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Test modules for flatness

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Abstract – A right R-module M is said to be a test module for flatness (shortly: an f-test module) provided for each left R-module N, $\mathrm{Tor}(M, N) = 0$ implies N is flat. f-test modules are a flat version of the Whitehead test modules for injectivity defined by Trlifaj. In this paper the properties of f-test modules are investigated and are used to characterize various families of rings. The structure of a ring over which every (finitely generated) right R-module is flat or f-test is investigated. Abelian groups that are Whitehead test modules for injectivity or f-test are characterized.

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1. Introduction

In this paper, we study and analyze the flatness of a module. It consists in evaluations of appropriate abelian group-valued functors Tor. A right R-module M is *flat* if and only if $\mathrm{Tor}_1(M, N) = 0$ for all left R-modules $N \in R - \mathrm{Mod}$. We would like to check the flatness of M by calculating a single Tor group using fixed module N.

Recall that a right R-module M is said to be *flat* (*projective*, *injective*, respectively) provided that the functor $M \otimes_R -$ ($\mathrm{Hom}_R(M, -)$, $\mathrm{Hom}_R(-, M)$, respectively) preserves short exact sequences. There are two basic results on testing flatness. First, by using Baer's Criterion, flatness can be tested using only all (finitely generated) ideals. Then Lazard's Theorem, saying that any flat module is a direct limit (over a directed index set) of finitely generated free modules.

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A right R-module N is called a *Whitehead test module for projectivity* (or a *p-test module*) if a right R-module M is projective whenever $\mathrm{Ext}_R(M,N) = 0$. Dually, a right R-module M is called a *Whitehead test module for injectivity* (or an *i-test module*) if a right R-module N is injective whenever $\mathrm{Ext}_R(M,N) = 0$. If $R = \mathbb{Z}$, then the question "Is $\mathbb{Z}$ a p-test $\mathbb{Z}$-module?" is exactly the well-known Whitehead problem. p-test modules were introduced and studied by Trlifaj in [15]. In that paper, the author calls a ring R right Ext-*ring* if each right module is either p-test or injective. Ext-rings have been studied further in [17, 16]. i-test modules were introduced and studied by Trlifaj in [17]. In that paper, the author also considered rings over which each (finitely generated) module is either i-test or projective, and referred to such rings as (*finitely saturated* rings) *fully saturated* rings. It is easy to see that fully saturated rings and Ext-rings are the same. Fully saturated rings have been studied further in [6].

In a recent paper [9], in contrast to the well-known notion of relative projectivity, Holston, Lopez-Permouth, Mastromatteo and Simental-Rodriguez introduced the notion of subprojectivity. Namely, a module M is said to be N-*subprojective* if for every epimorphism $g\colon B \to N$ and homomorphism $f\colon M \to N$ there exists a homomorphism $h\colon M \to B$ such that $g \circ h = f$. For a module M, *the subprojectivity domain* of M, $\underline{\mathfrak{Pr}}^{-1}(M)$, is defined to be the collection of all modules N such that M is N-subprojective, that is $\underline{\mathfrak{Pr}}^{-1}(M) = \{N \in \mathrm{Mod} - R \mid \text{M is N-subprojective}\}$. A module M_R is projective if and only if $\underline{\mathfrak{Pr}}^{-1}(M) = \mathrm{Mod} - R$. If N is projective, then M is vacuously N-subprojective. So, the smallest possible subprojectivity domain is the class of projective modules. A module with such a subprojectivity domain is defined in [9] to be *p-indigent*. p-indigent modules have been studied further in [5] where the author investigated the structure of rings over which every (simple) module is p-indigent or projective. In the same paper, the author proved that if a ring R is not von Neumann regular, then R is right fully saturated if and only if all non-projective modules are p-indigent (see [5, Corollary 3.1]).

This paper is inspired by similar ideas and notions given above. We say that M_R is ${}_RN$-*subflat* if for every exact sequence of left R-modules $0 \to H \to F \to N \to 0$, the sequence $0 \to M \otimes H \to M \otimes F \to M \otimes N \to 0$ is exact. *The subflat domain* of M_R, $\mathfrak{F}^{-1}(M)$, is defined to be the collection of all modules ${}_RN$ such that M is N-subflat. It is clear that a module M_R is flat if and only if $\mathfrak{F}^{-1}(M) = R - \mathrm{Mod}$. It is easy to see that every right R-module is subflat relative to all flat left R-modules, and one can show (Proposition 2.4) that flat modules are the only ones sharing the distinction of being in every single subflat domain. It is thus tempting to ponder the existence of modules whose subflat domain consists of only flat modules. To keep in line with [17], we refer to these modules as *Whitehead test module for flatness* (or *f-test module*).

In Section 2, subflatness and subflat domains are investigated. In Section 3, we show that a f-test module exists for an arbitrary ring. Moreover, i-test modules are f-test, and if R is a right Noetherian ring, then a finitely generated right R-module is f-test if and only if it is i-test.

In Section 4, we study rings possessing many f-test modules. A non von Neumann regular ring R is said to satisfy the property (F) provided that all non-flat modules are f-test. An Artinian serial ring R with unique singular simple module (up to isomorphism) satisfies (F). If R is a nonsemisimple left perfect ring which has at least one finitely generated left maximal ideal (for example, left Noetherian), then every finitely generated right R-module is flat or f-test if and only if R is a left Σ-CS ring and every finitely generated singular left R-module is p-indigent if and only if there is a ring direct sum $R \cong S \times T$, where S is semisimple Artinian ring and T is an indecomposable ring which is either (i) finitely saturated matrix ring over a local QF-ring; or, (ii) hereditary Artinian serial ring with $J^2 = 0$. Using this result, we show that R is a nonsingular left Artinian ring which satisfies (F) if and only if R is a right (or left) hereditary fully saturated ring if and only if $R \cong S \times T$, where S is semisimple Artinian ring and T is an indecomposable hereditary Artinian serial ring with $J(T)^2 = 0$. We show that, for hereditary Noetherian rings, a right R-module M is f-test if and only if M is i-test if and only if $\mathrm{Hom}(S, M) \neq 0$ for each singular simple right R-module S. An abelian group is f-test (or i-test) if and only if it contains a submodule isomorphic to $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, where p ranges over all primes.

We use the following notation and conventions: All rings are associative and with unit. Expressions like "a Noetherian ring" mean that the corresponding right and left conditions hold. Let R and S be rings. Then $S \times T$ denotes the ring direct sum of R and S. Further, $\mathrm{Mod}-R$ ($R-\mathrm{Mod}$) denotes the category of unitary right (left) R-modules. We often write M_R (${}_RM$) to denote M being a right (left) R-module. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*. We use the notation $E(M)$, $F(M)$, $\mathrm{Soc}(M)$, $\mathrm{Rad}(M)$, $Z(M)$ for the injective hull, flat cover, socle, Jacobson radical, singular submodule of M respectively. Also $J(R)$ denotes the Jacobson radical of a ring R. By $N \leq M$, we mean that N is a submodule of M. The i-th derived functor of the $\otimes_R$ (Hom_R) functor is denoted by Tor_i^R (Ext_R^i), and $\mathrm{Tor}_1^R = \mathrm{Tor}$ ($\mathrm{Ext}_R^1 = \mathrm{Ext}$). For all other basic or background material, we refer the reader to [12, 10, 2].

2. The subflat domain of a module

We will start in this section with some of the basic properties.

Definition 2.1. Let M_R be a R-module. An exact sequence of left R-modules $0 \to H \to F \to N \to 0$ is called *M-pure* if the sequence $0 \to M \otimes H \to M \otimes F \to M \otimes N \to 0$ is exact.

Let Γ be a complete set of representatives of finitely presented right R-modules. Set $M := \bigoplus_{S_i \in \Gamma} S_i$. M-pure exact sequences are called (Cohn) *pure exact* sequences.

Proposition 2.2. [10, Corollary 4.86] *Let R be a ring. Let $M \in$ Mod $- R$ and $0 \to N \hookrightarrow P \to M \to 0$ be a short exact sequence in* Mod $- R$ *such that P is flat. Then M is flat if and only if N is a pure submodule of P.*

The following proposition can be proved by using standard tensor product properties, so we omit its proof.

Proposition 2.3. *The following statements are equivalent for any given modules M_R and ${}_RN$.*

(1) *M is N-subflat.*

(2) *N is M-subflat.*

(3) *$N \cong P/K$, where P is a flat left R-module, and K is an M-pure submodule of P.*

(4) *Every exact sequence of left R-modules $0 \to K \to P \to N \to 0$ is M-pure.*

(5) $\mathrm{Tor}(M, N) = 0$.

It is clear that flat left R-modules are contained in the subflat domain of any right R-module, and that a right R-module is flat if and only if its subflat domain consists of all modules in $R -$ Mod. Moreover, we have the following fact.

Proposition 2.4. $\bigcap_{M \in \mathrm{Mod}-R} \mathfrak{F}^{-1}(M) = \{N \in R - \mathrm{Mod} \mid N \textit{ is flat}\}$

Definition 2.5. We say that an R-module ${}_RN$ is M-injective if for every exact sequence of left R-modules $0 \to H \to F \to M \to 0$, the sequence

$$0 \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(F, N) \longrightarrow \mathrm{Hom}(H, N) \longrightarrow 0$$

is exact, i.e. $\mathrm{Ext}(M, N) = 0$.

It is known that $\mathrm{Tor}(N, M)^+ \cong \mathrm{Ext}(N, M^+) \cong \mathrm{Ext}(M, N^+)$ by the First and Second Adjoint Isomorphism Theorems, (see [12]). Then, we get the following:

Proposition 2.6. *Let M be a right R-module and N a left R-module. Then, M is N-subflat if and only if N^+ is M-injective if and only if M^+ is N-injective.*

Let $(M_k)_{k\in K}$ be a family of left R-modules. It is well known that there is a natural isomorphism, $\mathrm{Tor}(A, \bigoplus_{k\in K} M_k) \cong \bigoplus_{k\in K} \mathrm{Tor}(A, M_k)$. Proposition 2.3 yields the following.

Proposition 2.7. *Let $(M_k)_{k\in K}$ be a family of left R-modules. Then $\mathfrak{F}^{-1}(\bigoplus_{k\in K} M_k) = \bigcap_{k\in K} \mathfrak{F}^{-1}(M_k)$.*

Proposition 2.8. *Let R be a ring which has the decomposition $R = R_1 \times R_2$. Then a right R-module M is N-subflat if and only if MR_i is NR_i-subflat for each $i = 1, 2$.*

Proof. By assumption, we have $K = KR_1 \oplus KR_2$ for any R-module K. A right R-module M is N-subflat if and only if N^+ is M-injective, i.e. $\mathrm{Ext}(M, N^+) = 0$. The claim follows from the isomorphism $\mathrm{Ext}_R(M, N^+) \cong \mathrm{Ext}_{R_1}(MR_1, N^+R_1) \oplus \mathrm{Ext}_{R_2}(MR_2, N^+R_2)$. □

3. Modules whose subflat domain consists of only flat modules

Since a right R-module is flat if and only if its subflat domain is $R - \mathrm{Mod}$, it makes sense to wonder about the extreme opposite: What are the modules which are subflat with respect to the smallest possible collection of modules? It is clear that such a smallest collection would have to consist precisely of the flat modules.

Definition 3.1. We will call a right R-module M a *test module for flatness* (or an *f-test* module) in case $\mathfrak{F}^{-1}(M) = \{A \in R - \mathrm{Mod} \mid A \text{ is flat}\}$, i.e. $\mathrm{Tor}(M, N) \neq 0$ for each non-flat left R-module N.

Considering that the notion of f-test modules is formally so similar to that of injective test modules, one would expect that many results in this theory will echo those of the other one. Certainly, the first problem that comes to mind with the introduction of the notion of f-test modules is whether such modules exist over all rings. The following proposition answers this question.

Proposition 3.2. *Every i-test module is f-test.*

Proof. Let M be a right R-module. Assume that M is N-subflat for a left R-module N. N^+ is M-injective by Proposition 2.6 and then, by the assumption, N^+ is injective. N is flat by [10, Theorem 4.9]. □

By Baer's Criterian for injectivity, a right R-module N is injective if and only if $\mathrm{Ext}(R/I, N) = 0$ for each right (essential) ideal I of R. Let $(I_k)_{k \in K}$ be a family of all right (essential) ideals of R. Set $M := \bigoplus_{k \in K} R/I_k$. It is clear that the right module M is i-test for any ring R.

Corollary 3.3. *Every ring has f-test right R-modules.*

The converse of Proposition 3.2 is not true by the following.

Proposition 3.4. *Let $(M_k)_{k \in K}$ be a family of all finitely presented right R-modules. Set $M := \bigoplus_{k \in K} M_k$. The following statements hold.*

(1) *M is f-test.*

(2) *M is i-test if and only if R is right Noetherian.*

Proof. (1) Suppose that M is N-subflat. Then, by Proposition 2.7, one has $\mathrm{Tor}(M_k, N) = 0$ for each M_k. This implies that N is flat by [7, Theorem 2.1.8].

(2) A right R-module N is FP-injective if and only if $\mathrm{Ext}(M_k, N) = 0$ for each M_k, i.e. $\mathrm{Ext}(M, N) = 0$. It is well known that FP-injective right R-modules are exactly the injectives if and only if R is right Noetherian ([7, p. 132]). □

A ring R is said to be a *von Neumann regular* ring if for each $a \in R$ there is an $r \in R$ such that $a = ara$. Every right (left) R-module is flat if and only if R is a von Neumann regular ring (see [12, Theorem 4.16]). The proof of the following is obvious from the definitions.

Proposition 3.5. *For an arbitrary ring R, the following conditions are equivalent.*

(1) *R is von Neumann regular.*

(2) *Every (non-zero) right (left) R-module is f-test.*

(3) *There exists a right (left) flat f-test R-module.*

From now on, unless otherwise stated, all rings will be non von Neumann regular.

The weak global dimension of R, $w.dim(R)$, is less than or equal 1 if and only if every submodule of a flat right (or left) R-module is flat if and only if every (finitely generated) right (or left) ideal is flat, (see [12, 9.24]).

Proposition 3.6. *Let R be a ring with $w.dim(R) \leq 1$. A right R-module M has a f-test submodule if and only if M is a f-test.*

Proof. Assume that $\operatorname{Tor}(M, N) = 0$ for a left R-module N, and M' is an f-test submodule of M. Consider the exact sequence $0 \to M' \hookrightarrow M \to K \to 0$. We have the sequence

$$\operatorname{Tor}_2(K, N) \longrightarrow \operatorname{Tor}_1(M', N) \longrightarrow \operatorname{Tor}_1(M, N) = 0.$$

By the hypothesis, $\operatorname{Tor}_2(K, N) = 0$. Then $\operatorname{Tor}_1(M', N) = 0$, and N is flat since M' is an f-test. The converse is clear. $\square$

By [12, Theorem 9.51], $\operatorname{Tor}(M, N^+) \cong \operatorname{Ext}(M, N)^+$ for any finitely presented right R-modules M and a right R-module N.

Proposition 3.7. *Let R be a right Noetherian ring. A finitely generated right R-module M is f-test if and only if M is i-test.*

Proof. Assume $\operatorname{Ext}(M, N) = 0$ for a module N_R. Then $\operatorname{Tor}(M, N^+) = 0$ and, by the assumption, N^+ is flat. Since R is right Noetherian, N is injective by [7, Corollary 3.2.17]. The converse follows from Proposition 3.2. $\square$

A ring R is said to be a right *C-ring* if $\operatorname{Soc}(R/I) \neq 0$ for each proper essential right ideal I of R. Left perfect rings and right semi-Artinian rings are right C-rings. A right R-module M is called *m-injective* if for any maximal right ideal I of R, any homomorphism $I \to M$ can be extended to a homomorphism $R \to M$, i.e. $\operatorname{Ext}(R/I, M) = 0$. Let Λ be a complete set of representatives of singular simple R-modules. A ring R is right C-ring if and only if every m-injective right R-module is injective, i.e. the module $\bigoplus_{S_i \in \Lambda} S_i$ is i-test (see, [14, Lemma 4]). Therefore, Proposition 3.2 yields the following.

Corollary 3.8. *Let R be a right C-ring. The right R-module $\bigoplus_{S_i \in \Lambda} S_i$ is f-test.*

4. Every module is flat or f-test

We know that there is an f-test module over any ring R by Proposition 3.3. This suggests the question of how close can the class of f-test right R-modules be to $\operatorname{Mod} - R$. By Proposition 3.5, every right R-module is f-test if and only if R is left von Neumann regular ring. If R is not von Neumann regular, then no flat module is f-test.

Proposition 4.1. *Let R be a ring. The following are equivalent.*

(1) *Every non-flat right R-module is f-test.*

(2) *Every non-flat left R-module is f-test.*

(3) $\mathrm{Tor}_1^R(M, N) \neq 0$ *for all non-flat right R-modules M and non-flat left R-modules N.*

Proof. (1) $\implies$ (2) Let M be a non-flat left R-module. Suppose that M is N-subflat. By the assumption, N is flat or f-test. If N is flat, then we are done. In the later case, M is flat, a contradiction. (2) $\implies$ (1) just right-left symmetry. (1) $\iff$ (3) is clear. □

Remark 4.2. Let M be a finitely presented right R-module, that is, M has a free presentation $F_1 \xrightarrow{f} F_0 \to M \to 0$ where F_0 and F_1 are finitely generated free modules. If we apply the functor $\mathrm{Hom}_R(-, R)$ to this presentation, we obtain the sequence

$$0 \to M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \mathrm{Tr}(M) \longrightarrow 0$$

where $\mathrm{Tr}(M)$ is the cokernel of the dual map $F_0^* \to F_1^*$. Note that, $\mathrm{Tr}(M)$ is a finitely presented left R-module. The left R-module $\mathrm{Tr}(M)$ is called *the Auslander–Biridger transpose* of the right R-module M. It is important to note that the right R-module $\mathrm{Tr}(M)$ is uniquely determined only up to splitting off or adding a projective direct summand ([3]).

Theorem 4.3 ([13, Theorem 8.3]). *Let M be a finitely presented right R-module. The following hold.*

(1) $\mathrm{Hom}(\mathrm{Tr}(M), \mathbb{E})$ *is exact if and only if the sequence* $M \otimes \mathbb{E}$ *is exact for any short exact sequence* $\mathbb{E}$ *of left R-modules.*

(2) $\mathrm{Hom}(M, \mathbb{E})$ *is exact if and only if the sequence* $\mathbb{E} \otimes \mathrm{Tr}(M)$ *is exact for any short exact sequence* $\mathbb{E}$ *of right R-modules.*

Proposition 4.4. *Let R be a ring and M a non-flat finitely presented right R-module. Then, M is f-test if and only if* $\underline{\mathfrak{Pr}}^{-1}(\mathrm{Tr}(M))$ *consists precisely of the flat left R-modules.*

Proof. Assume that $\mathrm{Tr}(M)$ is N-subprojective for a left R-module N. Consider the exact sequence $\mathbb{E}\colon 0 \to H \to F(N) \to N \to 0$. Since $\mathrm{Tr}(M)$ is N-subprojective, $\mathrm{Hom}(\mathrm{Tr}(M), \mathbb{E})$ is exact. Then, by Theorem 4.3, $M \otimes \mathbb{E}$ is exact, and so M is N-subflat by Proposition 2.3 . Therefore, by the assumption,

N is flat. For the converse, assume that M is an N-subflat for a left R-module N. Consider the exact sequence $\mathbb{E}\colon 0 \to H \to F \to N \to 0$. Since M is an N-subflat, $M \otimes \mathbb{E}$ is exact. Then, by Theorem 4.3, $\mathrm{Hom}(\mathrm{Tr}(M), \mathbb{E})$ is exact, and so $\mathrm{Tr}(M)$ is N-subprojective. Therefore N is flat by the assumption. □

Recall that a ring R is right *perfect* if and only if every flat right R-module is projective (see [2, Theorem 28.4]).

Corollary 4.5. *Let R be a left perfect ring. A non-flat finitely presented right R-module M is f-test if and only if* $\mathrm{Tr}(M)$ *is p-indigent.*

Recall that a ring R is called right *finitely saturated* if every non-projective finitely generated right R-module is i-test. Since i-test modules are f-test, by Proposition 3.2, every non-projective finitely generated (presented) right R-module is f-test in case R is finitely saturated. Proposition 3.7 yields the following.

Corollary 4.6. *Let R be a right Noetherian ring. The following statements are equivalent.*

(1) *Every non-flat finitely generated right R-module is f-test.*

(2) *Every non-flat finitely generated right R-module is i-test, i.e. R is right finitely saturated.*

Proposition 4.7. *Let R be a ring with $w.dim(R) \leq 1$. The following statements are equivalent.*

(1) *Every non-flat finitely generated right R-module is f-test.*

(2) *Every non-flat right R-module is f-test.*

Proof. (2) $\Longrightarrow$ (1) is obvious. To (1) $\Longrightarrow$ (2), let M be a non-flat right R-module. It is well known that a module is flat if its finitely generated submodules are flat, (see [12, Corollary 3.49]). Hence, M has a non-flat finitely generated submodule N. N is f-test by the hypothesis. Then, M is f-test by Proposition 3.6. □

For convenience, we will define the following condition for a ring R:

(F) Every right R-module is flat or f-test.

By Corollary 4.6 and Proposition 4.7 we have the following.

Corollary 4.8. *Let R be a right Noetherian right hereditary ring. R satisfies* (F) *if and only if every non-flat right R-module is i-test.*

Theorem 4.9 ([5, Theorem 3.1]). *The following conditions are equivalent for a nonsemisimple ring R.*

(1) *Every simple right R-module is projective or p-indigent.*

(2) *R is a right Σ-CS ring with a unique simple singular module* (*up to isomorphism*).

(3) *There is a ring direct sum $R \cong S \times T$, where S is semisimple Artinian ring and T is an indecomposable ring which is either*

  (a) *matrix ring over a local QF-ring, or*

  (b) *hereditary Artinian serial ring with $J(T)^2 = 0$.*

Note that (1) in Theorem 4.9 implies that every non-projective simple module is i-test by Corollary 3.8. But the converse is not true.

Example 4.10. Let R be a commutative local perfect ring which is not Artinian. R has a unique singular simple module, say S. S is i-test by Corollary 3.8. If S is p-indigent, then R is Artinian by Theorem 4.9, a contradiction.

Lemma 4.11. *Let R be a serial ring. R is fully saturated if and only if every simple right R-module is projective or p-indigent.*

Proof. Necessity is a consequence of [5, Corollary 3.1]. Conversely, by Theorem 4.9, $R \cong S \times T$, where S is semisimple Artinian ring and T is an indecomposable ring which is either matrix ring over a local QF-ring or hereditary Artinian serial ring with $J(T)^2 = 0$. In the former case, T has a homogeneous socle and, in the terminology of [1], every non-injective module is indigent by [1, Proposition 13]. Then, R is fully saturated by [1, Theorem 16]. In the later case, the claim follows by [5, Proposition 3.1, Corollary 3.1]. □

Corollary 4.12. *An Artinian serial ring R with unique singular simple module* (*up to isomorphism*) *is fully saturated. In particular, R satisfies* (F).

Proof. Note that an Artinian ring R is serial if and only if the transpose of every singular simple R-module is simple, (see [8, Theorem 1]). By our hypothesis, there is a unique singular simple left (right) R-module S (J), and so $\mathrm{Tr}(S) \cong J$. Since R is C-ring, we have S is f-test. By the isomorphism $\mathrm{Tr}(S) \cong J$ and Proposition 4.4 and Corollary 4.5, J is p-indigent. Then R is fully saturated by Lemma 4.11. □

Theorem 4.13. *Let R be a nonsemisimple left perfect ring which has at least one finitely generated left maximal ideal (e.g., let R be left Noetherian). The following statements are equivalent.*

(1) *Every finitely generated right R-module is flat or f-test.*

(2) *Every finitely generated left R-module is projective or p-indigent.*

(3) *R is a left Σ-CS ring and every finitely generated singular left R-module is p-indigent.*

(4) *There is a ring direct sum $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either*

  (a) *finitely saturated matrix ring over a local QF-ring, or*

  (b) *hereditary Artinian serial ring with $J(T)^2 = 0$.*

Proof. By the hypothesis, R has a finitely presented singular simple left R-module, say S.

(1) $\Longrightarrow$ (2). By Corollary 4.5, every finitely presented left R-module is p-indigent. In particular, S is p-indigent. Let S' be a singular simple left R-module which is not isomorphic to S. Then, S is clearly S'-subprojective. Since S is p-indigent, S' is then projective, contradicting the singularity of S'. Thus, R has a unique singular simple module up to isomorphism. Therefore, by Theorem 4.13, R is a left Σ-CS ring, and so R is left Noetherian by [11, Theorem 2.11]. The claim follows by Corollary 4.5.

(2) $\Longrightarrow$ (3). In particular, every simple left R-module is projective or p-indigent. Thus R is a right Σ-CS ring by Theorem 4.9. Every finitely generated singular left R-module is p-indigent by the assumption.

(3) $\Longrightarrow$ (1). Let M be a non-projective finitely generated left R-module. Since R is a left Σ-CS ring, $M = F \oplus D$ for some projective module F and a singular module D by [11, Theorem II]. Since M is finitely generated, D is also finitely generated. Then D is p-indigent by the assumption. Suppose that M is N-subprojective. Then, D is N-subprojective. But D is p-indigent, and so N is projective. Therefore M is p-indigent. By Corollary 4.5, every finitely generated right R-module is projective or f-test.

(2) $\Longrightarrow$ (4). By Theorem 4.9, $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either matrix ring over a local QF-ring or hereditary Artinian serial ring with $J(T)^2 = 0$. In the former case, R is finitely saturated by Corollary 4.6. This completes the proof.

(4) $\Longrightarrow$ (1). $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either finitely saturated matrix ring over a local QF-ring or a hereditary Artinian serial ring with $J(T)^2 = 0$. In the former case, (1) follows by Corollary 4.6. In the later case, it follows by Corollary 4.12 and Proposition 2.8. □

Corollary 4.14. *Let R be a nonsemisimple ring. Then the following are equivalent.*

(1) *R is a nonsingular left Artinian ring which satisfies* (F).

(2) *R is a right* (*or left*) *hereditary fully saturated ring.*

(3) *$R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable hereditary Artinian serial ring with $J(T)^2 = 0$.*

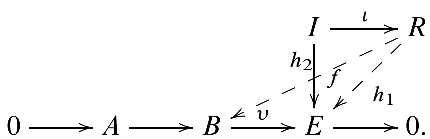
Proof. (1) $\Longrightarrow$ (3) is by Theorem 4.13, (3) $\Longrightarrow$ (1) is by Corollary 4.12 and Proposition 2.8, (2) $\Longleftrightarrow$ (3) is by [5, Theorem 3.2, Corollary 3.1]. □

Note that the assumption that R is a left perfect ring which has at least one finitely generated left maximal ideal is essential in Theorem 4.13.

Example 4.15. Let R be a Noetherian valuation domain which is not a field. Then R is finitely saturated, but is not right Artinian [17, Examples 4.6]. By Proposition 3.2, every finitely generated non-flat right module is f-test.

Lemma 4.16. *Let R be a nonsemisimple left coherent ring. If every finitely generated non-flat right R-module is f-test, then R is a left IF ring or a left semihereditary ring.*

Proof. In case every finitely generated right ideal of R is flat, every left ideal of R is flat by [12, 9.24]. Then, since R is left coherent, every finitely generated left ideal of R is projective by [12, Corollary 3.58]. Hence, R is a left semihereditary. If there is a non-flat finitely generated (presented) left ideal I of R, then $\mathrm{Tr}(I)$ is f-test by the assumption. By Proposition 4.4, $\underline{\mathfrak{Pr}}^{-1}(I)$ consists precisely of the flat left R-modules. Now, we show that every injective left R-module is flat, i.e. R is left IF ring. Let E be an injective left R-module. Consider the following diagram:



Since E is injective, there is a homomorphism $h_1 \colon R \to E$ such that $h_1 \circ \iota = f$. By projectivity of R, there is a homomorphism $h_2 \colon R \to B$ such that $\upsilon \circ h_2 = h_1$. Then, $f = h_1 \circ \iota = \upsilon \circ (h_2 \circ \iota)$, and hence E is I-subprojective. Therefore, E is flat by the first paragraph. □

LEMMA 4.17. *Let R be a left coherent ring such that R is not IF. Assume that every finitely generated non-flat right R-module is f-test. Then R satisfies* (F).

PROOF. By Lemma 4.16 and Proposition 4.7. □

Let R be a right $\aleph_0$-saturated ring. Then $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either (i) a matrix ring over a local QF-ring or (ii) Morita equivalent to a 2×2 upper triangular matrix ring over a skew-field (see [6, Theorem 13]). The characterization in case (i) can also be obtained under a weaker assumption by Corollary 4.6, Theorem 4.13 and Lemma 4.16, as follows.

COROLLARY 4.18. *Let R be a right finitely saturated left Noetherian ring. Then $R \cong S \times T$, where S is a semisimple Artinian ring and T is an indecomposable ring which is either* (i) *a matrix ring over a local QF-ring or* (ii) *a hereditary ring.*

Note that $\aleph_0$-saturated rings are Noetherian finitely saturated, but the converse is not true, see [17, Example 4.6].

Let R be a right semihereditary and M a finitely presented right R-module. Then, the Auslander-Bridger Transpose of M can be taken in the form $\mathrm{Tr}(M) = \mathrm{Ext}(M, R)$, see [13, Remark 5.2].

THEOREM 4.19. *Let R be a hereditary Noetherian ring. The following are equivalent for a non-flat left R-module M.*

(1) *M is i-test.*

(2) *M is f-test.*

(3) $\mathrm{Hom}(S, M) \neq 0$ *for each singular simple left R-module S.*

PROOF. (1) $\Longrightarrow$ (2) is by Proposition 3.2.

(2) $\Longrightarrow$ (3). Let S be a singular simple left R-module. Assume, contrarily, that $\mathrm{Hom}(S, M) = 0$. Consider the exact sequence $0 \to K \to P \to M \to 0$ with P projective. By the assumption, $0 \to \mathrm{Hom}(S, K) \to \mathrm{Hom}(S, P) \to \mathrm{Hom}(S, M) = 0$ is exact. Then, by Theorem 4.3, $0 \to \mathrm{Tr}(S) \otimes K \to \mathrm{Tr}(S) \otimes P \to \mathrm{Tr}(S) \otimes M \to 0$ is exact, and hence $\mathrm{Tor}(\mathrm{Tr}(S), M) = 0$. Since M is f-test, $\mathrm{Tr}(S)$ is

flat. But $\mathrm{Tr}(S)$ is finitely presented, and so it is projective by [10, Theorem 4.30]. Then, $S \cong \mathrm{Tr}(\mathrm{Tr}(S)) = \mathrm{Ext}(\mathrm{Tr}(S), R) = 0$, a contradiction.

(3) $\Longrightarrow$ (1). Let Λ be a complete set of representatives of singular simple R-modules. By (3), M contains a submodule isomorphic to the module $S = \bigoplus_{S_i \in \Lambda} S_i$. Note that hereditary Noetherian rings are C-rings by [4, p.98], and hence the module S is i-test by Corollary 3.8. Let $\mathrm{Ext}(M, N) = 0$ for some module N. Consider the exact sequence $0 \to S \hookrightarrow M \to M/S \to$. We have the following

$$\cdots \longrightarrow \mathrm{Ext}^1(M, N) = 0 \longrightarrow \mathrm{Ext}^1(S, N) \longrightarrow \mathrm{Ext}^2(M/S, N) = 0$$

Then, $\mathrm{Ext}^1(S, N) = 0$. But S is i-test, and so N is injective. □

Corollary 4.20. *An abelian group is f-test (i-test) if and only if it contains a submodule isomorphic to* $\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$, *where p ranges over all primes.*

Proposition 4.21. *Let R be a hereditary Noetherian ring and B a right R-module. An essential submodule A of B is f-test if and only if B is f-test.*

Proof. ($\Longrightarrow$) By Proposition 3.6.

($\Longleftarrow$) Assume, contrarily, that A is not f-test. Then, by Theorem 4.19, one gets $\mathrm{Hom}(S, A) = 0$ for some singular simple right R-module S. Since B is f-test, $\mathrm{Hom}(S, B) \neq 0$. Without loss of generality, we can assume that S is a submodule of B. We have, by essentiality, $A \cap S = S$. Then $\mathrm{Hom}(S, A) \neq 0$, a contradiction. Therefore, A is f-test by Theorem 4.19. □

The full characterization of nonsingular $\aleph_0$-saturated rings are given in [6, Theorem 13, Corollary 16]. But the full characterization of nonsingular finitely saturated rings is not known exactly. There exist rings that are finitely saturated, but are not $\aleph_0$-saturated [17, Examples 4.6 and 4.7]. In the following theorem, we present some equivalent conditions for the right finitely saturated ring which is not QF-ring.

Theorem 4.22. *Let R be a Noetherian ring such that R is not QF. The following are equivalent.*

(1) *R satisfies* (F).

(2) *Every right R-module is flat or i-test.*

(3) *R is right finitely saturated.*

(4) *R is right hereditary and every singular right R-module is f-test (i-test).*

(5) *R is right hereditary and every injective right R-module is flat or f-test (i-test).*

PROOF. (1) $\Longrightarrow$ (2) is by Lemma 4.16 and Corollary 4.8, (2) $\Longrightarrow$ (3) is obvious, (3) $\Rightarrow$ (1) is by Corollary 4.18, Corollary 4.6 and Proposition 4.7, and (1) $\Longrightarrow$ (4), (5) are by Lemma 4.16 and Theorem 4.19. For (4) $\Longrightarrow$ (1), note that nonsingular modules and flat modules are the same over hereditary Noetherian rings. Let M be a non-flat right R-module. Then M has a nonzero singular submodule, and hence it is f-test by Proposition 3.6. (5) $\Longrightarrow$ (1). Let M be a right R-module. By the assumption, $E(M)$ is flat or f-test. In first case, M is flat. In the later case, M is f-test by Proposition 4.21. □

Since condition (1) is left-right symmetric, it is also equivalent to the left-sided versions of (2) – (5).

A fully saturated ring R satisfies (F) by Proposition 3.2. We do not know if the converse is true in general but is true for the special case when R is an Artinian ring which is not QF by Theorem 4.22.

COROLLARY 4.23. *Let R be an Artinian ring such that R is not QF. R satisfies* (F) *if and only if R is* $\aleph_0$*-saturated* (*or fully saturated*) *ring.*

An indecomposable $\aleph_0$-saturated (or fully saturated) ring R which is not nonsingular is isomorphic to a matrix ring over a local QF-ring, ([6, Theorem 16]). We present the same characterization for rings which satisfy (F).

THEOREM 4.24. *Let R be a Noetherian ring such that R is not left* (*or right*) *nonsingular. If R satisfies* (F) (*or every finitely generated non-flat right* (*left*) *R-module is f-test*)*, then* $R \cong S \times T$*, where S is a semisimple Artinian ring and T is a indecomposable matrix ring over a local QF-ring.*

PROOF. By Lemma 4.16 and Theorem 4.13. □

Let us note that a QF-ring R is isomorphic to a matrix ring over a local ring if and only if it has homogeneous socle. A serial QF-ring with homogeneous socle satisfies (F) by Proposition 3.2, Theorem 4.9 and Lemma 4.11. We do not know precisely the structure of a QF-ring (isomorphic to a matrix ring over a local ring) over which every (finitely generated) right R-module is flat or f-test. The solution of this problem will give us a characterization of finitely saturated rings which are not nonsingular by Corollary 4.6.

A commutative local Artinian principal ideal ring R is fully saturated by [17, Example 4.5], and so it is a local QF-ring which satisfies (F) by Theorem 4.24. We do not know whether or not a local QF-ring which satisfies (F) is a principal ideal ring.

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