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On finite p-groups minimally of class greater than two

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In memory of Mario Curzio and Guido Zappa

Abstract – Let G be a finite nilpotent group of class three whose proper subgroups and proper quotients are nilpotent of class at most two. We show that G is either a 2-generated p-group or a 3-generated 3-group. In the first case the groups of maximal order with respect to a given exponent are all isomorphic except in the cases where $p = 2$ and $\exp(G) = 2^r$, $r \geq 4$. If G is 3-generated, then we show that there is a unique group of maximal order and exponent 3; but a similar result is not valid for exponent 9.

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1. Introduction

Let $\mathcal{K}$ be a class of finite groups. The finite group G is called a minimal non-$\mathcal{K}$-group (we write $G \in \mathrm{Min}(\mathcal{K})$), if $G \notin \mathcal{K}$ but every proper subgroup and every proper quotient of G belongs to $\mathcal{K}$.

For the class $\mathcal{K} = \mathcal{A}$ of all abelian groups, the structure of the groups in $\mathrm{Min}(\mathcal{A})$ can easily be derived from results of Miller-Moreno and Rédei (see [1, p. 281] and [1, p. 309]) and Lemma 2.1 below. Indeed, it is easy to see that such a group G

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is either a semidirect product $G = [N]Q$ of a minimal normal subgroup N by a complement Q of prime order, or it is one of the following groups:

i) $G_r = \langle a, b \mid a^{p^r} = b^p = 1,\ a^b = a^{1+p^{r-1}} \rangle$, $r \geq 2$;

ii) $G = \langle a, b \mid a^p = b^p = 1,\ [a, b] = c,\ c^p = 1,\ [a, c] = [b, c] = 1 \rangle$, p odd;

iii) the quaternion group Q_8.

G_r is of exponent p^r, and from i), ii), and iii) it follows that, for every exponent $p^r \neq 2, 4$, there exists precisely one p-group $G \in \mathrm{Min}(\mathcal{A})$ of exponent p^r. If $p^r = 4$ we get two groups: the dihedral group D_4 and the quaternion group Q_8, while the case $p^r = 2$ does not allow any such group.

In this paper, we discuss the minimal non-$\mathcal{N}_2$-groups, where $\mathcal{N}_2$ denotes the class of all nilpotent groups of class ≤ 2. The structure of non-nilpotent groups $G \in \mathrm{Min}(\mathcal{N}_2)$ follows immediately from the aforementioned results of Miller-Moreno and Rédei. Hence we will restrict attention to finite p-groups.

We prove that the p-groups in $\mathrm{Min}(\mathcal{N}_2)$ are either 2-generated or 3-generated 2-Engel. In order to give information on the p-groups in $\mathrm{Min}(\mathcal{N}_2)$ we determine the structure of the 2-generated free groups in the variety **W** of all nilpotent groups of exponent p^r $(r \geq 2)$ and class three, satisfying the law $[x, y, z]^p = 1$, and the structure of the 3-generated free groups in the variety **V** of all 2-Engel groups of exponent 3^r. We prove that there is a unique 2-generated group of exponent p in $\mathrm{Min}(\mathcal{N}_2)$: its order is p^4 with $p \geq 5$. If G is a 2-generated group in $\mathrm{Min}(\mathcal{N}_2)$of exponent p^r with $r \geq 2$ and p odd we see that $|G| \leq p^{3r}$; if $p = 2$, $r \geq 3$ then $|G| \leq 2^{3(r-1)}$; and if $p = 2$ and $r = 2$ then $|G| \leq 2^{3r-1}$. We give an explicit construction of the groups in $\mathrm{Min}(\mathcal{N}_2)$of exponent p^r and maximal order and we show that such groups are all isomorphic except in the case $p = 2$ and $r \geq 4$. If G is a 3-generated group of exponent 3 in $\mathrm{Min}(\mathcal{N}_2)$, we show that $|G| = 3^7$ and G is isomorphic to the 3-generated relatively free group in the variety of all groups of exponent 3 but the groups of exponent 9 of maximal order in $\mathrm{Min}(\mathcal{N}_2)$ are not isomorphic.

In the following the notation is standard. $G = [N]Q$ indicates the semidirect product of the normal subgroup N by the subgroup Q, and $d(G)$ indicates the minimal number of generators of G. Moreover $o(x)$ is the order of the element x. If **V** is a variety, $Fr_n(\mathbf{V})$ denotes the relatively free group of rank n in **V**.

All groups considered in this paper are finite.

2. Preliminaries

LEMMA 2.1. *A finite nilpotent group of class c ($c \geq 2$) has all of its proper quotients of class at most $c-1$ if and only if $Z(G)$ is cyclic and the c-th term of the lower central series $\Gamma_c(G)$ is of order p.*

PROOF. Suppose that G has class c and that all proper quotients of G are of class at most $c-1$. Then G is monolithic. Indeed, if N_1 and N_2 are two distinct minimal normal subgroups, then $G = G/N_1 \cap N_2$ is embedded in $G/N_1 \times G/N_2$ which is nilpotent of class at most $c-1$. As G is monolithic, $Z(G)$ is cyclic. If N is the minimal normal subgroup of G, then G/N is nilpotent of class at most $c-1$. So $\Gamma_c(G) = N$.

Conversely let G be a nilpotent group of class c and assume that $Z(G)$ is cyclic and $\Gamma_c(G)$ is of order p. Then for every normal subgroup K of G, we have $\Gamma_c(G) \subseteq K$. So

$$\Gamma_c(G/K) = \Gamma_c(G)K/K = 1. \qquad \square$$

LEMMA 2.2. *Let G be a nilpotent group such that all of its proper subgroups have class at most c but G has not class c. Then $Z(G) \subseteq \Phi(G)$.*

PROOF. Let M be a maximal subgroup of G. Then $M \trianglelefteq G$. Suppose that $Z(G) \nsubseteq M$. Then $G = Z(G)M$ and so G has class c, a contradiction. $\square$

LEMMA 2.3. *Let G be a p-group in* $\mathrm{Min}(\mathcal{N}_2)$. *Then either G can be generated by two elements, or G is a 2-Engel 3-group generated by three elements.*

PROOF. Suppose that G cannot be generated by two elements. Then for all $x, y \in G$ we have that $\langle x, y\rangle$ is a proper subgroup of G. So it is nilpotent of class 2. In particular G satisfies the 2-Engel condition. If $p \neq 3$ then G is nilpotent of class two ([1, p. 288]), a contradiction. So $p = 3$. Moreover G is generated by three elements, otherwise all subgroups generated by three elements would be proper subgroups of G, and G would be nilpotent of class two, a contradiction. $\square$

We now give a sufficient criterion for a p-group generated by two elements to have all of its proper subgroups of class two.

LEMMA 2.4. *Let G be a p-group which can be generated by two elements. Assume that $[\Phi(G), G] \leq Z(G)$. Then every proper subgroup of G is nilpotent of class two.*

Proof. It suffices to show that every maximal subgroup M of G is of class two. As G is generated by two elements, we have $G/\Phi(G) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. So $M = \langle \Phi(G), x\rangle$ for some x in M. We get $M' = \Phi(G)' \cdot [\Phi(G), x]$. By hypothesis, both factors are contained in $Z(G)$, so that the class of M is two. □

3. Min($\mathcal{N}_2$)-groups with two generators

We start with the smallest case:

Proposition 3.1. *Let $G \in \mathrm{Min}(\mathcal{N}_2)$ be a group of prime exponent p. If $d(G) = 2$, then $p \geq 5$, $|G| = p^4$ and $G \cong [N]\langle u\rangle$, where $N = \langle v_1\rangle \times \langle v_2\rangle \times \langle v_3\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and the action of u on N is given by*

$$v_1^u = v_1, \quad v_2^u = v_1 v_2, \quad v_3^u = v_2 v_3.$$

Proof. As $\exp(G) = p$, we infer that $p \neq 2$ and $|G/G'| = p^2$. Moreover $G'/\Gamma_3(G)$ is cyclic of order p and by Lemma 2.1, we have $|\Gamma_3(G)| = p$. So we get $|G| = p^4$. An inspection of the groups of order p^4 (see [1, p. 346]) yields the result . □

A group G in Min($\mathcal{N}_2$) of exponent p^r belongs to the variety $\mathbf{W}$ of all groups of exponent p^r and nilpotent of class three satisfying the law $[x, y, z]^p = 1$ (see Lemma 2.1).

We now collect some information of $Fr_2(\mathbf{W})$.

Proposition 3.2. *Let p^r be a power of a prime p and $r \geq 2$. Let $F = Fr_2(\mathbf{W})$ with free generators x, y. Then*

a) *$F/F' \simeq \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r}$ and either $|F'/\Gamma_3(F)| = p^r$ if $p \geq 3$ or $|F'/\Gamma_3(F)| = 2^{r-1}$. Moreover $\Gamma_3(F) \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ and hence $|F| = p^{3r+2}$ for $p \geq 3$; and $|F| = 2^{3r+1}$ if $p = 2$;*

b) $Z(F) \simeq \begin{cases} \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p & \text{if } p \geq 3, \\ \mathbb{Z}_{2^{r-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r \geq 3, \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } p = 2 \text{ and } r = 2; \end{cases}$

c) $[F^p, F] \leq Z(F)$;

d) *every proper subgroup of F is nilpotent of class two.*

Proof. a) As $\exp(F) = p^r$, we infer that $|F/F'| \leq p^{2r}$. Moreover, $F'/\Gamma_3(F) = \langle [x,y]\Gamma_3(F)\rangle$ is cyclic of exponent dividing p^r if $p \neq 2$ and 2^{r-1} otherwise (we have $1 = (xy)^{2^r} \equiv x^{2^r} y^{2^r} [y,x]^{\binom{2^r}{2}} \pmod{\Gamma_3(F)}$, so $[y,x]^{2^{r-1}} \equiv 1 \pmod{\Gamma_3(F)}$). Then $|F'/\Gamma_3(F)| \leq p^r$ if $p \neq 2$ or $\leq 2^{r-1}$ otherwise. Finally, we have $|\Gamma_3(F)| \leq p^2$, because there are only two basic commutators of weight 3. This implies $|F| \leq p^{3r+2}$ if $p \neq 2$, $|F| \leq 2^{3r+1}$ otherwise.

We now construct a group F_0, belonging to the variety $\mathbf{W}$, which has order either p^{3r+2} if $p \geq 3$, or 2^{3r+1}. So it will be $F_0 \simeq Fr_2(\mathbf{W})$.

Let $N = [A]\langle x\rangle$ be the semidirect product of the abelian group

$$A = \langle u\rangle \times \langle v_1\rangle \times \langle v_2\rangle,$$

with the cyclic group $\langle x\rangle$ of order p^r; where

$$o(v_1) = o(v_2) = p$$

and

$$o(u) = \begin{cases} p^r & \text{if } p \geq 3,\\ 2^{r-1} & \text{otherwise.}\end{cases}$$

The action of x on A is given by

$$u^x = uv_1, \quad v_1^x = v_1, \quad v_2^x = v_2.$$

Then we consider the group $F_0 = [N]\langle y\rangle$, where y is a cyclic group of order p^r and the action of y on N is given by

$$x^y = xu, \quad u^y = uv_2, \quad v_1^y = v_1, \quad v_2^y = v_2.$$

We can immediately verify that

$$u = [x,y], \quad v_1 = [u,x] = [x,y,x], \quad v_2 = [u,y] = [x,y,y].$$

So $F_0 = \langle x, y\rangle$. Moreover

$$F_0' = A, \quad F_0/F_0' = \langle xF_0'\rangle \times \langle yF_0'\rangle \simeq \mathbf{Z}_{p^r} \times \mathbf{Z}_{p^r}, \quad F_0'/\Gamma_3(F_0) = \langle u\Gamma_3(F_0)\rangle,$$

$$\Gamma_3(F_0) = \langle v_1\rangle \times \langle v_2\rangle \simeq \mathbf{Z}_p \times \mathbf{Z}_p, \qquad \Gamma_3(F_0) \leq Z(F_0).$$

We observe that, if $p \geq 3$, then $\langle u\Gamma_3(F_0)\rangle \simeq \mathbf{Z}_{p^r}$, while if $p = 2$, then $\langle u\Gamma_3(F_0)\rangle \simeq \mathbf{Z}_{2^{r-1}}$. By the above conditions we deduce that F_0 is nilpotent of class three with $|F_0| = p^{3r+2}$ if $p \geq 3$ while, if $p = 2$ then $|F_0| = 2^{3r+1}$.

It remains to show that the exponent of F_0 is p^r for all p.

First of all we prove that the exponent of N is p^r for all p. (We note that for $p \geq 3$ we have $\exp(N) = p^r$, and for $p \geq 5$ we have $\exp(F_0) = p^r$ by the regularity of these groups).

Let $w \in N$ where $w = ax^k$ with $a \in A$. Since N is of class two we have

$$w^n = (ax^k)^n = a^n x^{kn} [x^k, a]^{\binom{n}{2}}.$$

Since $[x^k, a] \in \Gamma_3(F_0)$ which has exponent p and $r \geq 2$, we have that $[x^k, a]^{\binom{p^r}{2}} = 1$. So $(ax^k)^{p^r} = 1$.

If w is an element of N we set

$$[w, y^h] = a_1 \in A, \quad [a_1, w] = c_1 \in \Gamma_3(F_0), \quad [a_1, y^h] = c_2 \in \Gamma_3(F_0).$$

For $n \geq 2$ it is easy to prove by induction the following results

$$[w, y^{hn}] = a_1^n c_2^{\binom{n}{2}} \tag{1}$$

and

$$(wy^h)^n = w^n y^{hn} a_1^{-\binom{n}{2}} c_1^{-\binom{n}{3}} c_2^{-2\binom{n}{3} - \binom{n}{2}}. \tag{2}$$

Since N is of exponent p^r and a has order 2^{r-1} for $p = 2$, we have by (2) that the exponent of F_0 is p^r for all p.

From now on we identify F_0 with F.

b) By the structure of F we can write an element $z \in F$ in the form

$$z = u^k v_1^l v_2^m x^i y^j.$$

We have $z \in Z(F)$ if and only if $[z, x] = [z, y] = 1$. So

$$\begin{aligned} 1 = [z, y] &= [u^k x^i y^j, y] \\ &= [u^k, y][u^k, y, x^i][x^i, y] \\ &= [u, y]^k [x, y]^i v_1^{\binom{i}{2}} \\ &= u^i v_1^{\binom{i}{2}} v_2^{k+ij}. \end{aligned} \tag{3}$$

Similarly we have

$$1 = [z, x] = u^{-j} v_1^k v_2^{-\binom{j}{2}}. \tag{4}$$

Therefore, for $p \geq 3$ we have $i \equiv j \equiv 0 \pmod{p^r}$ and $k \equiv 0 \pmod{p}$. It follows $z = u^{pk_1} v_i^l v_2^m$ with $k = pk_1$. This implies

$$Z(F) = \langle u^p \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \simeq \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p.$$

If $p = 2$ we must have $i \equiv j \equiv 0 \pmod{2^{r-1}}$ and $k \equiv 0 \pmod 2$. So we have $z = u^{2k_1} v_1^l v_2^m x^{2^{r-1} i_1} y^{2^{r-1} j_1}$ where $k = 2k_1,\ i = 2^{r-1} i_1,\ j = 2^{r-1} j_1$. Then, if $r \geq 3$ we get

$$Z(F) = \langle u^2 \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle x^{2^{r-1}} \rangle \times \langle y^{2^{r-1}} \rangle \simeq \mathbb{Z}_{2^{r-2}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

If $p = 2$ and $r = 2$, we have $u^2 = 1$, so $z = v_1^l v_2^m x^{2i_1} y^{2j_1}$ with $i = 2i_1$, $j = 2j_1$. But the condition $\binom{i}{2} \equiv 0 \pmod 2$ implies $i_1(2i_1 - 1) \equiv 0 \pmod 2$. So $i_1 \equiv 0 \pmod 2$. Similarly we obtain $j_1 \equiv 0 \pmod 2$. Therefore $z = v_1^l v_2^m$ and

$$Z(F) = \langle v_1 \rangle \times \langle v_2 \rangle = \Gamma_3(F) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

c) Observe that for all $a, b, c \in F$ we have $[a^p, b, c] = [a, b, c]^p = 1$. So $[F^p, F] \leq Z(F)$.

d) We have $[\Phi(F), F] = [F' F^p, F]$. Since $[F', F] = \Gamma_3(F) \leq Z(F)$ and $[F^p, F] \leq Z(F)$ by Part c), it follows that $[\Phi(F), F] \leq Z(F)$. So by Lemma 2.4 every proper subgroup of F is nilpotent of class two. $\square$

Theorem 3.3. *Let p be a prime and $r \geq 2$.*

a) *Let G be a 2-generator group in* $\mathrm{Min}(\mathcal{N}_2)$ *with* $\exp G = p^r$. *Then*

$$|G| \leq \begin{cases} p^{3r} & \text{if } p \geq 3, \\ 2^{3(r-1)} & \text{if } p = 2,\ r \geq 3, \\ 2^{3r-1} & \text{if } p = 2,\ r = 2. \end{cases}$$

b) *For each one of the above three cases, there is a group of exponent p^r in* $\mathrm{Min}(\mathcal{N}_2)$ *whose order attains the upper bound.*

Proof. a) Every 2-generator group $G \in \mathrm{Min}(\mathcal{N}_2)$ of exponent p^r is a quotient F/H of F where $H \cap Z(F)$ does not contain $\Gamma_3(F)$ because $G \simeq F/H$ is of class three. As $Z(G)$ is cyclic by Lemma 2.1, also $Z(F)/(H \cap Z(F))$ must be cyclic. Then $H \cap Z(F)$ is abelian of rank ≥ 2 if $p \neq 2$; of rank ≥ 4 if $p = 2$ and $r \geq 3$; of rank 1 if $p = 2$ and $r = 2$. Thus if $p \geq 3$ we have $|H| \geq p^2$ and $|G| \leq p^{3r}$; if $p = 2$ and $r \geq 3$ we have $|H| \geq 2^4$ and $|G| \leq 2^{3(r-1)}$. Finally we observe that, if $p = 2$ and $r = 2$, no quotient of F by a proper subgroup of $Z(F)$ is in $\mathrm{Min}(\mathcal{N}_2)$. In fact, there are only three proper subgroups of $Z(F)$, namely $H_1 = \langle v_1 \rangle$, $H_2 = \langle v_2 \rangle$, $H_3 = \langle v_1 v_2 \rangle$. We see that in each quotient F/H_i, $(i = 1, 2, 3)$ there are couples of independent elements of $Z(F/H_i)$: for example, $x^2 H_1, v_2 H_1$ in $Z(F/H_1)$; $y^2 H_2, v_1 H_2$ in $Z(F/H_2)$ and $(xy)^2 H_3, v_1 v_2 H_3$ in $Z(F/H_3)$. So no F/H_i belongs to $\mathrm{Min}(\mathcal{N}_2)$ and therefore $|G| \leq 2^{3r-1}$.

b) For the first two cases of a) we consider respectively the subgroups of $Z(F)$:

$$\begin{cases} R_1 = \langle v_2, u^{p^{r-1}} v_1 \rangle & \text{if } p \geq 3,\ r \geq 2, \\ R_2 = \langle v_2, v_1 u^{2^{r-2}}, v_1 x^{2^{r-1}}, v_1 y^{2^{r-1}} \rangle & \text{if } p = 2,\ r \geq 4, \\ R_3 = \langle v_2, u^2, x^4, v_1 y^4 \rangle & \text{if } p = 2, r = 3. \end{cases}$$

We want to show that $G_t = F/R_t \in \mathrm{Min}(\mathfrak{N}_2)$ $(t = 1, 2, 3)$. First, since R_t does not contain $\Gamma_3(F)$ it follows that G_t is of class three. Moreover, as every proper subgroup of F is of class two, the same holds for G_t. By definition of G_t, we also have $|\Gamma_3(G_t)| = p$. Therefore, by Lemma 2.1, it is sufficient to show that $Z(G_t)$ is cyclic.

Let us consider a typical element $zR_t \in G_t$ with $z = u^k v_1^l v_2^m x^i y^j \in F$. Then $zR_t \in Z(F/R_t)$ if and only if $[z, y] \in R_t$ and $[z, x] \in R_t$. By (3) and (4), this holds if and only if

$$u^i v_1^{\binom{i}{2}} v_2^{k+ij} \in R_t$$

and

$$u^{-j} v_1^k v_2^{-\binom{j}{2}} \in R_t$$

For $p \geq 3$ this happens if and only if there are $\alpha, \beta \in \mathbb{Z}$ such that

$$u^i v_1^{\binom{i}{2}} = (u^{p^{r-1}} v_1)^\alpha, \tag{5}$$

$$u^{-j} v_1^k = (u^{p^{r-1}} v_1)^\beta. \tag{6}$$

By equation (5) we obtain that $i \equiv \alpha p^{r-1} \pmod{p^r}$ and $\frac{i(i-1)}{2} \equiv \alpha \pmod{p}$. So

$$i\left(1 - \frac{i-1}{2} p^{r-1}\right) \equiv 0 \pmod{p^r} \tag{7}$$

which gives $i \equiv 0 \pmod{p^r}$.

By Equation (6) we get $-j \equiv p^{r-1}\beta \pmod{p^r}$ and $k \equiv \beta \pmod{p}$. So

$$j \equiv -p^{r-1} k \pmod{p^r}. \tag{8}$$

Therefore, we have that $zR_1 \in Z(F/R_1)$ if and only if

$$z = u^k v_1^l y^{-p^{r-1}k} = (uy^{-p^{r-1}})^k v_1^l.$$

We observe that

$$(uy^{-p^{r-1}})^{-p^{r-1}} = u^{-p^{r-1}} y^{p^{2r-2}} = u^{-p^{r-1}}.$$

Since $u^{p^{r-1}} v_1 \in R_1$, we have $v_1 R_1 = u^{-p^{r-1}} R_1 = (uy^{-p^{r-1}})^{-p^{r-1}} R_1$. Then $zR_1 = (uy^{-p^{r-1}})^{k-p^{r-1}l} R_1$. Thus $Z(F/R_1) = \langle uy^{-p^{r-1}} \rangle R_1$ is cyclic.

If $p = 2$ and $r \geq 3$ an analogous calculation yields

$$u^i v_1^{\binom{i}{2}} = (u^{2^{r-2}} v_1)^{\alpha} \tag{9}$$

and

$$u^{-j} v_1^k = (u^{2^{r-2}} v_1)^{\beta} \tag{10}$$

By (9) and (10) we obtain

$$i(1 - (i-1)2^{r-3}) \equiv 0 \pmod{2^{r-1}}$$

and

$$j \equiv -2^{r-2} k \pmod{2^{r-1}}.$$

So if $r \geq 4$, we obtain $i \equiv 0 \pmod{2^{r-1}}$; while if $r = 3$ we have $i \equiv 0 \pmod 2$.

In the case $p = 2$, $r \geq 4$ it follows that $zR_2 \in Z(F/R_2)$ if and only if $z = (uy^{-2^{r-1}})^k v_1^l x^{2^{r-1} i_1}$ with $i = 2^{r-1} i_1$. Since $(uy^{-2^{r-1}})^{-2^{r-2}} = u^{-2^{r-2}}$, we have $u^{-2^{r-2}} R_2 = v_1 R_2 = x^{2^{r-1}} R_2 = y^{2^{r-1}} R_2$. Thus $zR_2 = (uy^{-2^{r-1}})^{k-2^{r-2}(l+i_1)}$ and $Z(F/R_2) = \langle uy^{-2^{r-2}} \rangle R_2$ is cyclic.

In the case $p = 2$, $r = 3$ we have $zR_3 \in Z(F/R_3)$ if and only if $z = (uy^{-2})^k v_1^l$. Since $(uy^{-2})^{-2} = u^{-2} y^4 = u^{-2}$ and $u^{-2} R_3 = v_1 R_3$, we have $zR_3 = (uy^{-2})^{k-2l} R_3$. Thus, $Z(F/R_3) = \langle uy^{-2} R_3 \rangle$ is cyclic.

Finally, in the case $p = 2$ and $r = 2$, we consider the normal (non central) subgroup

$$R_4 = \langle v_2, y^2 \rangle.$$

Then $zR_4 \in Z(F/R_4)$ if and only if $z = u^k v_1^l x^i y^j$ with $k \equiv 0 \pmod 2$, $j \equiv 0 \pmod 2$, $i \equiv 0 \pmod 2$ and $\frac{i(i-1)}{2} \equiv 0 \pmod 2$. The last two conditions implies $i \equiv 0 \pmod 4$. Then $zR_4 = v_1^l R_4$ and thus $Z(F/R_4) = \langle v_1 \rangle R_4$ is cyclic. □

Theorem 3.4. *Let p be a prime and $r \geq 2$. If $p \geq 3$ or $p = 2$ and either $r = 3$ or $r = 2$, then all 2-generator groups in* $\mathrm{Min}(\mathcal{N}_2)$ *of exponent p^r and maximal order are isomorphic.*

Proof. Using the same notation as in the proof of Theorem 3.3, let $F/H \in \mathrm{Min}(\mathcal{N}_2)$ be of exponent p^r $(p \geq 3)$ and maximal order $|F/H| = p^{3r}$. By the proof of Theorem 3.3 it follows that $H \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. We will show that there exists an automorphism φ of F with $\varphi(H) = R_1$ and so $F/H \simeq F/R_1$. Since F/H is of nilpotency class three, we have that $\Gamma_3(F) \nleq H$. As $Z(F)$ is of rank three and $H \simeq \mathbb{Z}_p \times \mathbb{Z}_p$, we get $|H \cap \Gamma_3(F)| = p$. We construct the automorphism φ in two steps. First we give an automorphism which maps $H \cap \Gamma_3(F)$ onto the subgroup $\langle v_2 \rangle$ of R_1.

If $H \cap \Gamma_3(F) = \langle v_1 \rangle$, we consider the automorphism α of F with $\alpha(x) = y$ and $\alpha(y) = x$. In this case we have $\alpha([x, y, x]) = [y, x, y] = [x, y, y]^{-1}$, that is $\alpha(v_1) = v_2^{-1} \in R_1$.

If $H \cap \Gamma_3(F) = \langle v_2 v_1^h \rangle$ for some $h \in \mathbb{Z}$, we consider the automorphism β of F with $\beta(x) = x$ and $\beta(y) = x^{-h}y$. Then we have $\beta(v_1) = [x, x^{-h}y, x] = v_1$ and $\beta(v_2) = [x, x^{-h}y, x^{-h}y] = [x, y, y][x, y, x]^{-h} = v_2 v_1^{-h}$. So we have $\beta(v_2 v_1^h) = v_2 v_1^{-h} v_1^h = v_2 \in R_1$.

In both cases we have now found an automorphism of F which maps H onto a subgroup H^* of $Z(F)$ with

$$H^* \cap \Gamma_3(F) = \langle v_2 \rangle.$$

Therefore we may assume that $H^* = \langle v_2, v_1^m u^{np^{r-1}} \rangle$ with $m, n \in \mathbb{Z}$ and $n \not\equiv 0 \pmod p$. Since $n \not\equiv 0 \pmod p$, we have

$$H^* = \langle v_2, v_1^h u^{p^{r-1}} \rangle$$

with $h \equiv mn^{-1} \pmod p$. First let $h \not\equiv 0 \pmod p$. We consider the automorphism γ of F such that $\gamma(x) = x^h$ and $\gamma(y) = y$. We have $\gamma(v_2) = v_2^h \in H^*$ and

$$\gamma([x, y, x][x, y]^{p^{r-1}}) = [x, y, x]^{h^2}[x, y]^{hp^{r-1}} = ([x, y, x]^h [x, y]^{p^{r-1}})^h \in H^*.$$

So $\gamma(v_1 u^{p^{r-1}}) = (v_1^h u^{p^{r-1}})^h$ and $R_1^\gamma = H^*$.

Finally let $h \equiv 0 \pmod p$. So $H^* = \langle v_2, , u^{p^{r-1}} \rangle$. Since $[x^{p^{r-1}}, y] = u^{p^{r-1}} \in H^*$, we have that $x^{p^{r-1}} H^* \in Z(F/H^*)$. Similarly $y^{p^{r-1}} H^* \in Z(F/H^*)$. But the images of $x^{p^{r-1}}$ and $y^{p^{r-1}}$ under the canonical epimorphism of F/H^* onto $F/F' \simeq \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r}$ are independent, and so the center of F/H^* is not cyclic. This case does not occur.

Let $F/H \in \mathrm{Min}(\mathcal{N}_2)$ be of exponent 2^3 and maximal order 2^6. Then $|H| = 2^4$ and H must contain exactly one of the three subgroups $\langle v_1 \rangle$, $\langle v_2 \rangle$, $\langle v_1 v_2 \rangle$ of $\Gamma_3(F)$. The automorphism α of F, defined by $\alpha(x) = y$ and $\alpha(y) = x^{-1}y^{-1}$, is of order 3 and acts transitively on the non-identity elements of $\Gamma_3(F)$. So without loss of generality we may assume $H \cap \Gamma_3(F) = \langle v_2 \rangle$ and $v_1 \notin H$. Now consider the intersection of H with the subgroup $E = \langle v_1, v_2, u^2 \rangle = \Omega_1(F')$. Since $E/E \cap H \cong EH/H \leq Z(F/H)$ which is cyclic, we get $|E \cap H| = 2^2$.

The subgroups of E of order 2^2, that contain v_2 but not v_1 are precisely $L_1 = \langle v_2, u^2 \rangle$ and $L_2 = \langle v_2, u^2 v_1 \rangle$. If $L_2 \leq H$, then $v_1 L_2, x^2 L_2, u y^2 L_2 \in Z(F/L_2)$. So $Z(F/H)$ is not cyclic, because $Z(F/H) \cong Z((F/L_2)/(H/L_2))$ contains $Z(F/L_2)/(H/L_2)$ and $x^2 L_2, u y^2 L_2 \notin H/L_2$ since $H \leq Z(F)$. Therefore $L_1 \leq H$ and H/L_1 is a subgroup of rank 2 of $Z(F/L_1)$ that does not contain $v_1 L_1$. Since $|Z(F)/L_1| = 2^3$, we get the following four subgroups:

$$H_1 = \langle v_2, u^2, v_1 x^4, v_1 y^4 \rangle, \quad H_2 = \langle v_2, u^2, x^4, v_1 y^4 \rangle,$$
$$H_3 = \langle v_2, u^2, v_1 x^4, y^4 \rangle, \quad H_4 = \langle v_2, u^2, x^4, y^4 \rangle.$$

By a simple calculation, using the relations (3) and (4), we see that F/H_1 and F/H_2 have cyclic center, while the centers of the two remaining quotients are not cyclic. Finally, the theorem for the case $p = 2$ and $r = 3$ is proved by the automorphism β defined by $\beta(x) = xy$, $\beta(y) = y$ that fixes v_2 and u^2 and maps H_1 onto H_2.

Let $F/H \in \mathrm{Min}(\mathcal{N}_2)$ be of exponent 4 and maximal order 2^5. Then $|H| = 4$ and F/H is nilpotent of class 3 with cyclic center (see Lemma 2.1). Since $\Gamma_3(F) = \langle v_1, v_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, we must have $|H \cap \Gamma_3(F)| = 2$. As in the previous case, without loss we may assume $H \cap \Gamma_3(F) = \langle v_2 \rangle$. Let $L = \langle v_2 \rangle$. It is easy to see that $Z(F/L) = \langle v_1 L \rangle \times \langle y^2 L \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Now $H/L \trianglelefteq F/L$ and $|H/L| = 2$. If $v_1 L \in H/L$, then $\Gamma_3(F) = \langle v_1, v_2 \rangle \leq L$ and so F/L would be of class two, a contradiction. Hence $v_1 \notin H/L$, and hence either $H = \langle v_2, y^2 \rangle$ or $H = \langle v_2, v_1 y^2 \rangle$. But the automorphism γ of F, defined by $\gamma(x) = x$ and $\gamma(y) = x^2 y$, centralizes $\Gamma_3(F)$ and maps y^2 to $v_1 y^2$. Therefore all the quotients $F/H \in \mathrm{Min}(\mathcal{N}_2)$ of order 2^5 are isomorphic. $\square$

Remark 3.1. In the case $p = 2$ and $r \geq 4$, there are non-isomorphic groups in $\mathrm{Min}(\mathcal{N}_2)$ of exponent 2^r and maximal order $2^{3(r-1)}$. In fact, the two quotients F/R_2 and F/R_2^*, where $R_2 = \langle v_2, v_1 u^{2^{r-2}}, v_1 x^{2^{r-1}}, v_1 y^{2^{r-1}} \rangle$ and $R_2^* = \langle v_2, v_1 u^{2^{r-2}}, x^{2^{r-1}}, y^{2^{r-1}} \rangle$, have cyclic center but one can check that the power 2^{r-1} of an element $g = u^k v_1^l v_2^m x^i y^j$ in F is

$$g^{2^{r-1}} = (u^k x^i y^j)^{2^{r-1}} = (x^{2^{r-1}})^i (y^{2^{r-1}})^j (u^{-2^{r-2}(2^r-1)})^{ij};$$

so we have

$$g^{2^{r-1}} R_2 = v_1^{i+j+ij} R_2$$

and

$$g^{2^{r-1}} R_2^* = v_1^{ij} R_2^*.$$

It follows that the number of the elements of order 2^r is different in the two quotients and F/R_2, F/R_2^* are not isomorphic.

REMARK 3.2. The referee suggested to investigate the existence of groups in $\mathrm{Min}(\mathcal{N}_2)$ of exponent p^r and order p^k for all k with $r+2 \leq k < 3r$. He gave an example of minimal order p^{r+2}. Namely the group:

$$G_1 = \langle \bar{x},\ \bar{y},\ \bar{u} \mid \bar{x}^{p^r} = 1 = \bar{y}^p = \bar{u}^p,\ [\bar{x}, \bar{y}] = \bar{u},\ [\bar{u}, \bar{x}] = \bar{x}^{p^{r-1}},\ [\bar{u}, \bar{y}] = 1\rangle.$$

We have $G_1 = F/L_1$ where $L_1 = \langle v_2,\ u^p,\ x^{p^{r-1}} v_1^{-1},\ y^p\rangle$.

An other example of minimal order non-isomorphic to the previous one is given by

$$G_2 = \langle \bar{x},\ \bar{y},\ \bar{u} \mid \bar{x}^p = 1 = \bar{y}^{p^r} = \bar{u}^p,\ [\bar{x}, \bar{y}] = \bar{u},\ [\bar{u}, \bar{x}] = \bar{y}^{p^{r-1}},\ [\bar{u}, \bar{y}] = 1\rangle;$$

in fact, G_2 has an abelian maximal subgroup $\langle \bar{u}, \bar{y}\rangle$, while G_1 has no abelian maximal subgroup. This is the quotient of F by the subgroup:

$$L_2 = \langle v_2,\ u^p,\ x^p,\ y^{p^{r-1}} v_1^{-1}\rangle.$$

Other examples of order $p^{r+\frac{r+1}{2}}$, with $r = 2h+1$, are given by splitting metacyclic groups:

$$M_h = \langle \bar{x},\ \bar{y},\ \mid \bar{y}^{p^{2h+1}} = 1 = \bar{x}^{p^{h+1}},\ [\bar{y}, \bar{x}] = \bar{y}^{p^h}\ \rangle.$$

These are the quotients of F by the subgroups:

$$N_h = \langle v_2,\ u y^{p^h},\ x^{p^{h+1}},\ v_1 y^{p^{2h}}\rangle.$$

The problem of the existence of groups in $\mathrm{Min}(\mathcal{N}_2)$ of order other than of the maximal one seems of non easy solution. We have to construct quotients F/L of F with cyclic center. Considering the automorphisms α and β used in the proof of the Theorem 3.4, we can assume, W.L.O.G., that $L \geq H^* = \langle v_2,\ u^{p^{r-1}}\rangle$. We prove that the orders of such quotients cannot be greater than p^{2r+1}. Since $Z(F/H^*) \cong \mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, L has to contain a subgroup isomorphic to $\mathbb{Z}_{p^{r-1}} \times \mathbb{Z}_p \times \mathbb{Z}_p$. In fact

$$F/L \cong (F/H^*)/(L/H^*)$$

and

$$Z(F/L) \geq (Z(F/H^*)(L/H^*))/(L/H^*);$$

since both

$$Z(F/H^*) = \langle u^p H^*,\ v_1 H^*,\ x^{p^{r-1}} H^*,\ y^{p^{r-1}} H^*\rangle$$

and

$$(Z(F/H^*)(L/H^*))/(L/H^*)$$

has to be cyclic, it follows that L/H^* has to contain a complement of $\langle v_1 H^*\rangle$ in $Z(F/H^*)$. Thus $|L| \geq p^{r+1}$ and $|F/L| \leq p^{2r+1}$.

4. Min($\mathcal{N}_2$)-groups with three generators

It follows from Lemma 2.3 that a group $G \in \mathrm{Min}(\mathcal{N}_2)$, with three generators and exponent 3^r $(r \geq 1)$, belongs to the variety $\mathbf{V}$ of all 2-Engel groups of exponent 3^r. So G is a quotient of $Fr_3(\mathbf{V})$.

PROPOSITION 4.1. *Let $F = Fr_3(V)$ be the relatively free group with free generators x, y, z in the variety* $\mathbf{V}$.

a) $|\Gamma_3(F)| = 3$ *and* $|F| = 3^{6r+1}$.

b) $Z(F) \cong \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_{3^{r-1}} \times \mathbb{Z}_3$.

c) *Every proper subgroup of F is nilpotent of class two.*

d) *F belongs to* $\mathrm{Min}(\mathcal{N}_2)$ *if and only if $r = 1$.*

e) *Let F/H be a quotient of F of class three. Then $F/H \in$* $\mathrm{Min}(\mathcal{N}_2)$ *if and only if $Z(F/H)$ is cyclic.*

PROOF. a) Note that F/F' is a 3-generated group of exponent 3^r, so $|F/F'| \leq 3^{3r}$. Similarly, we have $|F'/\Gamma_3(F)| \leq 3^{3r}$. Now we show that $|\Gamma_3(F)| = 3$. In fact, $\Gamma_3(F)$ is generated by the basic commutators of weight three and, as F is 2-Engel, they are all equal to 1, except at most $[y, x, z]$ and $[z, x, y]$ (see, for example [2, p. 54]). Moreover, in a 2-Engel group G, for all $x_1, x_2, x_3 \in G$ the following conditions hold:

i) $[x_1, x_3, x_2] = [x_1, x_2, x_3]^{-1}$,

ii) $[x_1^{-1}, x_2] = [x_1, x_2^{-1}] = [x_1, x_2]^{-1}$

(see (2) and (3) in the proof of Satz 6.5 in [1, p. 288]).

So we get

$$\begin{aligned} [z, x, y] &= [[x, z]^{-1}, y] && \text{by ii)} \\ &= [x, z, y]^{-1} && \text{by i)} \\ = [x, y, z] &= [[y, x]^{-1}, z] && \text{by ii)} \\ &= [y, x, z]^{-1}. \end{aligned}$$

Hence $\Gamma_3(F) = \langle [x, y, z] \rangle$ is cyclic of order 3 (see [4, p. 358]) and $|F| \leq 3^{3r+1}$.

We now construct a group F_0, belonging to the variety $\mathbf{V}$, which has order 3^{3r+1}. Then it follows that $F_0 \cong F$ and $|F| = 3^{3r+1}$.

Let A be the abelian group of exponent 3^r defined by

$$A = \langle z \rangle \times \langle v_1 \rangle \times \langle v_2 \rangle \times \langle v_3 \rangle \cong \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r} \times \mathbb{Z}_{3^r} \times \mathbb{Z}_3$$

and let Q be the group of exponent 3^r and of nilpotency class 2 defined by

$$Q = \langle x, y \mid x^{3^r} = y^{3^r} = 1, u = [x, y], u^{3^r} = 1, [u, x] = [u, y] = 1 \rangle.$$

Let $F_0 = [A]Q$ be the semidirect product of A and Q with the action of Q on A defined by

$$(11) \qquad \begin{array}{llll} z^x = zv_2^{-1}, & v_1^x = v_1v_3, & v_2^x = v_2, & v_3^x = v_3, \\ z^y = zv_1, & v_1^y = v_1, & v_2^y = v_2v_3, & v_3^y = v_3. \end{array}$$

Since

$$u = [x, y], \quad v_1 = [z, y], \quad v_2 = [x, z], \quad v_3 = [v_1, x] = [v_2, y]$$

we obtain that $F_0 = \langle x, y, z\rangle$ and we have $|F_0| = |A||Q| = 3^{3r+1}3^{3r} = 3^{6r+1}$.

Also we have

$$(12) \qquad [z, u] = v_3, \quad [u, v_1] = [u, v_2] = [u, v_3] = 1.$$

So $F_0' = \langle u, v_1, v_2, v_3\rangle$ and $\Gamma_3(F_0) = \langle v_3\rangle$ is of order 3. Therefore F_0 is nilpotent of class 3.

To prove a) we only need to show that the group F_0 we have constructed belongs to the variety $\mathbf{V}$. In other words, we have to show that F_0 is a 2-Engel group of exponent 3^r. Since the right 2-Engel elements form a subgroup of a group (see [3]), it is sufficient to check that the generators $x,\ y,\ z$ of F_0 are right 2-Engel elements. In fact, by the definition of F_0, it is easy to see that the basic commutators of weight three on the generators, are the following:

$$\begin{array}{l} [x, y, y] = [x, y, x] = [z, x, x] = [z, y, z] = [z, y, y] = [z, x, z] = 1 \\ [x, y, z] = v_3^{-1}, \quad [z, y, x] = v_3. \end{array}$$

We observe that $v_3 \in Z(F_0)$ by (11) and (12). Then it follows that F_0 is nilpotent of class 3 and $\Gamma_3(F_0) = \langle v_3\rangle$ is of order 3.

Moreover, since A is abelian, the relations (11), (12) yield

$$(13) \qquad \begin{array}{lll} [x^\alpha, z^a] = v_2^{a\alpha}, & [v_1^b, x^\alpha] = v_3^{b\alpha}, & [z^a, y^\beta] = v_1^{a\beta} \\ [v_2^c, y^\beta] = v_3^{c\beta}, & [u^\gamma, z^a] = v_3^{-a\gamma} & \end{array}$$

where $a, b, c, \alpha, \beta, \gamma$ belong to $\mathbb{Z}_{3^r}$. Using the above relations, we can directly check that for all $g \in F_0$ we have

$$[x, g, g] = [y, g, g] = [z, g, g] = 1.$$

Write $g = vw$ with $v \in A$ and $w \in Q$. Since Q is of class 2 and A is abelian, we have $[x, w, w] = [x, v, v] = 1$. So

$$[x, g, g] = [x, v, w][x, w, v].$$

Letting $w = y^i s$, where $s \in \langle x, u\rangle$, and $v = z^j \widehat{v}$, where $\widehat{v} \in \langle v_1, v_2, v_3\rangle$, the relations displayed in (11), (12), and (13) yield

$$[x, v, w] = [x, z^j, w] = [v_2^j, y^i] = v_3^{ij}$$

and

$$[x, w, v] = [x, y^i, v] = [u^i, z^j] = v_3^{-ij}.$$

So $[x, g, g] = 1$.

The proof that y is right 2-Engel is analogous.

For z we observe that, since A is abelian and $[z, Q]$ is contained in A, we have

$$[z, v, v] = [z, v, w] = [z, w, v] = 1.$$

Moreover, letting $w = x^h y^i u^k$, by relations (11), (12), and (13) we have

$$[z, w] = [z, x^h y^i u^k] = [z, y^i][z, x^h]c, \quad c \in Z(F_0).$$

It follows that

$$[z, w, w] = [v_1^i v_2^{-h}, x^h y^i] = v_3^{hi} v_3^{-hi} = 1$$

It remains to check that the exponent of F_0 is 3^r. By the Hall-Petrescu identity (see [1, p. 317]) we have

$$g^{3^r} = (vw)^{3^r} = v^{3^r} w^{3^r} c_1^{\binom{3^r}{2}} c_2^{\binom{3^r}{3}},$$

where $c_1 \in F_0'$ and $c_2 \in \gamma_3(F_0) = \langle v_3\rangle$. Since Q, A, F_0' are of exponent 3^r and $|\Gamma_3(F_0)| = 3$, we have $(wv)^{3^r} = 1$.

We can now identify F with F_0.

b) By the relations (11) and (12) we have that u^3, v_1^3, v_2^3, $v_3 \in Z(F)$.

Conversely, computing the commutators between an element $g = vw = z^a v_1^b v_2^c v_3^d x^\alpha y^\beta u^\gamma$ and the generators x, y, z of F, we obtain

(14) $[x, vw] = [x, w][x, v][x, v, w] = u^\beta v_2^a v_3^{-b}[v_2^a, y^\beta] = v_2^a v_3^{a\beta - b} u^\beta,$

(15) $[y, vw] = [y, w][y, v][y, v, w] = u^{-\alpha} v_1^{-a} v_3^c [v_1^{-a}, x^\alpha] = v_1^{-a} v_3^{-c-a\alpha} u^{-\alpha},$

(16) $[z, vw] = [z, w] = [z, x^\alpha y^\beta u^\gamma] = [z, u^\gamma][z, y^\beta]^{x^\alpha}[z, x^\alpha] = v_1^\beta v_2^{-\alpha} v_3^{\gamma - \alpha\beta}.$

It follows that $g \in Z(F)$ only if $a \equiv \beta \equiv \alpha \equiv 0 \pmod{3^r}$ and $b \equiv c \equiv \gamma \equiv 0 \pmod 3$. So the elements of $Z(G)$ have the following form

$$g = v_1^{3b_1} v_2^{3c_1} v_3^d u^{3\gamma_1}$$

where $b_1,\ c_1,\ \gamma_1 \in \mathbb{Z}_{3^{r-1}}$ and $d \in \mathbb{Z}_3$. Thus

$$Z(F) = \langle v_1^3 \rangle \times \langle v_2^3 \rangle \times \langle v_3 \rangle \times \langle u^3 \rangle.$$

c) It is sufficient to show that every maximal subgroup M of F is of class two. As $F/\Phi(F) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, we have $M = \langle \Phi(F), x_1, x_2 \rangle$ for some $x_1, x_2 \in M$. We want to show that $M' = \langle [x_1, x_2], [x_i, F'], [x_i, F^3], \Phi(F)' \rangle$ $(i = 1, 2)$ is contained in $Z(M)$. In fact, F is nilpotent of class 3, so $[x_i, F'] \leq Z(F)$. We observe that $Z(F) = \langle v_3, F'^3 \rangle$ is contained in M and then $Z(F) \leq Z(M)$. Therefore $[x_i, F'] \leq Z(M)$. Since the identity $[g_1, g_2^n] = [g_1, g_2]^n$ holds in the 2-Engel group F, for all $n \in \mathbb{Z}$ and $g_1,\ g_2 \in F$, we have $[x_i, F^3] = [x_i, F]^3 \leq F'^3 \leq Z(M)$. In the same way we see that $\Phi(F)' \leq Z(M)$. Finally $[x_1, x_2] \in Z(M)$ because $[x_1, x_2, x_1] = [x_1, x_2, x_2] = 1$ holds in the 2-Engel group F.

d) Suppose $r > 1$, then $v_1^3,\ v_2^3,\ u^3$ belong to $Z(F)$ (see b)). So $Z(F)$ is not cyclic, contradicting Lemma 2.1.

Conversely, let $r = 1$, then $Z(F) = \langle v_3 \rangle$ is cyclic of order three. So, by Lemma 2.1 and c), F belongs to $\mathrm{Min}(\mathcal{N}_2)$.

e) Let $L = F/H$ be a quotient of F of class precisely three. If M/H is a maximal subgroup of L, then M is a maximal subgroup of F and, by c), it is nilpotent of class two. Since $\Gamma_3(L) = \Gamma_3(F)H/H$ is cyclic of order 3, by Lemma 2.1, $L \in \mathrm{Min}(\mathcal{N}_2)$ if and only if $Z(L)$ is cyclic. □

Proposition 4.2. a) *Let G be a 3-generated group in* $\mathrm{Min}(\mathcal{N}_2)$ *with* $\exp(G)$=9. *Then* $|G| \leq 3^7$.

b) *There are at least two non-isomorphic groups in* $\mathrm{Min}(\mathcal{N}_2)$ *of exponent* 9 *and order* 3^7.

Proof. a) Using the same notation as in the previous theorem, G has to be isomorphic to a quotient F/H of the relatively free group F with $\exp(F) = 3^2$. Since F/H has to be nilpotent of class 3, we have $v_3 \notin H$. As $Z(F/H)$ must be cyclic and $Z(F)$ is elementary abelian of rank 4, then H must contain a subgroup K of $Z(F)$ which is of rank 3 and $v_3 \notin H$. Now $Z(F)$ contains 40 subgroups of index 3. Among these, 13 contain v_3. So there are 27 subgroups of $Z(F)$ which do not contain $\langle v_3 \rangle$. The subgroup $K_1 = \langle v_1^3, v_2^3, u^3 \rangle = (F')^3$ is characteristic in F and the other 26 form a single orbit under the automorphism φ of F defined by

$$x^\varphi = y, \quad y^\varphi = z, \quad z^\varphi = x^{-1}y. \tag{17}$$

In fact consider the subgroup $K_2 = \langle v_1^3,\ v_2^3 v_3^{-1},\ u^3\rangle$ of $Z(F)$. We observe that $v_1^{\varphi} = v_1^{-1}v_2^{-1},\ v_2^{\varphi} = u,\ u^{\varphi} = v_1^{-1}$. A straightforward calculation shows that $K_2^{\varphi^{13}} = \langle v_1^3,\ v_2^3 v_3,\ u^3\rangle \neq K_2$. As φ is an automorphism of order 26, the orbit of K_2 has length 26. So we may assume that H contains one of the two subgroups $K_i,\ i = 1,2$. Consider F/K_1. A generic element of K_1 can be written in the form

$$v_1^{3l} v_2^{3m} u^{3n} \quad \text{with } l,m,n \in \{0,1,2\}.$$

From the relations (14), (15), and (16) we get that an element

$$gK_1 = z^a v_1^b v_2^c v_3^d x^{\alpha} y^{\beta} u^{\gamma} K_1$$

of F/K_1 belongs to $Z(F/K_1)$ if and only if

$$\begin{aligned} v_2^a v_3^{a\beta - b} u^{\beta} &= v_1^{3l_1} v_2^{3m_1} u^{3n_1},\\ v_1^{-a} v_3^{-c-a\alpha} u^{-\alpha} &= v_1^{3l_2} v_2^{3m_2} u^{3n_2},\\ v_1^{\beta} v_2^{-\alpha} v_3^{\gamma - \alpha\beta} &= v_1^{3l_3} v_2^{3m_3} u^{3n_3}, \end{aligned}$$

for some $l_i, m_i, n_i \in \{0,1,2\};\ i = 1,2,3$. It follows $a \equiv \beta \equiv b \equiv c \equiv \alpha \equiv \gamma \equiv 0 \pmod 3$. Let $a = 3a_1$, $\beta = 3\beta_1$, $b = 3b_1$, $c = 3c_1$, $\alpha = 3\alpha_1$ and $\gamma = 3\gamma_1$. Then

$$gK_1 = z^{3a_1} v_1^{3b_1} v_2^{3c_1} v_3^d x^{3\alpha_1} y^{3\beta_1} u^{3\gamma_1} K_1 = z^{3a_1} v_3^d x^{3\alpha_1} y^{3\beta_1} K_1.$$

In a similar way we see that $gK_2 \in Z(F/K_2)$ if and only if $\beta \equiv c \equiv \alpha \equiv 0 \pmod 3$ and $a \equiv 3b \pmod 9$ $\alpha \equiv 3\gamma \pmod 9$. If $\beta = 3\beta_1$, $c = 2c_1$ and $\alpha = 3\alpha_1$, we have that

$$gK_2 = z^{3b} v_1^b v_2^{3c_1} v_3^d x^{3\gamma} y^{3\beta_1} u^{\gamma} K_2 = (z^3 v_1)^b v_3^{d+c_1} (x^3 u)^{\gamma} y^{3\beta_1} K_2.$$

Therefore $Z(F/K_1)$ and $Z(F/K_2)$ are abelian groups which can be represented as direct product

$$Z(F/K_1) = \langle z^3 K_1\rangle \times \langle v_3 K_1\rangle \times \langle x^3 K_1\rangle \times \langle y^3 K_1\rangle$$

and

$$Z(F/K_2) = \langle z^3 v_1 K_2\rangle \times \langle v_3 K_2\rangle \times \langle x^3 u K_2\rangle \times \langle y^3 K_2\rangle$$

In order that a quotient $(F/K_i)/(H/K_i)$, $(i = 1,2)$ of F/K_i would be nilpotent of class 3 with cyclic center, we need that $v_3 K_i \notin Z(F/K_i)$ and that H/K_i would contains a subgroup of rank 3 of $Z(F/K_i)$. So the order of a group $F/H \in \mathrm{Min}(\mathcal{N}_2)$ is at most 3^7.

b) Consider the subgroups

$$H_1 = \langle v_1^3, v_2^3, u^3, x^3, y^3, z^3 v_3^{-1} \rangle \quad \text{and} \quad H_2 = \langle v_1^3, v_2^3 v_3^{-1}, u^3, x^3 u, y^3, z^3 v_1 \rangle$$

which contain K_1 and K_2, respectively. By the same argument used above to determine the center of F/K_i, one can check easily that the center of $Z(F/H_i)$ is cyclic. If $g = z^a v_1^b v_2^c v_3^d x^\alpha y^\beta u^\gamma$ is, as before, a generic element of F, we have

$$g^3 = z^{3a} v_1^{3(b-a\beta)} v_2^{3(c+a\alpha)} x^{3\alpha} y^{3\beta} u^{3(\gamma-\alpha\beta)}.$$

Using this relation we see that the exponent of F/H_i is 9. Moreover we see that the $\mho_1(F/H_1) = \langle v_3 H_1 \rangle$ while $\mho_1(F/H_2) = \langle v_1 H_2, v_3 H_2, u H_2 \rangle$ which is not cyclic. $\square$

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