and 5 divides $|H|$, then K/H is one of the simple groups A_n, where $n \in \{7, 8, 9\}$, $L_3(4)$, $L_2(7)$, $L_2(8)$, $U_3(3)$. Let P be a Sylow 5-subgroup H, then the Frattini argument implies that $G = HN_G(P)$. Since $7 \mid |G/H|$, $7 \mid |N_G(P)|$. Now we can see that $7 \mid |C_G(P)|$. Thus G has an element of order 35 and we can get a contradiction similar to Case 1.

  Case 3. If 7 divides $|K/H|$ and 5 does not divide $|H|$, then according to $|\mathrm{Out}(K/H)|$, $K/H = J_2, A_{10}$. Let $K/H = A_{10}$. According to $|G|$, we can easily conclude that $G \cong A_{10}$. Let $K/H = J_2$. Since $|G/K| \mid |\mathrm{Out}(K/H)| = 2$ and $|G|/|K/H| = 3$, we conclude that G is a central extension of a group of order 3 by J_2. Also, according to the order of the Schur Multiplier of J_2, we have this extension splits. Thus $G = C_3 \times J_2$, where C_3 is the cyclic group of order 3. It is easy to see that in this case $30 \in \mathrm{Vo}(C_3 \times J_2)$, which is a contradiction to the fact that $30 \notin \mathrm{Vo}(G)$.

- Let $n \geq 16$ and r_n be the largest prime not exceeding n. Since Remark 2.7 enables us to follow the proofs in [29] to conclude $G \cong M$, here we just mention the sketch of the proof in the following three steps.

Step 1. In this step, we prove that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group and $t_n(1) \mid |K/H|$. ($t_n(k) = \prod_{i=1}^{k}(\prod_{\frac{n}{i+1}<p\leq\frac{n}{i}} p_i)^i$, where p_j is defined as 1 if there is no prime between $\frac{n}{j+1}$ and $\frac{n}{j}$.)

Let $1 = H_0 < H_1 < \cdots < H_m = G$ be a chief series of G. Suppose p is a prime dividing $t_n(1)$. Since $p \parallel |G|$, we can assume $p \mid |H_{i+1}/H_i|$ and $p \nmid |H_i|$. Moreover, we can assume that $p' \nmid |H_i|$, for every $p' \mid t_n(1)$. Put $K := H_{i+1}$ and $H := H_i$. Since K/H is a direct product of isomorphic simple groups and $p \parallel |K/H|$, K/H is a group of order p or a nonabelian simple group. If K/H is cyclic, then $\frac{G/H}{C_{G/H}(K/H)}$ is embedded in the cyclic group of order $p-1$. Since $n \geq 16$, there is a prime q ($q \neq p$) such that $q \mid t_n(1)$. An easy calculation shows that $q \nmid (p-1)$ and $p \nmid (q-1)$. Thus $q \in \pi(|C_{G/H}(K/H)|)$ which implies $pq \in \omega(G)$. Since $p \parallel |G|$, Remark 2.7 yields $pq \in \mathrm{Vo}(G) = \mathrm{Vo}(A_n) = \omega(A_n)$, which is a contradiction to the fact that $p+q > n$. Therefore, K/H is a nonabelian simple group. To complete the proof of this step, let $p' \mid t_n(1)$ and $p' \nmid |K/H|$. Thus $p' \mid |G/K|$ and by the Frattini argument, we have $G = N_G(P)K$, where $P \in \mathrm{Syl}_{p'}(K)$. This implies that G has a subgroup of order pp' which is a contradiction, because $p' \nmid (p-1)$, $p \nmid (p'-1)$ and $pp' \notin \omega(G)$.

Step 2. Let $16 \leq n \leq 82$ and assume that $n, n-1$ and $n-2$ are not primes. According to step 1 and [33], we can see that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong A_m$, $r_n \leq m \leq n$. Let N be the inverse image of $C_{G/H}(K/H)$ in G. Thus $A_m \leq G/N \leq S_m$. Moreover, by an easy calculation, we can see that $G/N \cong A_n$ or S_n and hence, $G \cong A_n$. For instance, let $n = 27$. We have $A_m \leq G/N \leq S_m$, where $m \in \{23, 24, 25, 26, 27\}$. If $m = 27$, then since $|G| = |A_{27}|$, we can easily conclude that $G \cong M$, as desired. So, it is enough to get a contradiction for the case $m \neq 27$. In this case, we have $|N| \in \{3^3, 2.3^3.13, 3^3.13, 2.3^3.5^2.13, 3^3.5^2.13, 2^4.3^4.5^2.13, 2^3.3^4.5^2.13\}$. If $|N| = 3^3$, then since $8.17 \in \omega(A_{27}) = \mathrm{Vo}(A_{27})$, $8.17 \in \omega(G)$ and hence, we can easily see that $8.17 \in \omega(A_m)$ or $\omega(S_m)$, where $23 \leq m \leq 26$, a contradiction. Thus $13 \mid |N|$. If $N_{13} \in \mathrm{Syl}_{13}(N)$, then the Frattini argument shows that $19 \mid |N_G(N_{13})|$ and since $|N_G(N_{13})/C_G(N_{13})| \mid 12$, we conclude that $13.19 \in \omega(G)$. Now, Remark 2.7 implies that $13.19 \in \mathrm{Vo}(G) = \mathrm{Vo}(A_{27})$, a contradiction.

Step 3. Let $n \geq 83$ and $n, n-1$ and $n-2$ are not primes. According to Step 1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group. Also, by Remark 2.7, we can easily follow (Lemma 2.1, [29]) to prove that $t_n(6) \mid |K/H|$. This is the main key to show that there exists a normal subgroup N of G such that $G/N \cong A_m$ or S_m, $r_n \leq m \leq n$ in (Lemma 2.4, [29]). Now, it is enough to show that $n = m$. If $m \neq n$, then we derive a contradiction. Let q be the largest prime factor of $n!/m!$. In (Theorem 2.1, [29]), the following results are obtained:

(1) $q \geq 17$ and $q \geq n - m + 3$;

(2) if A_m contains the elements of order t, where $\gcd(t, q) = 1$, then $tq \in \omega(G)$.

Since the proof of the above statements relies on the fact that $G/N \cong A_m$ or S_m, we have the same statements here. Put $p_1 := r_m$. If $m - p_1 > 2$, then we take $p_2 = r_{m-p_1}$. Also, if $m-(p_1+p_2) > 2$, then take $p_3 = r_{m-(p_1+p_2)}$, and so on. Thus there exist certainly some odd primes $p_1 > p_2 > \cdots > p_k$ such that $m - 2 \leq p_1 + p_2 + \cdots + p_k \leq m$. If $q \neq p_i$, $1 \leq i \leq k$, then from the fact that A_m has an element of order $p_1 p_2 \ldots p_k$, we see that G has elements of order $q p_1 p_2 \ldots p_k$ from (2). According to (1), we have $p_1 + p_2 + \cdots + p_k + q \geq (m-2) + (n-m+3) > n$ which implies that $q p_1 p_2 \ldots p_k \notin \omega(A_n)$. But $q \parallel |G|$ and hence, $q p_1 p_2 \ldots p_k \notin \omega(G)$, a contradiction. Therefore, there exists $1 \leq i \leq k$ such that $q = p_i$. Put $l = p_1 + p_2 + \cdots + p_{i-1}$. Thus $q = r_{m-l}$ and hence, $17 \leq q = p_i \leq m - l \leq 2p_i$. We know that there exists another prime p_i', $\frac{1}{2}(m-l) < p_i' < m-l$ and $p_i' < p_i$. If $p_1 + p_2 + \cdots + p_{i-1} + p_i' \geq m-2$, then we can similarly get a contradiction. Thus $p_1 + p_2 + \cdots + p_{i-1} + p_i' < m-2$ and we can assume that $m' = m - (p_1 + p_2 + \cdots + p_{i-1} + p_i') < \frac{1}{2}(m-l)$. We take again $q_1 = r_{m'}, q_2 = r_{m'-q_1}, \ldots, q_s = r_{m'-(q_1+q_2+\cdots+q_{s-1})}$ such that $m'-2 \leq q_1+q_2+\cdots+p_{s-1} \leq m'$. Thus $p_1 > p_2 > \cdots > p_{i-1} > p_i' > q_1 > q_2 > \cdots > q_s$ and $m-2 \leq p_1+p_2+\cdots+p_{i-1}+p_i'+q_1+q_2+\cdots+q_s \leq m$. Moreover, $q_i \neq q$, $i = 1, 2, \ldots, s$, and hance, we can get a contradiction as above.

Part 3. Let $M = L_2(p)$, where p is an odd prime. Since $\mathrm{Vo}(G) = \mathrm{Vo}(L_2(p))$, according to Lemma 2.4(1) and Table 1, we have $\Gamma(G) = GK(L_2(p))$ and G is a nonsolvable group. Thus Lemma 2.9 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group and $|G/K| \mid |\mathrm{Out}(K/H)|$. According to $|G|$, we can conclude that $p \mid |G/K|$, $p \mid |H|$ or $p \mid |K/H|$.

If $p \mid |G/K|$, then as in the proof of Step 2 in [27], we can get a contradiction. If $p \mid |H|$, then the Frattini argument implies that $G = N_G(P)H$, where P is a Sylow p-subgroup of H. Also, since for every $k > 1$, pk is not an element of $\mathrm{Vo}(G)$, we have $C_G(P) = P$. Thus G/H is isomorphic to a homomorphic image of $N_G(P)/P$. But $N_G(P)/P$ is embedded in the cyclic group $\mathrm{Aut}(P)$. Thus G/H is cyclic, which is a contradiction to the fact that G/H is not solvable. Therefore, $p \mid |K/H|$ and according to $|G|$ and Lemma 2.10, we have $G \cong L_2(p)$, as desired. □

Proof of Theorem B. We have divided the proof of Theorem B into a sequence of cases.

Case 1. Let $M = S_6(2)$. According to Table 1 and Corollary 3.2, we can see that $\mathrm{OC}(G) = \mathrm{OC}(S_6(2))$. Thus Lemmas 2.1, 2.5, and 2.9 imply that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a nonabelian simple group, $|G/K| \mid |\mathrm{Out}(K/H)|$ and $K/H \le G/H \le \mathrm{Aut}(K/H)$. According to [33], K/H is isomorphic to one the following simple groups

$$A_5, A_6, A_7, A_8, A_9, U_4(2), L_2(7), L_2(8), U_3(3), L_3(4), S_6(2).$$

If $K/H \cong A_5, A_6, U_4(2)$, then 7 does not divide $|G/H|$. Since $5 \in \mathrm{Vo}(G/H)$, $\mathrm{Van}(G/H)$ contains an element xH of order 5. Without loss of generality we can assume that $o(x) = 5$. Thus xH is a subset of $\mathrm{Van}(G)$. Fix $L =< x > H$. If $R \in Syl_7(H)$, then Frattini argument implies that $L = N_L(R)H$. Since $5 \mid [L : H] = [N_L(R) : N_H(R)]$, we deduce that $5 \mid |N_L(R)|$. Thus there exist $h \in H$ and $1 \le i \le 4$ such that $x^i h \in N_L(R)$ has order 5. Since G does not contain any element of order 35, $< x^i h >$ acts fixed point freely on R and hence, $5 \mid 7-1$, a contradiction. If $K/H \cong L_2(7), L_2(8), U_3(3)$, then 5 does not divide $|G/H|$ and $7 \mid |G/H|$. Thus replacing the rules of 5 and 7 in the previous argument leads us to get a contradiction. If $K/H \cong A_7, A_8, L_3(4)$, then replacing 7 with 3 and 5 with 7 in the argument given in the above leads us to get a contradiction. Let $K/H \cong A_9$. If $G/H \cong S_9$, then $|H| = 2$ and if $G/H \cong A_9$, then $|H| = 8$. Now applying the previous argument for 2 and 7 shows that $7 \mid (|H| - 1)$ and hence, $G/H \cong A_9$ and $|H| = 8$. If H_1 is a normal minimal subgroup of G such that $H_1 \le H$, then applying the above argument shows that $7 \mid (|H_1| - 1)$ and hence, $|H_1| = 8$. Thus H is a normal minimal subgroup of G and hence, $H \cong Z_2 \times Z_2 \times Z_2$. Therefore, $G/C_G(H) \le \mathrm{Aut}(H) \cong GL_3(2)$. Therefore, $2^6.3^3.5 \mid |C_G(H)|$ and $|C_G(H)| \mid |G|/7$. Also, $C_G(H)/H$ is a normal subgroup of $G/H = K/H$ and hence, simplicity of K/H forces $C_G(H)/H = K/H$ or $C_G(H)/H = 1$, which is a contradiction. Therefore $K/H \cong S_6(2)$ which implies that $G \cong S_6(2)$.

Case 2. Let $M = U_5(2)$. According to Table 1 and Corollary 3.2, we have $\mathrm{OC}(M) = \mathrm{OC}(G)$. It follows from Lemma 2.5 that G is a 2-Frobenius group or $G \cong M$. We claim that G is not a 2-Frobenius group. Conversely, suppose that G is a 2-Frobenius group with normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$. Since $\mathrm{OC}(M) = \mathrm{OC}(G)$, according to Table 1 and Lemma 2.3, we have $\pi(|K/H|) = \pi_2(G) = \{11\}$, $|K/H| = 11$ and $|G/K| \mid 10$. Thus by $|U_5(2)| = 2^{10}.3^5.5.11$, $|H| \in \{2^9.3^5.5, 2^{10}.3^5, 2^9.3^5\}$.

Let $Q \in \mathrm{Syl}_{11}(K)$. Since Q acts fixed point freely on H, Thompson's nilpotency criterion shows that H is nilpotent. Thus if $P \in \mathrm{Syl}_p(H)$, where $p \mid |H|$, then $P \trianglelefteq K$ and hence, $11 \mid (|P| - 1)$. This forces $|H| = 2^{10}.3^5$ which implies that $|G/K| = 5$. According to [12], $4, 8 \in \omega(G)$. Thus if $P_2 \in \mathrm{Syl}_2(H)$, then P_2 is not an elementary abelian 2-group. Now, assume that N is a normal minimal subgroup of G such that $N \leq P_2$. Since G is solvable, we conclude that N is an elementary abelian 2-group of order 2^t, where $t > 0$. Thus our assumption on P_2 implies that $1 < 2^t < 2^{10}$. But K/H acts fixed point freely on N and hence, $11 \mid (2^t - 1)$, which is impossible by checking the different values of t. This shows that G is not 2-Frobenius and hence, $G \cong M$.

The proof for $M = U_4(2)$ is similar and we omit the details for the sake of convenience.

Case 3. Let $M = S_4(7)$. Note that $|S_4(7)| = |G| = 2^8.3^2.5^2.7^4$ and the components of $\Gamma(G)$ are $\{2, 3, 7\}$ and $\{5\}$. Let G be solvable and let $F(G)$ be the Fitting subgroup of G. According to Lemma 2.4(2-3), it is easy to see that G is a nearly 2-Frobenius group. If $5 \in \pi(|F(G)|)$, then since $25 \in \mathrm{Vo}(G)$, we deduce that $25 \in \omega(G)$ and hence, $P \in \mathrm{Syl}_5(F(G))$ is a cyclic normal subgroup of G. Therefore, $G/C_G(P)$ is a cyclic group which its order divides 4. Thus considering the components of $\Gamma(G)$, shows that 5 is an isolated point in $\Gamma(G)$, and Lemma 2.4(4) implies that $G/F(G)$ is a 2-group. Since G is nearly 2-Frobenius, $F(G)/F_2 \leq F(G/F_2)$ and hence, $G/F_2/F(G/F_2)$ is a 2-group. Thus $(G/F_2)/F(G/F_2)$ is not a Frobenius group and hence, G/F_2 is not a 2-Frobenius group, which contradicts to the fact that G is nearly 2-Frobenius. Thus $5 \notin \pi(|F(G)|)$. If there exists an element $x \in G$ such that $o(x) = 5r$, where $r \in \{2, 3, 7\}$, then since $5r \notin Vo(G)$, x is a non-vanishing element. Lemma 2.4(4) now implies that $o(xF(G)) \mid 2^i$ and hence, $5 \in \pi(|F(G)|)$, which is a contradiction. This shows that $GK(G) = \Gamma(G) = \Gamma(S_4(7)) = GK(S_4(7)))$. Therefore, $\mathrm{OC}(G) = \mathrm{OC}(S_4(7))$. Now according to [19] we have $G \cong S_4(7)$, this contradicts the fact that G is solvable. So G is not solvable and by Lemma 2.8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \leq |\mathrm{Out}(K/H)|$. Considering the orders of $S_4(7)$

and the finite simple K_3-groups and K_4-groups show that $K/H \cong A_5$, $A_5 \times A_5$, A_6, $L_2(7)$, $L_2(7) \times L_2(7)$, $L_2(8)$, A_7, A_8, $L_2(49)$, $L_3(4)$ or $S_4(7)$.

If $K/H \ncong A_5$, $A_5 \times A_5$, A_6, $L_2(7)$, $L_2(7) \times L_2(7)$, $L_2(8)$ and $S_4(7)$, then G/H contains an element xH of order 5. Also, for $P \in \mathrm{Syl}_7(H)$, considering the order of G/H forces $1 < |P| \leq 7^3$. Since $G = N_G(P)H$, without loss of generality, we can assume that $x \in N_G(P)$ and x is a 5-element. Also, since G/H does not contain any normal 5-subgroup, we can assume by (Theorem C, [15]) that $xH \in \mathrm{Van}(G/H)$ and hence, Lemma 2.4(iv) shows that $xH \subseteq \mathrm{Van}(G)$. Thus $xP \subseteq \mathrm{Van}(G)$. On the other hand, 5 is an isolated point in $\Gamma(G)$, so $\langle x \rangle$ acts fixed point freely on P. Thus $5 \mid |P| - 1$, which is impossible. If $K/H \cong L_2(7)$, $L_2(7) \times L_2(7)$ or $L_2(8)$, then replacing the roles of 5 and 7 in the previous argument and if $K/H \cong A_5$ and A_6, then replacing 5 with 3 and 7 with 5 and the relative subgroups in the previous argument lead us to get a contradiction. Also, since $25 \in \mathrm{Vo}(G)$, $25 \in \omega(G)$, so $K/H \ncong A_5 \times A_5$.This shows that $K/H \cong S_4(7)$ and hence, $G \cong S_4(7)$, as claimed.

Case 4. Let $M = L_2(49)$. According to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Since $|L_2(49)| = 2^4.3.5^2.7^2$, Remark 2.7 implies that G has an irreducible character of 3-defect zero. Thus by Lemma 2.8, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K/H \cong S_1 \times \cdots \times S_t$, where S_i, $1 \leq i \leq t$, is a simple K_3-group or a simple K_4-group and for every $1 \leq i, j \leq t$, we have $S_i \cong S_j$. Since $3 \mid |S_i|$, where $1 \leq i \leq t$, and $3 \parallel |G|$, we conclude that $K/H \cong S$, where S is a simple K_3-group or a simple K_4-group and $K/H \leq G/H \leq \mathrm{Aut}(K/H)$.

Subcase 1. Let K/H be a simple K_3-group. Since $|K/H| \mid |G|$ and $3 \parallel |G|$, checking the orders of simple K_3-groups shows that $K/H \in \{A_5, L_2(7)\}$. If $K/H \cong A_5$, then $A_5 \leq G/H \leq S_5$. It follows that $2^2 \cdot 3 \cdot 5 \mid |G/H|$ and $|G/H| \mid 2^3 \cdot 3 \cdot 5$. Thus $|H| = 7^2 \cdot 2 \cdot 5$ or $|H| = 7^2 \cdot 2^2 \cdot 5$. Let $P \in \mathrm{Syl}_5(H)$. By Farttini's argument, we have $G = N_G(P)H$. Thus $G/H \cong N_G(P)/N_H(P)$ and $3 \mid |N_G(P)|$. Put $Q \in \mathrm{Syl}_3(N_G(P))$. Since G has an irreducible character of 3-defect zero and $15 \notin \mathrm{Vo}(M) = \mathrm{Vo}(G)$, we deduce that $15 \notin \omega(G)$. Thus Q acts fixed point freely on P and hence, $3 = |Q| \mid (|P|-1) = 5-1$, which is impossible. If $K/H \cong L_2(7)$, then we conclude that $G/H \leq \mathrm{Aut}(L_2(7))$. Thus $2^3.3.7 \mid |G/H|$ and $|G/H| \mid 2^4.3.7$. Therefore, $|H| = 5^2 \cdot 2 \cdot 7$ or $|H| = 5^2 \cdot 7$, which implies that $n_5(H) = 1$. If $P \in \mathrm{Syl}_5(H)$, then $P \trianglelefteq G$ and we have $P \cong \mathbb{Z}_{25}$, because $25 \in \mathrm{Vo}(M) = \mathrm{Vo}(G)$ and $P \in \mathrm{Syl}_5(G)$. Since $P \leq C_G(P)$, $\dfrac{|N_G(P)|}{|C_G(P)|} \mid 4$. Thus $\dfrac{|G|}{|C_G(P)|} \mid 4$ and hence, $\dfrac{|G|}{4} \mid |C_G(P)|$, which implies that $3.25 \in \omega(G)$. But G

has an irreducible character of 3-defect zero and hence, $3.25 \in \mathrm{Vo}(G) = \mathrm{Vo}(M)$, a contradiction.

SUBCASE 2. Assume that K/H is a simple K_4-group. If K/H is isomorphic to one of the groups listed in Lemma 1.1 (2), then comparing the orders of these groups and K/H forces $K/H \cong L_2(49)$ and hence, $H = 1$ and $K = G \cong L_2(49)$, as desired. If $K/H \cong L_2(r)$, then $r \in \{2,3,5,7\}$, which is impossible. If $K/H \cong L_2(2^m)$, where $m \geq 5$, $2^m - 1 = u$ and $(2^m + 1)/3 = t$ are primes, then since $u, t \in \pi(|G|) = \{2,3,5,7\}$, we get a contradiction. Finally, assume that $K/H = L_2(3^m)$, where m and $(3^m + 1)/4 = t$ are odd primes. But $t \in \pi(|G|) = \{2,3,5,7\}$, which is a contradiction.

CASE 5. Let $M = L_2(2^m)$, where $2^m + 1/3 = t$ and $2^m - 1 = u$, are primes greater than 3. Then according to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Thus Lemma 2.3 implies that G is not a 2-Frobenius group. Also, if G is a Frobenius group with kernel F and complement H, then according to Lemma 2.2, we have $\mathrm{OC}(G) = \{|F|, |H|\}$. Since $u \parallel |G|$ and $u \in \mathrm{OC}(M)$, we obtain $u \in \mathrm{OC}(G)$, by Lemma 3.1. If $u = |F|$, then $|H| \mid (u-1)$. Thus $2^m(2^m+1) \mid (2^m - 2)$, which is impossible. If $u = |H|$ and $P \in \mathrm{Syl}_t(F)$, then since F is nilpotent, we see that $P \trianglelefteq G$ and hence, H acts fixed point freely on P. Thus, $(2^m - 1) = |H| \mid (|P| - 1) = 2(2^{m-1} - 1)/3$, which is impossible. Thus according to Lemma 2.1, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ and K/H is a nonabelian simple group such that $u \mid |K/H|$.

SUBCASE 1. If K/H is a simple K_3-group, then $K/H \in \{A_5, L_2(7)\}$, because $3 \parallel |G|$ and hence, $3 \parallel |K/H|$. We have $u \in \pi(|K/H|)$ and hence, $u = 5$ or $u = 7$. Since $u = 2^m - 1$, we deduce that $u \neq 5$ and hence $K/H \ncong A_5$. If $u = 7$, then $m = 3$, which is a contradiction.

SUBCASE 2. If K/H is a simple K_4-group, then since $3 \parallel |G|$, we deduce that $3 \parallel |K/H|$ and hence, $K/H \in \{L_2(16), L_2(25), L_2(49), L_3(5), U_3(7), L_2(2^{m'}), L_2(r)\}$, under conditions of Lemma 1.1(2). If $K/H \cong L_2(16)$ or $L_2(25)$, then $2^m - 1 \in \{5, 13, 17, 31, 43\}$, which is impossible. If $K/H \cong L_2(49)$, then $u = 3$, which is impossible. Now, if $K/H \cong L_2(r)$, then $r \in \{u, t\}$. If $r = u = 2^m - 1$, then $|L_2(r)| = r(r^2-1)/2 = (2^m - 1)2^m(2^{m-1} - 1) \mid (2^m - 1)2^m(2^m + 1)$, and hence $(2^{m-1} - 1) \mid (2^m + 1)$. It follows that $m = 2$ or $m = 3$, which is impossible. If $r = t$, then $r = t = 2^m + 1/3$. Since $u \mid |L_2(r)| = r(r-1)(r+1)/2$, we have $2^m - 1 = u \mid (t-1)/2 = (2^{m-1} - 1)/3$ or $2^m - 1 = u \mid (t+1)/2 = 2(2^{m-2} + 1)/3$, which is impossible. Finally, if $K/H \cong L_2(2^{m'})$, then $2^{m'} - 1$ is a prime number.

Thus $2^{m'} - 1 = u$ or $2^{m'} - 1 = t$. But $t \mid (2^m + 1)$, and $2^{m'} - 1 = u$. From this, we have $2^{m'} - 1 = u = 2^m - 1$, and hence $m' = m$. It shows that $G \cong L_2(2^m)$, as claimed.

CASE 6. Let $M = L_2(25)$. According to Table 1, we obtain that $\Gamma(L_2(25))$ has three components. Thus Lemmas 2.4 and 3.1 show that G is a nonsolvable group and $13 \in \mathrm{OC}(G)$. Since G is nonsolvable, G is not 2-Frobenius. Also, Lemma 2.2 and checking the orders imply that G is not a Frobenius group. Thus according to Lemma 2.1, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ such that $13 \in \pi(|K/H|)$. Furthermore, $|K/H| \in \{13 \cdot p^\alpha \cdot q^\beta, 13 \cdot 2^\alpha \cdot 3^\beta \cdot 5^\gamma\}$, where $p, q \in \{2, 3, 5\}$ and $\alpha, \beta, \gamma \in \mathbb{N}$. If $|K/H| = 13 \cdot p^\alpha \cdot q^\beta$, then by checking the orders of simple K_3-groups in Lemma 1.1(1), we can easily get a contradiction. Thus $|K/H| = 13 \cdot 2^\alpha \cdot 3^\beta \cdot 5^\gamma$ and K/H is one of the groups listed in Lemma 1.1(2). If K/H is a group listed in Lemma 1.1(2-a), then checking the orders of the groups shows that $K/H \cong L_2(25)$. Thus $|G| = |K/H|$ which implies that $G = K \cong L_2(25)$, as desired. If K/H is a group listed in Lemma 1.1(2-b,d), then we can see that $7 \in \pi(|K/H|)$, a contradiction. Also, if K/H is a group listed in Lemma 1.1(2-c), then $K/H \cong L_2(2^m)$, where $m \geq 5$ and $2^m - 1 = u$ is prime. Thus $u \notin \{3, 5, 13\}$, which is a contradiction.

If $M = L_2(81)$, then replacing 13 with 41 in the argument given for $L_2(25)$ leads us to see that $G \cong L_2(81)$.

CASE 7. Let $M = L_2(3^m)$, under conditions of Lemma 1.1(2-d). According to Table 1 and Lemma 2.4(1), we obtain that G is nonsolvable. Thus by Lemma 2.8, G has a normal series $1 \trianglelefteq K \trianglelefteq H \trianglelefteq G$ such that $K/H \cong S_1 \times \cdots \times S_l$, where S_i, $1 \leq i \leq l$, is a simple K_3-group or a simple K_4-group and for every $1 \leq i, j \leq t$, we have $S_i \cong S_j$. Since $|L_2(3^m)| = 3^m.(3^m - 1).(3^m + 1)/2$, conditions of Lemma 1.1(2-d) show that $4 \parallel |G|$ and G has an irreducible character of u-defect zero, where $t = (3^m + 1)/4$. Since $4 \parallel |G|$ and $4 \mid |S_i|$, we deduce that $l = 1$ and $4 \parallel |K/H|$. Therefore, K/H is a simple K_3-group or a simple K_4-group. Let $u \mid (3^m - 1)/2$, under conditions of Lemma 1.1(2-d).

SUBCASE 1. Let K/H be a simple K_3-group. Since $4 \parallel |K/H|$, we deduce that $K/H \cong A_5$, by checking the orders of simple K_3-groups. Thus $5 \in \pi(|G|) = \{2, 3, u, t\}$. Therefore, $5 \mid (3^m - 1)$ or $5 \mid (3^m + 1)$. This shows that $2 \mid m$, which is a contradiction with conditions of Lemma 1.1(2-d).

SUBCASE 2. Assume that K/H is a simple K_4-group. Since $4 \parallel |K/H|$, we deduce that $K/H \cong L_2(3^e)$ or $L_2(r)$ satisfying conditions of Lemma 1.1(2-b,d).

First let $K/H \cong L_2(3^e)$. Since $\pi(|K/H|) = \pi(|G|)$ and $|K/H| \mid |G|$, we deduce that $e \leq m$ and $u, t \in \pi(|K/H|)$. If $u \mid (3^m - 1)/2$ and $u \mid (3^e - 1)/2$, then $e = m$ and hence, $K/H \cong M$. Since $|G| = |M| = |K/H|$, we deduce that $H = 1$ and $K = G$ and hence, $G \cong M$, as desired. Also, if $t \mid (3^m + 1)/4$ and $t \mid (3^e - 1)/2$, then $t \mid \gcd((3^m + 1)/4, (3^e - 1)/2)$ and hence, $2m \mid e$. This forces e is even, which is a contradiction.

If $K/H \cong L_2(r)$, then we can see at once that $r \in \{t, u\}$. If $r = u$, then since $|K/H| = u(u^2 - 1)/2$ and either $3^m - 1 = 2u$ or 2.11^2, we deduce that $|K/H| = 3(3^m - 1)(3^m + 1)(3^{m-1} - 1)/8 \mid |G|$ or $|K/H| = 2^2.3.5.11$. Thus either $(3^{m-1} - 1)/4 \mid 3^{m-1}$ or $t = (3^5 + 1)/4 \mid 2^2.3.5$, which is impossible. If $r = t$, then since $|K/H| = t(t^2 - 1)/2$, we deduce that $u \mid (t - 1)$ or $(t + 1)$, which is a contradiction, because $3^m + 1 = 4t$.

If $M \in \{L_3(4), L_2(8), Sz(8), Sz(32)\}$, then according to [17], we have $G \cong M$. Thus it remains to consider the case in which M is one of the groups $L_2(16)$, $L_3(q)$, where $q \in \{3, 5, 7, 8, 17\}$, $U_3(q)$, where $q \in \{3, 4, 5, 7, 8, 9\}$, $S_4(q)$, where $q \in \{4, 5, 9\}$, $L_4(3)$, $U_4(3)$, $D_4(2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$. Thus M satisfies the conditions of Corollary 3.2 and hence, we have $\mathrm{OC}(G) = \mathrm{OC}(M)$. If $M \in \{L_3(3), U_3(3), U_3(4), U_3(5), {}^2F_4(2)'\}$, then similar argument for the group $U_4(2)$ in Lemma 2.5 shows that $G \cong M$. Moreover, according to [7], [9], [10], [13], [19], [20], [21] , [22], [26], [34] the remaining groups are characterizable by their order components and hence the proof of Theorem B is complete. □

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