

Quotient graphs for power graphs

DANIELA BUBBOLONI (*) – MOHAMMAD A. IRANMANESH (**) –
SEYED M. SHAKER (***)

ABSTRACT – In a previous paper of the first author a procedure was developed for counting the components of a graph through the knowledge of the components of one of its quotient graphs. Here we apply that procedure to the proper power graph $\mathcal{P}_0(G)$ of a finite group G , finding a formula for the number of its components which is particularly illuminative when $G \leq S_n$ is a fusion controlled permutation group. We make use of the proper quotient power graph $\tilde{\mathcal{P}}_0(G)$, the proper order graph $\mathcal{O}_0(G)$ and the proper type graph $\mathcal{T}_0(G)$. All those graphs are quotient of $\mathcal{P}_0(G)$. We emphasize the strong link between them determining number and typology of the components of the above graphs for $G = S_n$. In particular, we prove that the power graph $\mathcal{P}(S_n)$ is 2-connected if and only if the type graph $\mathcal{T}(S_n)$ is 2-connected, if and only if the order graph $\mathcal{O}(S_n)$ is 2-connected, that is, if and only if either $n = 2$ or none of $n, n - 1$ is a prime.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 05C25, 20B30.

KEYWORDS. Quotient graph, power graph, permutation groups.

(*) *Indirizzo dell'A.*: Dipartimento di Matematica e Informatica U. Dini, Università degli Studi di Firenze, Viale Morgagni 67/a, 50134 Firenze, Italy
E-mail: daniela.bubboloni@unifi.it

(**) *Indirizzo dell'A.*: Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
E-mail: iranmanesh@yazd.ac.ir

(***) *Indirizzo dell'A.*: Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
E-mail: seyed_shaker@yahoo.com

CONTENTS

1. Introduction and main results	62
2. Graphs	66
3. Power graphs	68
4. Order graphs	70
5. Quotient graphs of power graphs associated with permutation groups.	72
6. Quotient graphs of power graphs associated with fusion controlled permutation groups	77
7. The number of components in $\tilde{\mathcal{P}}_0(S_n)$, $\mathcal{T}_0(S_n)$, $\mathcal{O}_0(S_n)$	80
REFERENCES	89

1. Introduction and main results

Kelarev and Quinn [10] defined the directed power graph $\overrightarrow{\mathcal{P}(S)}$ of a semigroup S as the directed graph in which the set of vertices is S and, for $x, y \in S$, there is an arc (x, y) if $y = x^m$, for some $m \in \mathbb{N}$. The *power graph* $\mathcal{P}(S)$ of a semigroup S , was defined by Chakrabarty, Ghosh and Sen [6] as the corresponding underlying undirected graph. They proved that for a finite group G , the power graph $\mathcal{P}(G)$ is complete if and only if G is a cyclic group of prime power order. In [4, 5] Cameron and Ghosh obtained interesting results about power graphs of finite groups, studying how the group of the graph automorphisms of $\mathcal{P}(G)$ affects the structure of the group G . Mirzargar, Ashrafi and Nadjafi [13] considered some further graph theoretical properties of the power graph $\mathcal{P}(G)$, such as the clique number, the independence number and the chromatic number and their relation to the group theoretical properties of G . Even though young, the theory of power graphs seems to be a very promising research area. The majority of its beautiful results dating before 2013 are collected in the survey [1].

In this paper we deal with the connectivity of $\mathcal{P}(G)$, where G is a group. All the groups considered in this paper are finite. Since it is obvious that $\mathcal{P}(G)$ is connected of diameter at most 2, the focus is on 2-connectivity. Recall that a graph $X = (V_X, E_X)$ is 2-connected if, for every $x \in V_X$, the x -deleted subgraph of X is connected. Thus $\mathcal{P}(G)$ is 2-connected if and only if $\mathcal{P}_0(G)$, the 1-deleted subgraph of $\mathcal{P}(G)$, is connected. $\mathcal{P}_0(G)$ is called the *proper power graph* of G and our main aim is to find a formula for the number $c_0(G)$ of its components. We denote its vertex set $G \setminus \{1\}$ by G_0 . Recently Curtin, Pourgholi and Yousefi-Azari [7] considered the properties of the diameter of $\mathcal{P}_0(G)$ and characterized the groups G for which $\mathcal{P}_0(G)$ is Eulerian. Moghaddamfar, Rahbariyan, and Shi [14],

found many relations between the group theoretical properties of G and the graph theoretical properties of $\mathcal{P}_0(G)$. Here we apply the theory developed in [2] to get control on number and nature of the components of $\mathcal{P}_0(G)$ through those of some of its quotient graphs.

Throughout the paper, n always indicates a natural number. The symmetric group S_n and the alternating group A_n are interpreted as naturally acting on the set $N = \{1, \dots, n\}$ and with identity element id . We are going to use notation and definitions given in [2] about graphs and homomorphisms. In particular, every graph is finite, undirected, simple and reflexive, so that there is a loop on each vertex. The assumption about loops, which is not common for power graphs, is clearly very mild in treating connectivity and does not affect any result about components.

Up to now, the 2-connectivity of $\mathcal{P}(G)$ has been studied for nilpotent groups and for some types of simple groups in [15]; for groups admitting a partition and for the symmetric and alternating groups, with a particular interest on the diameter of $\mathcal{P}_0(G)$, in [9]. In those papers the results are obtained through ingenious ad hoc arguments, without developing a general method. The arguments often involve element orders and, when $G \leq S_n$, the cycle decomposition of permutations. We observed that what makes those ideas work is the existence of some quotient graphs for $\mathcal{P}_0(G)$. For $\psi \in S_n$, let T_ψ denote the type of ψ , that is, the partition of n given by the lengths of the orbits of ψ . Then there exists a quotient graph $\mathcal{O}_0(G)$ of $\mathcal{P}_0(G)$ having vertex set $\{o(g) : g \in G_0\}$ and, when G is a permutation group, there exists a quotient graph $\mathcal{T}_0(G)$ of $\mathcal{P}_0(G)$ having vertex set $\mathfrak{T}(G_0)$ given by the types T_ψ of the permutations $\psi \in G_0$ (Sections 4 and 5).

Recall that a homomorphism f from the graph X to the graph Y is called complete if it maps both the vertices and edges of X onto those of Y ; tame if vertices with the same image are connected; locally surjective if it maps the neighborhood of each vertex of X onto the neighborhood in Y of its image; orbit if the sets of vertices in V_X sharing the same image coincide with the orbits of a group of automorphisms of X . The starting point of our approach is to consider the *quotient power graph* $\tilde{\mathcal{P}}_0(G)$, obtained from $\mathcal{P}_0(G)$ by the identification of the vertices generating the same cyclic subgroup (Section 3). The projection π of $\mathcal{P}_0(G)$ on $\tilde{\mathcal{P}}_0(G)$ is tame and thus the number $\tilde{c}_0(G)$ of components of $\tilde{\mathcal{P}}_0(G)$ is equal to $c_0(G)$. Moreover, both $\mathcal{O}_0(G)$ and $\mathcal{T}_0(G)$ may be seen also as quotients of $\tilde{\mathcal{P}}_0(G)$, with a main difference between them. The projection \tilde{o} on $\mathcal{O}_0(G)$ is not, in general, locally surjective (Example 4.4) while, for any $G \leq S_n$, the projection \tilde{i} on $\mathcal{T}_0(G)$ is complete and locally surjective (Propositions 5.4 and 6.3). As a consequence, while finding $c_0(G)$ through $\mathcal{O}_0(G)$ can be hard, it is manageable

through $\mathcal{T}_0(G)$. Call now $G \leq S_n$ *fusion controlled* if, for every $\psi \in G$ and $x \in S_n$ such that $\psi^x \in G$, there exists $y \in N_{S_n}(G)$ such that $\psi^x = \psi^y$. Obviously S_n and A_n are both fusion controlled, but they are not the only examples. For instance, if $n = mr$, with $m, r \geq 2$, then the base group G of the wreath product $S_m \wr S_r = N_{S_n}(G)$ is fusion controlled. If G is fusion controlled, then \tilde{t} is a complete orbit homomorphism (Proposition 6.2) and hence [2, Theorem B] applies to $X = \tilde{\mathcal{P}}_0(G)$ and $Y = \mathcal{T}_0(G)$, giving an algorithmic method to get $c_0(G)$.

In order to state our main results we need some further notation. Denote by $\tilde{\mathcal{C}}_0(G)$ the set of components of $\tilde{\mathcal{P}}_0(G)$; by $\mathcal{C}_0(\mathcal{T}(G))$ the set of components of $\mathcal{T}_0(G)$ and by $c_0(\mathcal{T}(G))$ their number; by $c_0(\mathcal{O}(G))$ the number of components of $\mathcal{O}_0(G)$. For $T \in \mathcal{T}(G_0)$, denote by $\mu_T(G)$ the number of permutations of type T in G ; by $o(T)$ the order of any permutation having type T ; by $C(T)$ the component of $\mathcal{T}_0(G)$ containing T ; by $\tilde{\mathcal{C}}_0(G)_T$ the set of components of $\tilde{\mathcal{P}}_0(G)$ in which there exists at least one vertex of type T . Finally, for $C \in \tilde{\mathcal{C}}_0(G)_T$, let $k_C(T)$ be the number of vertices in C having type T , and let ϕ denote the Euler totient function.

THEOREM A. *Let $G \leq S_n$ be fusion controlled. For $1 \leq i \leq c_0(\mathcal{T}(G))$, let $T_i \in \mathcal{T}(G_0)$ be such that $\mathcal{C}_0(\mathcal{T}(G)) = \{C(T_i) : i \in \{1, \dots, c_0(\mathcal{T}(G))\}\}$, and pick $C_i \in \tilde{\mathcal{C}}_0(G)_{T_i}$. Then*

$$(1.1) \quad c_0(G) = \sum_{i=1}^{c_0(\mathcal{T}(G))} \frac{\mu_{T_i}(G)}{\phi(o(T_i))k_{C_i}(T_i)}.$$

The connectivity properties of the graphs $\tilde{\mathcal{P}}_0(G)$, $\mathcal{T}_0(G)$ and $\mathcal{O}_0(G)$ are strictly linked when $G \leq S_n$, especially when G is fusion controlled. In the last section of the paper we consider $G = S_n$, finding $c_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$. In particular we find, with a different approach, the values of $c_0(S_n)$ in [9, Theorem 4.2]. Throughout the paper we denote by P the set of prime numbers and put $P + 1 = \{x \in \mathbb{N} : x = p + 1 \text{ for some } p \in P\}$.

THEOREM B. *The values of $c_0(S_n) = \tilde{c}_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ are as follows.*

- (i) *For $2 \leq n \leq 7$, they are given in Table 1 below.*

Table 1. $c_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ for $2 \leq n \leq 7$

n	2	3	4	5	6	7
$c_0(S_n)$	1	4	13	31	83	128
$c_0(\mathcal{T}(S_n))$	1	2	3	3	4	3
$c_0(\mathcal{O}(S_n))$	1	2	2	2	2	2

- (ii) For $n \geq 8$, they are given by Table 2 below, according to whether n is a prime, one greater than a prime, or neither.

Table 2. $c_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ for $n \geq 8$

n	$n \in P$	$n \in P + 1$	$n \notin P \cup (P + 1)$
$c_0(S_n)$	$(n - 2)! + 1$	$n(n - 3)! + 1$	1
$c_0(\mathcal{T}(S_n)) = c_0(\mathcal{O}(S_n))$	2	2	1

COROLLARY C. Let $n \geq 2$. The following facts are equivalent:

- (i) $\mathcal{P}(S_n)$ is 2-connected;
- (ii) $\mathcal{P}_0(S_n)$ is connected;
- (iii) $\tilde{\mathcal{P}}_0(S_n)$ is connected;
- (iv) $\mathcal{T}_0(S_n)$ is connected;
- (v) $\mathcal{O}_0(S_n)$ is connected;
- (vi) $n = 2$ or $n \in \mathbb{N} \setminus [P \cup (P + 1)]$.

Observe that $\mathcal{T}_0(S_n)$ has also a purely arithmetic interest, because $\mathcal{T}(S_n)$ is the whole set of the partition of n . Going beyond a mere counting, we describe the components of the graphs belonging to $\mathcal{G}_0 = \{\mathcal{P}_0(S_n), \tilde{\mathcal{P}}_0(S_n), \mathcal{T}_0(S_n), \mathcal{O}_0(S_n)\}$. To start with, note that in the connected case $n \in \mathbb{N} \setminus [P \cup (P + 1)]$, no $X_0 \in \mathcal{G}_0$ is a complete graph because X_0 admits as quotient $\mathcal{O}_0(S_n)$ which is surely incomplete having as vertices at least two primes. For $\psi \in S_n \setminus \{id\}$ vertex of $\mathcal{P}_0(S_n)$, let $[\psi]$ denote the corresponding vertex of $\tilde{\mathcal{P}}_0(S_n)$.

THEOREM D. Let $n \geq 8$, with $n \in \{p, p + 1\}$ for some prime p . Let $k = (n - 2)!$ if $n = p$ and let $k = n(n - 3)!$ if $n = p + 1$. Let Δ_n be the set of every nonidentity permutation in S_n which is not a p -cycle.

- (i) $\mathcal{P}_0(S_n)$ consists of the main component induced on Δ_n and k -many complete components, each comprised of $(p - 1)$ -many p -cycles.
- (ii) $\tilde{\mathcal{P}}_0(S_n)$ consists of the main component induced on $\{[\psi]: \psi \in \Delta_n\}$ and k -many isolated vertices.
- (iii) $\mathcal{T}_0(S_n)$ consists of the main component induced on $\{T_\psi: \psi \in \Delta_n\}$ and the component containing the type of a p -cycle which is an isolated vertex.
- (iv) $\mathcal{O}_0(S_n)$ consists of the main component induced on $\{o(\psi): \psi \in \Delta_n\}$ and the component given by the isolated vertex p .

In all the above cases, the main component is never complete.

Complete information about the components of the graphs in \mathcal{G}_0 , for $3 \leq n \leq 7$, can be found within the proof of Theorem B (i), taking into account Lemma 3.7 for $\mathcal{P}_0(S_n)$. In particular, looking at the details, one easily checks that all the components of $X_0 \in \mathcal{G}_0$ apart from one are isolated vertices (complete graphs when $X_0 = \mathcal{P}_0(S_n)$) if and only if $n \geq 8$ or $n = 2$.

In paper [3] we treated the alternating group A_n , computing $c_0(A_n)$, $c_0(\mathcal{T}(A_n))$ and $c_0(\mathcal{O}(A_n))$. We also corrected some mistakes about $c_0(A_n)$ found in [9]. We believe that the algorithmic method [2, Theorem B] may help, more generally, to obtain $c_0(G)$ where G is simple and almost simple. This, in particular, could give an answer to the interesting problem of classifying all the simple groups with 2-connected power graph, posed in [15, Question 2.10]. About that problem, in [3] we show that there exist infinite examples of alternating groups with 2-connected power graph and that A_{16} is that of smaller degree.

2. Graphs

For a finite set A and $k \in \mathbb{N}$, let $\binom{A}{k}$ be the set of the subsets of A of size k . In this paper, as in [2], a graph $X = (V_X, E_X)$ is a pair of finite sets such that $V_X \neq \emptyset$ is the set of vertices, and E_X is the set of edges which is the union of the set of loops $L_X = \binom{V_X}{1}$ and a set of proper edges $E_X^* \subseteq \binom{V_X}{2}$. We usually specify the edges of a graph X giving only E_X^* .

Paper [2] is the main reference for the present paper. For the general information about graphs see [2, Section 2]. Recall that, for a graph X , $\mathcal{C}(X)$ denotes the set of components of X and $c(X)$ their number. If $x \in V_X$, the component of X containing x is denoted by $C_X(x)$ or more simply, when there is no risk of misunderstanding, by $C(x)$. For $s \in \mathbb{N} \cup \{0\}$, a subgraph γ of X such that $V_\gamma = \{x_i: 0 \leq i \leq s\}$ with distinct $x_i \in V_X$ and $E_\gamma^* = \{\{x_i, x_{i+1}\}: 0 \leq i \leq s-1\}$, is called a path of length s between x_0 and x_s , and will be simply denoted by the ordered list x_0, \dots, x_s of its vertices.

For the formal definitions of surjective, complete, tame, locally surjective, pseudo-covering, orbit and component equitable homomorphism and the notation for the corresponding sets of homomorphisms see [2, Section 4.2, Definitions 5.3, 5.7, 4.4, 6.4].

By [2, Propositions 5.9 and 6.9], we have that

$$(2.1) \quad \text{O}(X, Y) \cap \text{Com}(X, Y) \subseteq \text{LSur}(X, Y) \cap \text{CE}(X, Y) \cap \text{Com}(X, Y).$$

The content of [2] is all we need to conduce our arguments up to just a couple of definitions and related results.

DEFINITION 2.1. Let X and Y be graphs, and $f \in \text{Hom}(X, Y)$. Then f is called a *2-homomorphism* if, for every $e \in E_X^*$, $f(e) \in E_Y^*$. We denote the set of the 2-homomorphisms from X to Y by $2\text{Hom}(X, Y)$.

From that definition we immediately deduce the following lemma.

LEMMA 2.2. *Let $f \in 2\text{Hom}(X, Y)$ and $x \in V_X$. If $f(x)$ is isolated in Y , then x is isolated in X .*

DEFINITION 2.3. Let X be a graph. If $x_0 \in V_X$, then the *x_0 -deleted subgraph* $X - x_0$ is defined as the subgraph of X with vertex set $V_X \setminus \{x_0\}$ and edge set given by the edges in E_X not incident to x_0 . X is called *2-connected* if, for every $x_0 \in V_X$, $X - x_0$ is connected.

To deal with vertex deleted subgraphs and quotient graphs, we will use the following lemma several times.

LEMMA 2.4. *Let $f \in \text{Hom}(X, Y)$.*

- (i) *Suppose $x_0 \in V_X$ is such that $f^{-1}(f(x_0)) = \{x_0\}$. Then f induces naturally $f_{x_0} \in \text{Hom}(X - x_0, Y - f(x_0))$. Moreover, if f is surjective (complete, pseudo-covering), then also f_{x_0} is surjective (complete, pseudo-covering).*
- (ii) *Let \sim be an equivalence relation on V_X such that, for each $x_1, x_2 \in V_X$, $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$. Then the map $\tilde{f}: [V_X] \rightarrow V_Y$, defined by $\tilde{f}([x]) = f(x)$ for all $[x] \in [V_X]$, is a homomorphism from X/\sim to Y such that $\tilde{f} \circ \pi = f$. If f is surjective (complete, pseudo-covering), then also \tilde{f} is surjective (complete, pseudo-covering).*

PROOF. (i) Since $f(V_X \setminus \{x_0\}) \subseteq V_Y \setminus \{f(x_0)\}$, we can consider the map $f_{x_0}: V_X \setminus \{x_0\} \rightarrow V_Y \setminus \{f(x_0)\}$, defined by $f_{x_0}(x) = f(x)$ for all $x \in V_X \setminus \{x_0\}$. We show that f_{x_0} defines a homomorphism. Pick $e \in E_{X-x_0}$, so that $e = \{x_1, x_2\} \in E_X$ for suitable $x_1, x_2 \in V_X$ with $x_1, x_2 \neq x_0$. By $f^{-1}(f(x_0)) = \{x_0\}$, we get $f(x_1), f(x_2) \neq f(x_0)$ and thus, since f is a homomorphism, we get $\{f(x_1), f(x_2)\} \in E_{Y-f(x_0)}$. If f is surjective, clearly f_{x_0} is surjective. Assume now that f is complete and show that f_{x_0} is complete. Let $\{f_{x_0}(x_1), f_{x_0}(x_2)\} \in E_{Y-f(x_0)}$. Then $\{f(x_1), f(x_2)\} \in E_Y$, with $f(x_1), f(x_2) \neq f(x_0)$. Since f is complete, there exists $\bar{x}_1, \bar{x}_2 \in V$ such that $f(\bar{x}_1) = f(x_1)$, $f(\bar{x}_2) = f(x_2)$ and $\{\bar{x}_1, \bar{x}_2\} \in E_X$. From $f(x_1), f(x_2) \neq f(x_0)$ we deduce that $\bar{x}_1, \bar{x}_2 \neq x_0$. Thus $\bar{x}_1, \bar{x}_2 \in V_{X-x_0}$ and $\{\bar{x}_1, \bar{x}_2\} \in E_{X-x_0}$. An obvious adaptation of this argument works also in the pseudo-covering case.

(ii) The fact that \tilde{f} is a homomorphism such that $\tilde{f} \circ \pi = f$ is the content of [11, Theorem 1.6.10]. Assume that f is surjective. Then, by $\tilde{f} \circ \pi = f$, \tilde{f} is surjective too. Assume now that f is complete and show that \tilde{f} is complete. By what observed above, \tilde{f} is surjective. Let $e = \{\tilde{f}([x_1]), \tilde{f}([x_2])\} = \{f(x_1), f(x_2)\} \in E_Y$. Since f is complete, there exists $\bar{x}_1, \bar{x}_2 \in V_X$ such that $f(\bar{x}_1) = f(x_1)$, $f(\bar{x}_2) = f(x_2)$ and $\{\bar{x}_1, \bar{x}_2\} \in E_X$. Then also $e' = \{[\bar{x}_1], [\bar{x}_2]\} \in [E_X]$ and $e = \tilde{f}(e')$. An obvious adaptation of this argument works also in the pseudo-covering case. \square

When no ambiguity arises, we denote the map f_{x_0} of the above lemma again by f .

In the following, whatever the deleted vertex $x_0 \in V_X$ is, we write $X_0 = ((V_X)_0, (E_X)_0)$ for the x_0 -deleted subgraph. Moreover we write $\mathcal{C}_0(X)$ for the set of the components of X_0 as well as $c_0(X)$ for their number. This helps to standardize the notation throughout the paper. Some abbreviations will sometimes be introduced. The terminology not explicitly introduced is standard and can be found in [8].

3. Power graphs

Throughout the next sections, let G be a finite group with identity element 1 and let $G_0 = G \setminus \{1\}$. For $x \in G$, denote by $o(x)$ the order of x .

DEFINITION 3.1. The *power graph* of G is the graph $\mathcal{P}(G) = (G, E)$ where, for $x, y \in G$, $\{x, y\} \in E$ if there exists $m \in \mathbb{N}$ such that $x = y^m$ or $y = x^m$. The *proper power graph* $\mathcal{P}_0(G) = (G_0, E_0)$ is defined as the 1-deleted subgraph of $\mathcal{P}(G)$.

To deal with the graphs $\mathcal{P}(G)$ and $\mathcal{P}_0(G)$ and simplify their complexity, we start considering the corresponding quotient graphs in which the elements of G , generating the same cyclic subgroup, are identified in a unique vertex.

DEFINITION 3.2. Define for $x, y \in G$, $x \sim y$ if $\langle x \rangle = \langle y \rangle$. Then \sim is an equivalence relation on G and the equivalence class of $x \in G$, $[x] = \{x^m : 1 \leq m \leq o(x), \gcd(m, o(x)) = 1\}$ has size $\phi(o(x))$. The quotient graph $\mathcal{P}(G)/\sim = ([G], [E])$ is called the *quotient power graph* of G and denoted by $\tilde{\mathcal{P}}(G)$.

By definition of quotient graph, the vertex set of $\mathcal{P}(G)/\sim$ is $[G] = \{[x] : x \in G\}$ and $\{[x], [y]\} \in [E]$ is an edge in $\tilde{\mathcal{P}}(G)$ if there exist $\tilde{x} \in [x]$ and $\tilde{y} \in [y]$ such that $\{\tilde{x}, \tilde{y}\} \in E$, that is, \tilde{x}, \tilde{y} are one the power of the other.

LEMMA 3.3. For every $x, y \in G$, $\{[x], [y]\} \in [E]$ if and only if $\{x, y\} \in E$.

PROOF. Let $x, y \in G$ such that $\{[x], [y]\}$ is an edge in $\tilde{\mathcal{P}}(G)$. Then there exist $\tilde{x} \in [x]$ and $\tilde{y} \in [y]$ such that one of them is a power of the other. To fix the ideas, let $\tilde{x} = (\tilde{y})^m$, for some $m \in \mathbb{N}$. Since $x \in \langle \tilde{x} \rangle$ and $\tilde{y} \in \langle y \rangle$, there exist $a, b \in \mathbb{N}$ such that $x = (\tilde{x})^a$ and $\tilde{y} = y^b$. It follows that $x = y^{abm}$ and thus $\{x, y\} \in E$. The converse is trivial. \square

The above lemma may be thought of as saying that the projection of $\mathcal{P}(G)$ onto its quotient $\tilde{\mathcal{P}}(G)$ is a strong homomorphism in the sense of [12, Definition 1.5].

DEFINITION 3.4. The [1]-deleted subgraph of $\tilde{\mathcal{P}}(G)$ is called the *proper quotient power graph* of G and denoted by $\tilde{\mathcal{P}}_0(G) = ([G]_0, [E]_0)$.

Since $[x] = [1]$ if and only if $x = 1$, applying Lemma 2.4 and [2, Lemma 4.3] to the projection of $\mathcal{P}(G)$ onto $\tilde{\mathcal{P}}(G)$ gives that $\tilde{\mathcal{P}}_0(G)$ is equal to the quotient graph $\mathcal{P}_0(G)/\sim$. For short, we denote the set of components of $\mathcal{P}_0(G)$ by $\mathcal{C}_0(G)$ and their number by $c_0(G)$. Similarly we denote the set of components of $\tilde{\mathcal{P}}_0(G)$ by $\tilde{\mathcal{C}}_0(G)$ and their number by $\tilde{c}_0(G) = c(\tilde{\mathcal{P}}_0(G))$. Lemma 3.3 immediately extends to $\tilde{\mathcal{P}}_0(G)$.

LEMMA 3.5. For every $x, y \in G_0$, $\{[x], [y]\} \in [E]_0$ if and only if $\{x, y\} \in E_0$.

LEMMA 3.6. The graph $\tilde{\mathcal{P}}_0(G)$ is a tame and pseudo-covered quotient of $\mathcal{P}_0(G)$. In particular $c_0(G) = \tilde{c}_0(G)$.

PROOF. Let $x, y \in G_0$ such that $x \sim y$. Then $y = x^m$ for some $m \in \mathbb{N}$ and thus $\{x, y\} \in E$. This shows that the quotient graph $\tilde{\mathcal{P}}_0(G)$ is tame ([2, Definition 3.1]). Then, [2, Proposition 3.2] applies giving $c_0(G) = \tilde{c}_0(G)$. The fact that the quotient is pseudo-covered ([2, Definition 5.14]) is an immediate consequence of Lemma 3.5. \square

LEMMA 3.7. Let π be the projection of $\mathcal{P}_0(G)$ on $\tilde{\mathcal{P}}_0(G)$. Then, the map from $\mathcal{C}_0(G)$ to $\tilde{\mathcal{C}}_0(G)$ which associates, with every $C \in \mathcal{C}_0(G)$, the component $\pi(C)$ is a bijection. Given $\tilde{C} \in \tilde{\mathcal{C}}_0(G)$, the set of vertices of the unique $C \in \mathcal{C}_0(G)$ such that $\pi(C) = \tilde{C}$ is given by $\pi^{-1}(V_{\tilde{C}})$.

PROOF. By Lemma 3.6, π is tame and pseudo-covering. Thus we apply [2, Corollary 5.13] to π . \square

In the recent paper [16] the second and third author have investigated the groups G for which $\tilde{\mathcal{P}}(G)$ is a tree and for which $\tilde{\mathcal{P}}_0(G)$ is a path or a bipartite graph.

4. Order graphs

Let $O(G) = \{o(g) : g \in G\}$. The map $o: G \rightarrow O(G)$, associating with every $x \in G$ its order $o(x)$, is called the *order map* on G . We say that $m \in \mathbb{N}$ is a proper divisor of $n \in \mathbb{N}$ if $m \mid n$ and $m \notin \{1, n\}$.

DEFINITION 4.1. The *order graph* of G is the graph $\mathcal{O}(G)$ with vertex set $O(G)$ and edge set $E_{\mathcal{O}(G)}$ where, for every $m, n \in O(G)$, $\{m, n\} \in E_{\mathcal{O}(G)}$ if $m \mid n$ or $n \mid m$. The *proper order graph* $\mathcal{O}_0(G)$ is defined as the 1-deleted graph of $\mathcal{O}(G)$. Its vertex set is then $O_0(G) = O(G) \setminus \{1\}$.

Note that $\{m, n\} \in E_{\mathcal{O}_0(G)}^*$ only if one of m and n is a proper divisor of the other. Clearly $\mathcal{O}(G)$ is always connected and it is 2-connected if and only if $\mathcal{O}_0(G)$ is connected.

PROPOSITION 4.2. *Let G be a group. The order map defines a complete homomorphism $o: \mathcal{P}(G) \rightarrow \mathcal{O}(G)$ which induces a complete homomorphism $o: \mathcal{P}_0(G) \rightarrow \mathcal{O}_0(G)$, and a complete 2-homomorphism $\tilde{o}: \tilde{\mathcal{P}}_0(G) \rightarrow \mathcal{O}_0(G)$. If G is cyclic, \tilde{o} is an isomorphism.*

PROOF. For every $m \in \mathbb{N}$, $o(x^m)$ is a divisor of $o(x)$, so o is a surjective homomorphism. We show that o is complete. Let $e = \{o(x), o(y)\} \in E_{\mathcal{O}(G)}$, for some $x, y \in G$. Then, without loss of generality, we may assume that $o(y) \mid o(x)$. Since in $\langle x \rangle$ there exist elements of each order dividing $o(x)$, there exists $\bar{y} \in \langle x \rangle$ with $o(\bar{y}) = o(y)$. Let $m \in \mathbb{N}$ be such that $\bar{y} = x^m$. Then $\{x, \bar{y}\} \in E$ and $o(\{x, \bar{y}\}) = e$. Now, since $o(x) = 1$ if and only if $x = 1$, Lemma 2.4 applies giving the desired result both for the vertex deleted graph and for the quotient graph. We are left to check that \tilde{o} is a 2-homomorphism. Let $\{[x], [y]\} \in [E]_0^*$ and show that $\tilde{o}([x]) \neq \tilde{o}([y])$. Assume the contrary, that is, $o(x) = o(y)$. By Lemma 3.5, we have $\{x, y\} \in E_0$ so that x and y are one the power of the other. It follows that $\langle x \rangle = \langle y \rangle$, against $[x] \neq [y]$.

Finally let G be cyclic. To prove that \tilde{o} is an isomorphism, it is enough to show that \tilde{o} is injective. Assume that for some $[x], [y] \in [G]_0$ we have $\tilde{o}([x]) = \tilde{o}([y])$, that is, $o(x) = o(y) = m$. Since in a cyclic group there exists exactly one subgroup for each $m \mid |G|$, we deduce that $\langle x \rangle = \langle y \rangle$ and so $[x] = [y]$. \square

An application of [2, Lemma 4.3] gives, in particular, the following result.

COROLLARY 4.3. *For each finite group G , the graph $\mathcal{O}_0(G)$ is a quotient of the graph $\tilde{\mathcal{P}}_0(G)$.*

We exhibit now an example showing that, in general, \tilde{o} is not pseudo-covering.

EXAMPLE 4.4. Let G be the 2-Sylow subgroup of S_4 given by

$$G = \{id, (1\ 3), (2\ 4), (1\ 3)(2\ 4), (1\ 2)(3\ 4), (1\ 4)(2\ 3), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}.$$

Then $\mathcal{O}_0(G)$ is reduced to a path of length 1 between the only two vertices 2 and 4, while $\tilde{\mathcal{P}}_0(G)$ has 6 vertices and 5 components because the vertices $[(1\ 3)]$, $[(2\ 4)]$, $[(1\ 2)(3\ 4)]$, $[(1\ 4)(2\ 3)]$ are isolated while $\{[(1\ 2\ 3\ 4)], [(1\ 3)(2\ 4)]\}$ is an edge. \tilde{o} takes the component having as only vertex $[(1\ 3)]$ onto the subgraph $(2, \{2\})$, which is not a component of $\mathcal{O}_0(G)$. By [2, Theorem A (i)], this guarantees that \tilde{o} is not pseudo-covering.

The above example indicates that the reduction of complexity obtained passing from the proper power graph to the proper order graph, is usually too strong. For instance, if G is the group in the previous example and C_4 the cyclic group of order 4, we have $\mathcal{O}_0(G) \cong \mathcal{O}_0(C_4)$. In particular we cannot hope, in general, to count the components of $\tilde{\mathcal{P}}_0(G)$ relying on those of $\mathcal{O}_0(G)$. Anyway, taking into account the graph $\mathcal{O}_0(G)$, we get useful information on the isolated vertices of $\tilde{\mathcal{P}}_0(G)$.

LEMMA 4.5. *Let $x \in G_0$.*

- (i) *If $o(x) \in O_0(G)$ is isolated in $\mathcal{O}_0(G)$, then $[x]$ is isolated in $\tilde{\mathcal{P}}_0(G)$.*
- (ii) *If $[x]$ is isolated in $\tilde{\mathcal{P}}_0(G)$, then $o(x)$ is a prime and the component of $\mathcal{P}_0(G)$ containing x is a complete graph on $p - 1$ vertices.*
- (iii) $c_0(\mathcal{O}(G)) \leq |\{p \in P : p \mid |G|\}|$.

PROOF. (i) Apply Lemma 2.2 to the 2-homomorphism \tilde{o} of Proposition 4.2.

(ii) Let $[x]$ be isolated in $\tilde{\mathcal{P}}_0(G)$. We first show that $o(x) = p$, for some prime p . Assume, by contradiction, that $o(x)$ is composite. Then there exists $k \in \mathbb{N}$ such that $1 \neq \langle x^k \rangle \neq \langle x \rangle$ and so $\{[x], [x^k]\} \in [E]_0^*$, contradicting $[x]$ isolated. Let next $C = C_{\mathcal{P}_0(G)}(x)$. By [2, Theorem A (i)] applied to the pseudo-covering projection $\pi: \mathcal{P}_0(G) \rightarrow \tilde{\mathcal{P}}_0(G)$, we have that $\pi(C) = C_{\tilde{\mathcal{P}}_0(G)}([x])$ and thus $\pi(C)$ is reduced to the vertex $[x]$. So, if $x' \in V_C$ we have that $[x'] = [x]$ that is $\langle x' \rangle = \langle x \rangle$. It follows that V_C is the set of generators of the cyclic group $\langle x \rangle$ of order p . Thus $|V_C| = p - 1$ and C is a complete graph on $p - 1$ vertices.

(iii) Let $m \in O_0(G)$. If p is a prime dividing m , then $p \in O_0(G)$ and $p \mid |G|$. Moreover, $\{m, p\} \in E_{\mathcal{O}_0(G)}$ so that $m \in C_{\mathcal{O}_0(G)}(p)$. \square

5. Quotient graphs of power graphs associated with permutation groups

Let $G \leq S_n$ be a permutation group of degree $n \in \mathbb{N}, n \geq 2$ acting naturally on $N = \{1, \dots, n\}$. We want to determine $c_0(G)$ from $X = \tilde{\mathcal{P}}_0(G)$ by suitable quotients. We need to find a graph Y and a homomorphism $f \in \text{PC}(X, Y) = \text{LSur}(X, Y) \cap \text{Com}(X, Y)$ to which to apply Formula (1.1) in [2, Theorem A] or, better, a homomorphism $f \in \text{O}(X, Y) \cap \text{Com}(X, Y)$ to which we can apply the Procedure [2, 6.10]. To start with, we need to associate with every permutation an arithmetic object.

5.1 – Partitions

Let $n, r \in \mathbb{N}$, with $n \geq r$. An r -partition of n is an unordered r -tuple $T = [x_1, \dots, x_r]$, with $x_i \in \mathbb{N}$ for every $i \in \{1, \dots, r\}$, such that $n = \sum_{i=1}^r x_i$. The x_i are called the *terms* of the partition; the *order* of T is defined by $o(T) = \text{lcm}\{x_i\}_{i=1}^r$. We denote by $\mathfrak{T}_r(n)$ the set of the r -partitions of n and we call each element in $\mathfrak{T}(n) = \bigcup_{r=1}^n \mathfrak{T}_r(n)$ a *partition* of n . Given $T \in \mathfrak{T}(n)$, let $m_1 < \dots < m_k$ be its $k \in \mathbb{N}$ distinct terms; if m_j appears $t_j \geq 1$ times in T we use the notation $T = [m_1^{t_1}, \dots, m_k^{t_k}]$. Moreover, we say that $[m_1^{t_1}, \dots, m_k^{t_k}]$ is the *normal form* of T and that t_j is the *multiplicity* of the term m_j . We will accept, in some occasions, the multiplicity $t_j = 0$ simply to say that a certain natural number m_j does not appear as a term in T . We usually do not write the multiplicities equal to 1. For instance the partition $[1, 1, 3]$ can be written $[1^2, 3^1]$ or $[1^2, 3]$ or $[1^2, 2^0, 3]$. The partition $[1^n]$ is called the *trivial partition*. We put $\mathfrak{T}_0(n) = \mathfrak{T}(n) \setminus \{[1^n]\}$.

5.2 – Types of permutations

Let $\psi \in S_n$. The type of ψ is the partition of n given by the unordered list $T_\psi = [x_1, \dots, x_r]$ of the sizes x_i of the $r \in \mathbb{N}$ orbits of ψ on N . Note that the fixed points of ψ correspond to the terms $x_i = 1$, while the lengths of the disjoint cycles in which ψ uniquely splits are given by the terms $x_i \geq 2$. For instance $(1\ 2\ 3) \in S_3$ has type $[3]$, while $(1\ 2\ 3) \in S_4$ has type $[1, 3]$. Note also that $T_{id} = [1^n]$. For $\psi \in S_n$, we denote by $M_\psi = \{i \in N: \psi(i) \neq i\}$ the support of ψ . Thus $|M_\psi|$ is the sum of the terms different from 1 in T_ψ . The permutations of type $[1^{n-k}, k]$, for some $k \geq 2$, are the k -cycles; the 2-cycles are also called transpositions. Note that for every $\psi \in S_n$ and $s \in \mathbb{N}$, $T_\psi = T_{\psi^s}$ if and only if $\langle \psi \rangle = \langle \psi^s \rangle$. Note also that $o(T_\psi) = o(\psi)$. Recall that $\psi, \varphi \in S_n$ are conjugate in S_n if and only if $T_\psi = T_\varphi$. The map $t: S_n \rightarrow \mathfrak{T}(n)$, defined by $t(\psi) = T_\psi$ for all $\psi \in S_n$, is surjective. In other words, each partition of n may be viewed as the

type of some permutation in S_n . We call t the *type map*. If $X \subseteq S_n$, then $t(X)$ is the set of types *admissible* for X in the sense of [2, Section 4.1], and it is denoted by $\mathfrak{T}(X)$. For $T \in \mathfrak{T}(n)$, we denote by $\mu_T(G)$ the number of permutations of type T in $G \leq S_n$. If the normal form of T is $[m_1^{t_1}, \dots, m_k^{t_k}]$, then it is well known that

$$(5.1) \quad \mu_T(S_n) = \frac{n!}{m_1^{t_1} \dots m_k^{t_k} t_1! \dots t_k!}.$$

5.3 – Powers of partitions

Given $T = [m_1^{t_1}, \dots, m_k^{t_k}] \in \mathfrak{T}(n)$ in normal form, the *power* of T of exponent $a \in \mathbb{N}$, is defined as the partition

$$(5.2) \quad T^a = \left[\left(\frac{m_1}{\gcd(a, m_1)} \right)^{t_1 \gcd(a, m_1)}, \dots, \left(\frac{m_k}{\gcd(a, m_k)} \right)^{t_k \gcd(a, m_k)} \right].$$

Note that T^a is not necessarily in normal form. Moreover, for each $\psi \in S_n$ and each $a \in \mathbb{N}$, we have $T_{\psi^a} = (T_\psi)^a$. As a consequence, the power notation for partitions is consistent with a typical property of the powers: if $a, b \in \mathbb{N}$, then $(T^a)^b = T^{ab} = T^{ba} = (T^b)^a$. We say that T^a is a *proper power* of T if $[1^n] \neq T^a \neq T$. Throughout the section, we will use the notation in (5.2) without further reference.

LEMMA 5.1. *Let $T \in \mathfrak{T}(n)$ and $a \in \mathbb{N}$.*

- (i) $T^a = T$ if and only if $\gcd(a, o(T)) = 1$.
- (ii) $T^a = [1^n]$ if and only if $o(T) \mid a$.
- (iii) T^a is a proper power of T if and only if $\gcd(a, o(T))$ is a proper divisor of $o(T)$.
- (iv) If T^a is a proper power of T , then $o(T^a)$ is a proper divisor of $o(T)$.

PROOF. (i) Let $T^a = T$. Then the number of terms in T^a and in T is the same, that is, $\sum_{i=1}^k t_i \gcd(a, m_i) = \sum_{i=1}^k t_i$, which implies $\gcd(a, m_i) = 1$ for all $i \in \{1, \dots, k\}$ and thus $\gcd(a, \text{lcm}\{m_i\}_{i=1}^k) = \gcd(a, o(T)) = 1$. Conversely, if $\gcd(a, o(T)) = 1$, then $\gcd(a, \text{lcm}\{m_i\}_{i=1}^k) = 1$ and so $\frac{m_i}{\gcd(a, m_i)} = m_i$, for all $i \in \{1, \dots, k\}$ so that $T^a = T$.

(ii) $T^a = [1^n]$ is equivalent to $\frac{m_i}{\gcd(a, m_i)} = 1$, and thus to $m_i \mid a$ for all $i \in \{1, \dots, k\}$, that is to $o(T) = \text{lcm}\{m_i\}_{i=1}^k \mid a$.

(iii) It follows by the definition of proper power and by (i) and (ii).

(iv) Assume that T^a is a proper power of T . Then, by (iii), $\gcd(a, o(T)) \neq 1, o(T)$. Let $\psi \in S_n$ such that $T = T_\psi$. Then $T^a = T_{\psi^a}$ and $o(T^a) = o(\psi^a) \notin \{1, o(\psi) = o(T)\}$ is a proper divisor of $o(T)$. \square

LEMMA 5.2. (i) Let $T \in \mathfrak{T}(n)$, $a \in \mathbb{N}$ and $T' = T^a$ be a proper power of T . Then the following facts hold:

- (a) T' admits at least one term with multiplicity at least 2;
- (b) if there exists a term x of T' appearing with multiplicity 1, then a is coprime with x and x is a term of T .

(ii) Let $h \in \mathbb{N}$, with $1 \leq h < n/2$. Then there exists no type in $\mathfrak{T}(n)$ having $[h, n - k]$ or $[n]$ as a proper power.

PROOF. (i)(a) Since $T' = T^a$ is a proper power of T , by Lemma 5.1, there exists $j \in \{1, \dots, k\}$ with $\gcd(a, m_j) \geq 2$. Thus the term $\frac{m_j}{\gcd(a, m_j)}$ in T' has multiplicity at least $t_j \gcd(a, m_j) \geq 2$.

(i)(b) If a term x in T' appears with multiplicity 1, then there exists $j \in \{1, \dots, k\}$ with $t_j \gcd(a, m_j) = 1$ and $x = \frac{m_j}{\gcd(a, m_j)}$. It follows that $t_j = 1$ and $\gcd(a, m_j) = 1$. Moreover $x = m_j$ is a term of T .

(ii) This is a consequence of (i) and of $k \neq n - k$. \square

5.4 – The type graph of a permutation group

DEFINITION 5.3. Let $G \leq S_n$. We define the *type graph* of G , as the graph $\mathcal{T}(G)$ having as vertex set $\mathfrak{T}(G)$ and $\{T, T'\} \in E_{\mathcal{T}(G)}$ if $T, T' \in \mathfrak{T}(G)$ are one the power of the other. We define also the *proper type graph* of G , as the $[1^n]$ -deleted subgraph of $\mathcal{T}(G)$ and denote it by $\mathcal{T}_0(G)$. Its vertex set is then $\mathfrak{T}_0(G) = \mathfrak{T}(G) \setminus \{[1^n]\}$ and its edge set is denoted by $E_{\mathcal{T}_0(G)}$. The set of its components is denoted by $\mathcal{C}_0(\mathcal{T}(G))$ and their number by $c_0(\mathcal{T}(G))$.

Note that $\{T, T'\} \in E_{\mathcal{T}_0(G)}^*$ if and only if $T, T' \in \mathfrak{T}_0(G)$ are one the proper power of the other. Clearly $\mathcal{T}(G)$ is always connected and it is 2-connected if and only if $\mathcal{T}_0(G)$ is connected. Since from now on we intend to focus on proper graphs, we will tacitly assume $n \geq 2$.

PROPOSITION 5.4. Let $G \leq S_n$. Then the following facts hold:

- (i) the type map on G induces a complete homomorphism $t: \mathcal{P}_0(G) \rightarrow \mathcal{T}_0(G)$ and a complete 2-homomorphism $\tilde{t}: \tilde{\mathcal{P}}_0(G) \rightarrow \mathcal{T}_0(G)$;
- (ii) $\mathcal{T}_0(G)$ is a quotient of $\tilde{\mathcal{P}}_0(G)$. In particular, if $\tilde{\mathcal{P}}_0(G)$ is connected, then $\mathcal{T}_0(G)$ is connected too.

PROOF. (i) We begin showing that the map $t: G \rightarrow \mathfrak{T}(G)$ defines a complete homomorphism $t: \mathcal{P}(G) \rightarrow \mathcal{T}(G)$. Let $\{\psi, \varphi\} \in E$. Thus $\psi, \varphi \in G$ are one the power of the other, say $\psi = \varphi^a$ for some $a \in \mathbb{N}$. Then $t(\psi) = T_\psi = T_{\varphi^a} = (T_\varphi)^a = t(\varphi)^a$ and thus $\{t(\psi), t(\varphi)\} \in E_{\mathcal{T}(G)}$. This shows that t is a homomorphism. We show that t is complete, that is, $t(\mathcal{P}(G)) = \mathcal{T}(G)$. Note that $t(G) = \mathfrak{T}(G)$ trivially holds by definition of $\mathfrak{T}(G)$. Let $e = \{t(\psi), t(\varphi)\} \in E_{\mathcal{T}(G)}$ so that $t(\psi), t(\varphi)$ are one the power of the other. Let, say, $t(\psi) = t(\varphi)^a$ for some $a \in \mathbb{N}$. Thus we have $T_\psi = T_{\varphi^a}$. Now, define $\bar{\psi} = \varphi^a$ and note that $t(\bar{\psi}) = t(\psi)$. Hence $\bar{e} = \{\bar{\psi}, \varphi\} \in E$ and $t(\bar{e}) = e$. Since the only permutation having type $[1^n]$ is id , by Lemma 2.4(i), we also get a homomorphism $t: \mathcal{P}_0(G) \rightarrow \mathcal{T}_0(G)$. Moreover, applying Lemma 2.4(ii), t also induces the complete homomorphism $\tilde{t}: \tilde{\mathcal{P}}(G) \rightarrow \mathcal{T}(G)$ defined by $\tilde{t}([\psi]) = T_\psi$ for all $[\psi] \in [G]$ and a corresponding complete homomorphism $\tilde{t}: \tilde{\mathcal{P}}_0(G) \rightarrow \mathcal{T}_0(G)$. It remains to show that \tilde{t} is a 2-homomorphism. Let $\{[\psi], [\varphi]\} \in [E]_0^*$ and assume that $\tilde{t}([\psi]) = \tilde{t}([\varphi])$. Then $[\psi] \neq [\varphi]$ and, by Lemma 3.5, ψ and φ are one the power of the other. Moreover, we have $T_\psi = T_\varphi$. Hence $\langle \psi \rangle = \langle \varphi \rangle$, against $[\psi] \neq [\varphi]$.

(ii) It follows by (i), [2, Lemma 4.3], and [2, Proposition 3.2]. □

Let $G \leq S_n$. The homomorphism \tilde{t} , defined in the above proposition, transfers to $[G]$ all the concepts introduced for G in terms of type. In particular we define the type $T_{[\psi]}$ of $[\psi] \in [G]$, by T_ψ . Moreover, we say that $[\psi]$ is a k -cycle (a transposition) if ψ is k -cycle (a transposition). For $X \subseteq S_n$ consider $[X] = \{[x] \in [S_n]: x \in X\} \subseteq [S_n]$. Then, according to [2, Section 4.1], $\tilde{t}([X]) = t(X) = \mathfrak{T}(X)$ is the set of *types admissible* for $[X]$. Since every subset of $[S_n]$ is given by $[X]$, for a suitable $X \subseteq S_n$, that defines the concept of admissibility for all the subsets of $[S_n]$. If \hat{X} is a subgraph of $\tilde{\mathcal{P}}_0(G)$ the set of types admissible for \hat{X} , denoted by $\mathfrak{T}(\hat{X})$, is given by the set of types admissible for $V_{\hat{X}}$. In particular, for $C \in \tilde{\mathcal{C}}_0(G)$, we have $\mathfrak{T}(C) = \{T \in \mathfrak{T}_0(G): \text{there exists } [\psi] \in V_C \text{ with } T_\psi = T\}$.

It is useful to isolate a fact contained in Proposition 5.4.

COROLLARY 5.5. *Let $G \leq S_n$. If $\varphi, \psi \in G_0$ are such that $\{[\varphi], [\psi]\} \in [E]_0^*$, then one of T_φ and $T_\psi \in \mathfrak{T}_0(G)$ is a proper power of the other. In particular $T_\varphi \neq T_\psi$.*

Observe that the converse of the above corollary does not hold. Consider, for instance, $\varphi = (1\ 2\ 3\ 4)$, $\psi = (1\ 2)(3\ 4) \in S_4$. We have that $T_\psi = [2^2]$ is the power of exponent 2 of $T_\varphi = [4]$, but there is no edge between $[\varphi]$ and $[\psi]$ in $\tilde{\mathcal{P}}_0(S_4)$, because no power of $(1\ 2\ 3\ 4)$ is equal to $(1\ 2)(3\ 4)$.

5.5 – The order graph of a permutation group

In this section we show that, for any permutation group, the graph $\mathcal{O}_0(G)$ is a quotient of $\mathcal{T}_0(G)$.

Define the map

$$(5.3) \quad o_{\mathcal{T}}: \mathfrak{T}_0(G) \longrightarrow O_0(G), \quad o_{\mathcal{T}}(T) = o(T), \quad \text{for all } T \in \mathfrak{T}_0(G),$$

and recall the map \tilde{o} defined in Proposition 4.2.

PROPOSITION 5.6. *For every $G \leq S_n$, the map $o_{\mathcal{T}}$ defines a complete 2-homomorphism $o_{\mathcal{T}}: \mathcal{T}_0(G) \rightarrow \mathcal{O}_0(G)$ such that $o_{\mathcal{T}} \circ \tilde{i} = \tilde{o}$. In particular $\mathcal{O}_0(G)$ is a quotient of $\mathcal{T}_0(G)$ and $c_0(\mathcal{O}(G)) \leq c_0(\mathcal{T}(G))$.*

PROOF. First note that the map $o_{\mathcal{T}}$ in (5.3) is well defined. Indeed, if $T \in \mathfrak{T}_0(G)$, then there exists $\psi \in G_0$ such that $T = T_\psi$ and so $o(T) = o(T_\psi) = o(\psi) \in O_0(G)$. The same argument shows that $o_{\mathcal{T}} \circ \tilde{i} = \tilde{o}$. By Proposition 4.2, the map \tilde{o} is a complete homomorphism from $\tilde{\mathcal{P}}_0(G)$ in $\mathcal{O}_0(G)$. In particular \tilde{o} is surjective and so also $o_{\mathcal{T}}$ is surjective.

We show that $o_{\mathcal{T}}$ is a 2-homomorphism. Let $\{T, T'\}$ be an edge in $\mathcal{T}_0(G)$. Then T, T' are one the power of the other. Let, say, $T' = T^a$ for some $a \in \mathbb{N}$. Then $o(T')$ is a divisor of $o(T)$, which says $\{o(T), o(T')\} \in E_{\mathcal{O}_0(G)}$. Moreover, if $T \neq T'$ we have $[1^n] \neq T^a \neq T$ and thus Lemma 5.1 (iv) implies that $o(T')$ is a proper divisor of $o(T)$.

We finally show that $o_{\mathcal{T}}$ is complete. Let $e = \{m, m'\} \in E_{\mathcal{O}_0(G)}$. Since \tilde{o} is complete, there exist $[\varphi], [\varphi'] \in [G]_0$ such that $\tilde{o}([\varphi]) = m, \tilde{o}([\varphi']) = m'$ and $\{[\varphi], [\varphi']\} \in [E]_0$. By Proposition 5.4 (i), the map \tilde{i} is a homomorphism and so $\{\tilde{i}([\varphi]), \tilde{i}([\varphi'])\} = \{T_\varphi, T_{\varphi'}\}$ is an edge in $\mathcal{T}_0(G)$. Now it is enough to observe that $o_{\mathcal{T}}(T_\varphi) = o(\varphi) = m$ and $o_{\mathcal{T}}(T_{\varphi'}) = o(\varphi') = m'$. To close, we apply [2, Lemma 4.3 and Proposition 3.2]. \square

COROLLARY 5.7. *Let $G \leq S_n$. If $m \in O_0(G)$ is isolated in $\mathcal{O}_0(G)$, then each type of order m is isolated in $\mathcal{T}_0(G)$.*

PROOF. By Proposition 5.6, $o_{\mathcal{T}}$ is a 2-homomorphism. Thus we conclude applying Lemma 2.2 to $o_{\mathcal{T}}$. \square

6. Quotient graphs of power graphs associated with fusion controlled permutation groups

DEFINITION 6.1. Let $G \leq S_n$.

- (a) G is called *fusion controlled* if $N_{S_n}(G)$ controls the fusion in G , with respect to S_n , that is, if for every $\psi \in G$ and $x \in S_n$ such that $\psi^x \in G$, there exists $y \in N_{S_n}(G)$ such that $\psi^x = \psi^y$.
- (b) For each $x \in N_{S_n}(G)$, define the map

$$(6.1) \quad F_x: [G]_0 \longrightarrow [G]_0, \quad F_x([\psi]) = [\psi^x], \quad \text{for all } [\psi] \in [G]_0.$$

Note that F_x is well defined, that is, for every $[\psi] \in [G]_0$, $F_x([\psi])$ does not depend on the representative of $[\psi]$, and $F_x([\psi]) \in [G]_0$. Those facts are immediate considering that conjugation is an automorphism of the group S_n and that $x \in N_{S_n}(G)$.

Recalling now the homomorphism \tilde{t} defined in Proposition 5.4 and the definition of \mathfrak{G} -consistency given in [2, Definition 4.4 (b)], we get the following result.

PROPOSITION 6.2. *Let $G \leq S_n$. Then the following hold.*

- (i) *For every $x \in N_{S_n}(G)$, the map F_x is a graph automorphism of $\tilde{\mathcal{P}}_0(G)$ which preserves the type.*
- (ii) *If G is fusion controlled, then $\mathfrak{G} = \{F_x: x \in N_{S_n}(G)\}$ is a subgroup of $\text{Aut}(\tilde{\mathcal{P}}_0(G))$ and \tilde{t} is a \mathfrak{G} -consistent 2-homomorphism from $\tilde{\mathcal{P}}_0(G)$ to $\mathcal{T}_0(G)$. In particular \tilde{t} is a complete orbit 2-homomorphism.*

PROOF. (i) Since, for every $x \in N_{S_n}(G)$, we have $F_x \circ F_{x^{-1}} = F_{x^{-1}} \circ F_x = id_{[G]_0}$ we deduce that F_x is a bijection. We show that F_x is a graph homomorphism. Let $\{[\varphi], [\psi]\} \in [E]_0$. Then, by Lemma 3.5, φ and ψ are one the power of the other. Let, say, $\varphi = \psi^m$ for some $m \in \mathbb{N}$. Thus, since conjugation is an automorphism of S_n , we have $\varphi^x = (\psi^m)^x = (\psi^x)^m$ and so $\{[\varphi^x], [\psi^x]\} \in [E]_0$. Next we see that F_x is complete. Let $e = \{[\varphi^x], [\psi^x]\} \in [E]_0$. Then by Lemma 3.5, φ^x and ψ^x are one the power of the other. Let, say $\varphi^x = (\psi^x)^m$ for some $m \in \mathbb{N}$. Now, considering $\bar{e} = \{[\psi], [\psi^m]\} \in [E]_0$, we have $F_x(\bar{e}) = e$.

Finally $T_{F_x([\psi])} = T_{[\psi]}$ because ψ^x is a conjugate of ψ and thus has its same type. In other words, for every $x \in N_{S_n}(G)$, we have

$$(6.2) \quad \tilde{t} \circ F_x = \tilde{t}.$$

(ii) The map $N_{S_n}(G) \rightarrow \text{Aut}(\tilde{\mathcal{P}}_0(G))$ associating F_x to $x \in N_{S_n}(G)$, is a group homomorphism and so its image \mathfrak{G} is a subgroup of $\text{Aut}(\tilde{\mathcal{P}}_0(G))$. We show that \tilde{t} is \mathfrak{G} -consistent checking that the conditions (a) and (b) in [2, Lemma 4.5] are satisfied. Condition (a) is just (6.2). To get condition (b), pick $[\varphi], [\psi] \in [G]_0$ with $T_\varphi = T_\psi$. Then φ and ψ are elements of G conjugate in S_n . Thus, as G is fusion controlled, they are conjugate also in $N_{S_n}(G)$. So there exists $x \in N_{S_n}(G)$ such that $\varphi = \psi^x$, which gives $[\varphi] = [\psi^x] = F_x([\psi])$. \square

We say that $C, \hat{C} \in \tilde{\mathcal{C}}_0(G)$ are *conjugate* if there exists $x \in N_{S_n}(G)$ such that $\hat{C} = F_x(C)$.

PROPOSITION 6.3. *Let $G \leq S_n$ be fusion controlled. Then the following hold.*

- (i) $\mathcal{T}_0(G)$ is an orbit quotient of $\tilde{\mathcal{P}}_0(G)$.
- (ii) $T \in \mathfrak{T}_0(G)$ is isolated in $\mathcal{T}_0(G)$ if and only if each $[\psi] \in [G]_0$ of type T is isolated in $\tilde{\mathcal{P}}_0(G)$.
- (iii) If $T \in \mathfrak{T}_0(G)$, then the components of $\tilde{\mathcal{P}}_0(G)$ admissible for T are conjugate.

PROOF. Keeping in mind the definition of orbit quotient given in [2, Definition 5.14], (i) is a rephrasing of Proposition 6.2 (ii). To show (ii) recall that, by [2, Proposition 5.9 (ii)], every orbit homomorphism is locally surjective and then apply [2, Corollary 5.12] and Lemma 2.2 to the orbit 2-homomorphism \tilde{t} . (iii) is an application of [2, Proposition 6.9 (i)]. \square

LEMMA 6.4. *Let G be fusion controlled, $C \in \tilde{\mathcal{C}}_0(G)$ and $T \in \mathfrak{T}(C)$. Then $\mathfrak{T}(C) = V_C(T)$.*

PROOF. Apply [2, Theorem A (i)] to \tilde{t} recalling that, by (2.1), every orbit homomorphism is locally surjective. \square

Proposition 6.2 guarantees that when G is fusion controlled, we have

$$(6.3) \quad \tilde{t} \in \text{O}(\tilde{\mathcal{P}}_0(G), \mathcal{T}_0(G)) \cap \text{Com}(\tilde{\mathcal{P}}_0(G), \mathcal{T}_0(G)).$$

Thus the machinery for counting the components of $\tilde{\mathcal{P}}_0(G)$ by those in $\mathcal{T}_0(G)$ can start provided that we control the numbers $k_{\tilde{\mathcal{P}}_0(G)}(T)$ and $k_C(T)$, for $T \in \mathfrak{T}(G_0)$ and $C \in \tilde{\mathcal{C}}_0(G)$. We see first that $k_{\tilde{\mathcal{P}}_0(G)}(T)$ is easily determined, for any permutation group G , through $\mu_T(G)$. Recall that ϕ denotes the Euler totient function.

LEMMA 6.5. *Let $G \leq S_n$. If $T \in \mathfrak{T}(G_0)$, then*

$$k_{\tilde{\mathcal{P}}_0(G)}(T) = \frac{\mu_T(G)}{\phi(o(T))}.$$

PROOF. If $T = [m_1^{t_1}, \dots, m_k^{t_k}] \in \mathfrak{T}(G_0)$, then $G_T = \{\sigma \in G : \sigma \text{ is of type } T\}$ is nonempty and each element in G_T has the same order given by $\text{lcm}\{m_i\}_{i=1}^k = o(T)$. Consider the equivalence relation \sim which defines the quotient graph $\tilde{\mathcal{P}}_0(G)$. Since generators of the same cyclic subgroup of G share the same type, it follows that G_T is union of \sim -classes, each of them of size $\phi(o(T))$. On the other hand, $[\sigma] \in [G]$ has type T if and only if $\sigma \in G_T$. This means that $k_{\tilde{\mathcal{P}}_0(G)}(T)$ is the number of \sim -classes contained in G_T , and so $k_{\tilde{\mathcal{P}}_0(G)}(T) = \frac{\mu_T(G)}{\phi(o(T))}$. \square

Let $T \in \mathfrak{T}_0(G)$. Accordingly to [2, Definition 6.1], we consider the set $\tilde{\mathcal{C}}_0(G)_T$ of the components of $\tilde{\mathcal{P}}_0(G)$ admissible for T and denote by $\tilde{c}_0(G)_T$ its order. Thus $\tilde{c}_0(G)_T$ counts the number of components of $\tilde{\mathcal{P}}_0(G)$ in which there exists at least one vertex $[\psi]$ of type T .

LEMMA 6.6. *Let $G \leq S_n$ be fusion controlled and $T \in \mathfrak{T}_0(G)$.*

- (i) $\tilde{c}_0(G)_T = \frac{k_{\tilde{\mathcal{P}}_0(G)}(T)}{k_C(T)} = \frac{\mu_T(G)}{\phi(o(T))k_C(T)}$ for all $C \in \tilde{\mathcal{C}}_0(G)_T$.
- (ii) *If T is isolated in $\mathfrak{T}_0(G)$, then $\tilde{c}_0(G)_T = k_{\tilde{\mathcal{P}}_0(G)}(T) = \frac{\mu_T(G)}{\phi(o(T))}$.*

PROOF. (i) By (6.3) and (2.1), we can apply [2, Proposition 6.8] to \tilde{t} and then use Lemma 6.5 to make the computation explicit.

(ii) is a consequence of (i) and of Proposition 6.3 (ii). \square

PROOF OF THEOREM A. Applying [2, Theorem A] to \tilde{t} , we get

$$c_0(G) = \tilde{c}_0(G) = \sum_{i=1}^{c_0(\mathfrak{T}(G))} \tilde{c}_0(G)_{T_i}$$

and, by Lemma 6.6 (i), we obtain

$$\sum_{i=1}^{c_0(\mathfrak{T}(G))} \tilde{c}_0(G)_{T_i} = \sum_{i=1}^{c_0(\mathfrak{T}(G))} \frac{\mu_{T_i}(G)}{\phi(o(T_i))k_{C_i}(T_i)}. \quad \square$$

We now observe some interesting limitations to the types which can appear in a same component of $\tilde{\mathcal{P}}_0(G)$.

COROLLARY 6.7. *Let $G \leq S_n$ be fusion controlled and $C \in \tilde{\mathcal{C}}_0(G)$.*

- (i) *If $T, T' \in \mathfrak{T}(C)$, then $\frac{k_{\tilde{\mathcal{P}}_0(G)}(T)}{k_C(T)} = \frac{k_{\tilde{\mathcal{P}}_0(G)}(T')}{k_C(T')}$.*
- (ii) *$C \cong \tilde{t}(C)$ if and only if there exists $T \in \mathfrak{T}(C)$ such that $k_C(T) = 1$ and, for every $T' \in \mathfrak{T}(C)$, $k_{\tilde{\mathcal{P}}_0(G)}(T) = k_{\tilde{\mathcal{P}}_0(G)}(T')$.*
- (iii) *If C contains all the vertices of $[G]_0$ of a certain type $T \in \mathfrak{T}(C)$, then C also contains all the vertices of $[G]_0$ of type T' for all $T' \in \mathfrak{T}(C)$.*
- (iv) *If there exists $T \in \mathfrak{T}(C)$ such that $k_C(T) = k_{\tilde{\mathcal{P}}_0(G)}(T) > 1$, then $C \not\cong \tilde{t}(C)$.*

PROOF. An immediate application of [2, Proposition 7.2]. □

Finally we state a useful criterion of connectedness for $\tilde{\mathcal{P}}_0(G)$.

COROLLARY 6.8. *Let G be fusion controlled and let $\mathcal{T}_0(G)$ be connected. Then $\tilde{\mathcal{P}}_0(G)$ is a union of conjugate components. Moreover, if there exist $T \in \mathfrak{T}_0(G)$ and $C \in \tilde{\mathcal{C}}_0(G)$, with C containing all the vertices of $[G]_0$ of type T , then $\tilde{\mathcal{P}}_0(G)$ is connected.*

PROOF. Let $C \in \tilde{\mathcal{C}}_0(G)$ and consider $\tilde{t}(C)$. Then $\tilde{t}(V_C) = \mathfrak{T}(C)$. By [2, Theorem A (i)], $\tilde{t}(C)$ is a component of $\mathcal{T}_0(G)$ and since that graph is connected, we get $\tilde{t}(V_C) = \mathfrak{T}_0(G)$ and so $\tilde{t}^{-1}(\tilde{t}(V_C)) = [G]_0$. By [2, Theorem A (iii)], this implies that $\tilde{\mathcal{P}}_0(G)$ is the union of the components in $\mathcal{C}(\tilde{\mathcal{P}}_0(G))_{\tilde{t}(C)}$ which, by [2, Lemma 6.3 (i)] are equal to the components admissible for any $T \in \mathfrak{T}(G_0)$. So Proposition 6.3 (iii) applies giving $\tilde{\mathcal{P}}_0(G)$ as a union of conjugate components. The last part follows from [2, Corollary 5.15]. □

7. The number of components in $\tilde{\mathcal{P}}_0(S_n)$, $\mathcal{T}_0(S_n)$, $\mathcal{O}_0(S_n)$

In this section we apply Procedure [2, 6.10] to S_n (which is fusion controlled) to make the computation of Formula (1.1) concrete. Exploiting the strong link among the graphs $\mathcal{P}_0(S_n)$, $\tilde{\mathcal{P}}_0(S_n)$, $\mathcal{T}_0(S_n)$, $\mathcal{O}_0(S_n)$, we determine simultaneously $c_0(S_n) = \tilde{c}_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$. On the route, we give a description of the components of those four graphs. Recall that $\mathfrak{T}_0(n) = \mathfrak{T}(S_n) \setminus \{[1^n]\}$ and that, for $T \in \mathfrak{T}_0(n)$, the numbers $\mu_T(S_n)$ are computed by (5.1). Also recall that, if $T \in \mathfrak{T}_0(n)$ and C is a component of $\tilde{\mathcal{P}}_0(S_n)$, then $k_C(T)$ counts the number of vertices in C having type T .

By Lemma 6.4 and Theorem A, the procedure to get $\tilde{c}_0(S_n)$ translates into the following.

PROCEDURE 7.1
COMPUTATION OF $\tilde{c}_0(S_n)$
(I) Selection of T_i and C_i .

Start. Pick arbitrary $T_1 \in \mathfrak{T}_0(n)$ and choose any $C_1 \in \tilde{\mathcal{C}}_0(S_n)_{T_1}$.

Basic step. Given $T_1, \dots, T_i \in \mathfrak{T}_0(n)$ and $C_1, \dots, C_i \in \tilde{\mathcal{C}}_0(S_n)$ such that

$$C_j \in \tilde{\mathcal{C}}_0(S_n)_{T_j} \quad (1 \leq j \leq i), \text{ choose any } T_{i+1} \in \mathfrak{T}_0(n) \setminus \bigcup_{j=1}^i \mathfrak{T}(C_j) \text{ and any } C_{i+1} \in \tilde{\mathcal{C}}_0(S_n)_{T_{i+1}}.$$

Stop. The procedure stops in $c_0(\mathcal{T}(S_n))$ steps.

(II) The value of $\tilde{c}_0(S_n)$.

Compute the integers

$$\tilde{c}_0(S_n)_{T_j} = \frac{\mu_{T_j}(S_n)}{\phi(o(T_j))k_{C_j}(T_j)} \quad (1 \leq j \leq c_0(\mathcal{T}(S_n)))$$

and sum them up to get $\tilde{c}_0(S_n)$.

The complete freedom in the choice of the $C_j \in \tilde{\mathcal{C}}_0(S_n)_{T_j}$ allows us to compute each $\tilde{c}_0(S_n)_{T_j} = \frac{\mu_{T_j}(S_n)}{\phi(o(T_j))k_{C_j}(T_j)}$ selecting C_j as the component containing $[\psi]$, for $[\psi]$ chosen as preferred among the $\psi \in S_n \setminus \{id\}$ with $T_\psi = T_j$. We will apply this fact with no further mention. Remarkably, the number $c_0(\mathcal{T}(S_n))$ counts the steps of the procedure.

7.1 – Preliminary lemmas and small degrees

We start by summarizing what we know about isolated vertices by Proposition 6.3 (ii), Lemma 4.5, and Corollary 5.7.

LEMMA 7.2. *The following properties hold.*

- (i) *The type $T \in \mathfrak{T}_0(S_n)$ is isolated in $\mathcal{T}_0(S_n)$ if and only if each $[\psi] \in [S_n]_0$ of type T is isolated in $\tilde{\mathcal{P}}_0(S_n)$.*
- (ii) *If $m \in O_0(S_n)$ is isolated in $\mathcal{O}_0(S_n)$, then each vertex of order m is isolated in $\tilde{\mathcal{P}}_0(S_n)$ and each type of order m is isolated in $\mathcal{T}_0(G)$.*
- (iii) *If, for some $\psi \in S_n$, $[\psi]$ is isolated in $\tilde{\mathcal{P}}_0(S_n)$, then $o(\psi)$ is prime and the component of $P_0(S_n)$ containing ψ is a complete graph on $p - 1$ vertices.*

As a consequence, we are able to analyse the prime or prime plus 1 degrees.

LEMMA 7.3. *Let $n \in \{p, p+1\}$ for some $p \in P$. Then the following facts hold:*

- (i) *p is isolated in $\mathcal{O}_0(S_n)$. The type $[1^{n-p}, p]$ is isolated in $\mathcal{T}_0(S_n)$;*
- (ii) *each vertex of $[S_n]_0$ of order p is isolated in $\tilde{\mathcal{P}}_0(S_n)$;*
- (iii) *the number of components of $\tilde{\mathcal{P}}_0(S_n)$ containing the elements of order p in $[S_n]_0$ is given by $\tilde{c}_0(S_p)_{[p]} = (p-2)!$ if $n = p$, and by $\tilde{c}_0(S_{p+1})_{[1,p]} = (p+1)(p-2)!$ if $n = p+1$.*

PROOF. (i)–(ii) Since $n \in \{p, p+1\}$, we have $p \leq n$ so that S_n admits elements of order p . Since there exists no element with order kp for $k \geq 2$, p is isolated in $\mathcal{O}_0(S_n)$. Thus Lemma 7.2 applies.

(iii) The counting follows from Lemma 6.6 (ii) and Formula (5.1) after having observed that the only type of order p in $\mathcal{T}_0(S_p)$ is $[p]$ and that the only type of order p in $\mathcal{T}_0(S_{p+1})$ is $[1, p]$. \square

LEMMA 7.4. *For $n \geq 6$, the transpositions of $\tilde{\mathcal{P}}_0(S_n)$ lie in the same component $\tilde{\Delta}_n$ of $\tilde{\mathcal{P}}_0(S_n)$. Moreover $\mathfrak{T}(\tilde{\Delta}_n) \supseteq \{[1^{n-2}, 2][1^{n-5}, 2, 3], [1^{n-3}, 3]\}$.*

PROOF. Let $[\varphi_1]$ and $[\varphi_2]$ be two distinct transpositions in S_n . Then their supports $M_{\varphi_1} M_{\varphi_2}$ are distinct. If $|M_{\varphi_1} \cap M_{\varphi_2}| = 1$, then there exist distinct $a, b, c \in N$ such that $\varphi_1 = (a b)$ and $\varphi_2 = (a c)$. Moreover, as $n \geq 6$, there exist distinct $e, f, g \in N \setminus \{a, b, c\}$ and we have the path

$$[(a b)], [(a b)(d e f)], [(d e f)], [(a c)(d e f)], [(a c)]$$

between $[\varphi_1]$ and $[\varphi_2]$. If $|M_{\varphi_1} \cap M_{\varphi_2}| = 0$, then there exist distinct $a, b, c, d \in N$ such that $\varphi_1 = (a b)$ and $\varphi_2 = (c d)$. Let $\varphi_3 = (a c)$. By the previous case, there exists a path between $[\varphi_1]$ and $[\varphi_3]$ and a path between $[\varphi_2]$ and $[\varphi_3]$. Therefore there exists also a path between $[\varphi_1]$ and $[\varphi_2]$. This shows that all the transpositions of $\tilde{\mathcal{P}}_0(S_n)$ lie in the same component $\tilde{\Delta}_n$. Next, collecting the types met in the paths, we get $\mathfrak{T}(\tilde{\Delta}_n) \supseteq \{[1^{n-2}, 2][1^{n-5}, 2, 3], [1^{n-3}, 3]\}$. \square

We note now an interesting immediate fact.

LEMMA 7.5. *Let X, Y be graphs and $f \in \text{Hom}(X, Y)$. If \hat{X} is a complete subgraph of X , then $f(\hat{X})$ is a complete subgraph of Y .*

COROLLARY 7.6. *Let $n \geq 6$ and Δ_n be the unique component of $\mathcal{P}_0(A_n)$ such that $\pi(\Delta_n) = \tilde{\Delta}_n$. Then neither one of the components $\Delta_n, \tilde{\Delta}_n, \tilde{i}(\tilde{\Delta}_n)$ of the graphs $\mathcal{P}_0(S_n), \tilde{\mathcal{P}}_0(S_n), \mathcal{T}_0(S_n)$ respectively, nor the connected subgraph $\tilde{o}(\tilde{\Delta}_n)$ of $\mathcal{O}_0(S_n)$ is a complete graph.*

PROOF. First note that the existence of a unique component Δ_n of $\mathcal{P}_0(A_n)$ such that $\pi(\Delta_n) = \tilde{\Delta}_n$ is guaranteed by [2, Corollary 5.13], because π is pseudo-covering and tame due to Lemma 3.6. Moreover, by Proposition 6.2, \tilde{t} is a complete orbit homomorphism and thus locally surjective. Hence [2, Theorem A (i)] guarantees that $\tilde{t}(\tilde{\Delta}_n)$ is a component of $\tilde{\mathcal{P}}_0(S_n)$ with $V_{\tilde{t}(\tilde{\Delta}_n)} = \mathfrak{T}(\tilde{\Delta}_n)$. On the other hand, by Proposition 5.6, we have the complete graph homomorphism $o_{\mathcal{T}}: \mathcal{T}_0(S_n) \rightarrow \mathcal{O}_0(S_n)$ such that $o_{\mathcal{T}} \circ \tilde{t} = \tilde{o}$. In particular $\tilde{o}(\tilde{\Delta}_n) = o_{\mathcal{T}}(\tilde{t}(\tilde{\Delta}_n))$, so that we can interpret the sequence of graphs

$$\Delta_n, \quad \tilde{\Delta}_n, \quad \tilde{t}(\tilde{\Delta}_n), \quad \tilde{o}(\tilde{\Delta}_n)$$

as

$$(7.1) \quad \Delta_n, \quad \pi(\Delta_n), \quad (\tilde{t} \circ \pi)(\Delta_n), \quad (o_{\mathcal{T}} \circ \tilde{t} \circ \pi)(\Delta_n).$$

By Lemma 7.4, $\tilde{o}(\tilde{\Delta}_n)$ admits as vertices the integer 2 and 3 which are not adjacent in $\mathcal{O}_0(S_n)$. Thus $\tilde{o}(\tilde{\Delta}_n)$ is not a complete graph. Then to deduce that no graph in the sequence (7.1) is complete we start from the bottom and apply three times Lemma 7.5. \square

Note that, in general, $\tilde{o}(\tilde{\Delta}_n)$ is not a component of $\mathcal{O}_0(S_n)$ because \tilde{o} is not pseudo-covering. For instance, $\tilde{o}(\tilde{\Delta}_6)$ is not a component of $\mathcal{O}_0(S_6)$ because $4 \notin V_{\tilde{o}(\tilde{\Delta}_6)}$ while 4 belongs to the component of $\mathcal{O}_0(S_6)$ containing $\tilde{o}(\tilde{\Delta}_6)$. An argument in the proof of Theorem B (ii) shows that $\tilde{o}(\tilde{\Delta}_n)$ is indeed a component at least for $n \geq 8$.

PROOF OF THEOREM B (i). For each $n \in \mathbb{N}$, with $2 \leq n \leq 7$, we compute $\tilde{c}_0(S_n)$, $c_0(\mathcal{T}(S_n))$ and $c_0(\mathcal{O}(S_n))$ separately. As usual we view S_n as naturally acting on $N = \{1, \dots, n\}$.

Since $[S_2]_0 = \{[(1\ 2)]\}$, we immediately have $\tilde{c}_0(S_2) = c_0(\mathcal{T}(S_2)) = c_0(\mathcal{O}(S_2)) = 1$. Since $[S_3]_0 = \{[(1\ 2)], [(1\ 3)], [(2\ 3)], [(1\ 2\ 3)]\}$, we have $\mathfrak{T}_0(S_3) = \{[1, 2], [3]\}$ and $\mathcal{O}_0(S_3) = \{2, 3\}$. Thus, by Lemma 7.3, we get $\tilde{c}_0(S_3) = 4$ and $c_0(\mathcal{T}(S_3)) = c_0(\mathcal{O}(S_3)) = 2$.

Let $n = 4$. We start considering the type $T_1 = [4]$ and the cycle $\psi = (1\ 2\ 3\ 4) \in S_4$. By Corollary 5.5 and Lemma 5.2, the only vertex distinct from $[\psi]$ adjacent to $[\psi]$ is $\varphi = [(1\ 3)(2\ 4)]$ and no other vertex can be adjacent to $[\psi]$ or $[\varphi]$. Thus the component C_1 of $\tilde{\mathcal{P}}_0(S_4)$ having as a vertex $[\psi]$ is a path of length one, $k_{C_1}(T_1) = 1$ and $\tilde{c}_0(S_4)_{T_1} = \frac{\mu_{[4]}(S_4)}{\phi(4)} = 3$. Note that $\mathfrak{T}(C_1) = \{[4], [2^2]\}$. By Lemma 7.3, a vertex of type $T_2 = [1, 3]$ is isolated and thus $\tilde{c}_0(S_4)_{T_2} = \frac{\mu_{[1,3]}(S_4)}{\phi(3)} = 4$.

Consider now the type $T_3 = [1^2, 2]$. T_3 is not a proper power and has no proper power. So, by Corollary 5.5, a component admissible for T_3 is again reduced to a single vertex. Thus $\tilde{c}_0(S_4)_{T_3} = \mu_{[1^2, 2]}(S_4) = 6$. Since all the possible types in S_4 have been considered, the Procedure 7.1 stops giving $c_0(\mathcal{T}(S_4)) = 3$ and $\tilde{c}_0(S_4) = 3+4+6 = 13$. Since $O_0(S_4) = \{2, 4, 3\}$ we instead have $c_0(O(S_4)) = 2$.

Let $n = 5$. By Lemma 7.3 (ii), the vertices of type $T_1 = [5]$ in $\tilde{\mathcal{P}}_0(S_5)$ are in $3! = 6$ components which are isolated vertices. Let C_1 be one of those components. Consider for the type $T_2 = [1, 4]$, the cycle $\psi = (1\ 2\ 3\ 4) \in S_5$. By Corollary 5.5 and Lemma 5.2, the component C_2 containing $[\psi]$ admits as vertices just $[(1\ 2\ 3\ 4)]$ and $[(1\ 3)(2\ 4)]$. Thus $k_{C_2}(T_2) = 1$ and $\tilde{c}_0(S_5)_{T_2} = \frac{\mu_{[1, 4]}(S_5)}{\phi(4)} = 15$. Moreover, $\mathfrak{T}(C_1) \cup \mathfrak{T}(C_2) = \{[5], [1, 4], [1, 2^2]\}$. We next consider $T_3 = [2, 3]$ and $\psi = (1\ 2)(3\ 4\ 5) \in S_5$. By Lemma 5.2, T_3 is not a power and so there exists no $[\varphi] \in [S_5]_0$ such that $\varphi^s = \psi$. On the other hand, to get a power of $T_3 = T_\psi$ different from $[1^5]$, we must consider T_{ψ^a} where $\gcd(a, o(\psi)) \neq 1$, that is, ψ^a for $a \in \{2, 3, 4\}$. Thus the component C_3 containing $[\psi]$ contains the path $[(1\ 2)], [(1\ 2)(3\ 4\ 5)], [(3\ 4\ 5)]$. We show that C_3 is indeed that path. If there exists a proper edge $\{[(3\ 4\ 5)], [\varphi]\}$, then, by Corollary 5.5, $T_\varphi \notin \mathfrak{T}(C_1) \cup \mathfrak{T}(C_2) \cup \{[1^2, 3]\}$ and thus $T_\varphi \in \{[1^3, 2], [2, 3]\}$. But $[1^3, 2]$ and $[1^2, 3]$ are not one the power of the other and thus, by Corollary 5.5, we must have $T_\varphi = [2, 3]$, say $\varphi = (a\ b)(c\ d\ e)$ where $\{a, b, c, d, e\} = N$. So $[(a\ b)(c\ d\ e)]^2 = (c\ e\ d)$ is a generator of $\langle (3\ 4\ 5) \rangle$. It follows that $(a\ b) = (1\ 2)$ and $(c\ e\ d) \in \{(3\ 4\ 5), (3\ 5\ 4)\}$. Thus $\varphi \in \{\varphi_1 = (1\ 2)(3\ 4\ 5), \varphi_2 = (1\ 2)(3\ 5\ 4)\}$. But it is immediately checked that $[\varphi_1] = [\varphi_2] = [\psi]$. Similarly one can check that the only proper edge $\{[(1\ 2)], [\varphi]\}$ is given by the choice $\varphi = \psi$. So we have $\tilde{c}_0(S_5)_{T_3} = \frac{\mu_{[2, 3]}(S_5)}{\phi(6)} = 10$ and, since all the possible types in S_4 have been considered, we get $c_0(\mathcal{T}(S_5)) = 3$ and $\tilde{c}_0(S_5) = 31$. On the other hand there are only two components for $O_0(S_5)$: one reduced to the vertex 5 and the other one having as set of vertices $\{2, 3, 4, 6\}$.

Let $n = 6$. In $\tilde{\mathcal{P}}_0(S_6)$, by Lemma 7.3 (iii), the elements of type $T_1 = [1, 5]$ are inside 36 components which are isolated vertices. Let C_1 be one of them. Consider the type $T_2 = [2, 4]$ and $\psi = (1\ 2)(3\ 4\ 5\ 6)$. The component C_2 containing $[\psi]$ is the path

$$[(1\ 2)(3\ 4\ 5\ 6)], \quad [(3\ 5)(4\ 6)], \quad [(3\ 4\ 5\ 6)].$$

This is easily checked, by Corollary 5.5, taking into account that the only proper power of $[2, 4]$ is $[1^2, 2^2]$ and that, by Lemma 5.2, no type admits $[2, 4]$ as proper power. Moreover, $[1^2, 2^2]$ admits no proper power and is the proper power only of $[2, 4]$ and $[1^2, 4]$. It follows that $k_{C_2}(T_2) = 1$ and $\tilde{c}_0(S_6)_{T_2} = \frac{\mu_{[2, 4]}(S_6)}{\phi(4)} = 45$. Since

$$\mathfrak{T}(C_1) \cup \mathfrak{T}(C_2) = \{[1, 5], [2, 4], [1^2, 2^2], [1^2, 4]\},$$

we consider $T_3 = [1^4, 2]$ and $\psi = (1\ 2) \in S_6$. By Lemma 7.4, all the vertices of type T_3 are in $C_3 = \tilde{\Delta}_6$ and, using Corollary 6.7(iii), we see that C_3 contains also all the vertices of type $[1, 2, 3]$ and $[1^3, 3]$. But it is easily checked that no further type exists having as power one of the types $[1^4, 2], [1, 2, 3], [1^3, 3]$, so that $\mathfrak{T}(C_3) = \{[1^4, 2], [1, 2, 3], [1^3, 3]\}$. We claim that all the elements of type $T_4 = [2^3]$ are in a same component C_4 , so that $\tilde{c}_0(S_6)_{T_4} = 1$. Let $[\varphi_1]$ and $[\varphi_2]$ be distinct elements in $[S_6]_0$ of type $[2^3]$. Note that, since $[\varphi_1], [\varphi_2]$ are distinct they share at most one transposition. Let $\varphi_1 = (a\ b)(c\ d)(e\ f)$, with $\{a, b, c, d, e, f\} = \{1, 2, 3, 4, 5, 6\}$. Since the 2-cycles in which φ_1 splits commute and also the entries in each cycle commute, we can restrict our analysis to $\varphi_2 = (a\ b)(c\ e)(d\ f)$, if φ_1, φ_2 have one cycle in common, and to $\varphi_2 = (a\ c)(b\ e)(d\ f)$, if φ_1, φ_2 have no cycle in common. In the first case we have the following path of length 8 between $[\varphi_1]$ and $[\varphi_2]$:

$$[\varphi_1], [(a\ e\ d\ b\ f\ c)], [(a\ d\ f)(e\ b\ c)], [(e\ a\ b\ d\ c\ f)], [(e\ d)(a\ c)(b\ f)], \\ [(d\ a\ b\ e\ c\ f)], [(d\ b\ c)(a\ e\ f)], [(a\ d\ e\ b\ f\ c)], [\varphi_2].$$

In the second case we have the following path of length 4 between $[\varphi_1]$ and $[\varphi_2]$:

$$[\varphi_1], [(a\ f\ d\ b\ e\ c)], [(a\ d\ e)(f\ b\ c)], [(f\ a\ b\ d\ c\ e)], [\varphi_2].$$

Collecting the types met in those paths, we see that $\mathfrak{T}(C_4) \supseteq \{[2^3], [6], [3^2]\}$ and since all the other possible types in S_6 have been considered we get that $\mathfrak{T}(C_4) = \{[2^3], [6], [3^2]\}$. Thus the Procedure 7.1 stops giving $c_0(\mathcal{T}(S_6)) = 4$ and $\tilde{c}_0(S_6) = 83$. Moreover $c_0(\mathcal{O}(S_6)) = 2$ with the two components of $\mathcal{O}_0(S_6)$ having as vertex sets $\{5\}$ and $\{2, 3, 4, 6\}$.

Let $n = 7$. In $\tilde{\mathcal{P}}_0(S_7)$, by Lemma 7.3, the elements of type $T_1 = [7]$ are in $\tilde{c}_0(S_7)_{T_1} = 120$ components which are isolated vertices. Let C_1 be one of them. By Lemma 7.4 and Corollary 6.7(iii), all the vertices of type $T_2 = [1^5, 2]$ and those of types $[1^2, 2, 3], [1^4, 3]$ are in the same component $C_2 = \tilde{\Delta}_7$. In particular $\tilde{c}_0(S_7)_{T_2} = 1$. We show that also the types $[1^3, 4], [1^3, 2^2], [2^2, 3], [1, 2, 4], [2, 5]$ and $[1^2, 5]$ are admissible for C_2 . From the path

$$[(1\ 2\ 3\ 4)], [(1\ 3)(2\ 4)], [(1\ 3)(2\ 4)(5\ 6\ 7)], [(5\ 6\ 7)]$$

we deduce that $[1^3, 4], [1^3, 2^2]$ and $[2^2, 3]$ are admissible for C_2 , because $[(5\ 6\ 7)]$ is a vertex of C_2 . Then it is enough to consider the path $[(1\ 2)(3\ 4\ 5\ 6)], [(3\ 5)(4\ 6)]$ for getting the type $[1, 2, 4]$ and the path $[(1\ 2\ 3\ 4\ 5)], [(1\ 2\ 3\ 4\ 5)(6\ 7)], [(6\ 7)]$ for getting the types $[2, 5]$ and $[1^2, 5]$.

We turn now our attention to the type $T_3 = [1, 6]$ and to $\psi_0 = (1\ 2\ 3\ 4\ 5\ 6) \in S_7$. Let C_3 be the component of $\tilde{\mathcal{P}}_0(S_7)$, containing $[\psi_0]$. By Lemma 5.2 (ii), T_3 is not a power and its only power are the types $[1, 2^3]$, $[1, 3^2]$, which in turn admit no powers and are only the power of T_3 . It follows that $\mathfrak{T}(C_3) = \{[1, 2^3], [1, 3^2], [1, 6]\}$. In particular, if $[\psi] \in V_{C_3}$, then ψ admits a unique fixed point. Moreover, by what shown for $[S_6]$, all the vertices in $[S_7]$ of type T_3 fixing 7 are contained in C_3 . We show that, indeed, each $[\psi] \in V_{C_3}$ is such that $\psi(7) = 7$. By contradiction, assume $\psi(7) \neq 7$, for some $[\psi] \in V_{C_3}$. Then, since there is a path between $[\psi]$ and $[\psi_0]$, there exist $[\varphi], [\psi'] \in V_{C_3}$ with $\varphi(7) = 7$, $\psi'(j) = j$, for some $j \neq 7$ and an edge $\{[\varphi], [\psi']\} \in [E]_0^*$. But either φ is a power of ψ' and so φ admits j as a fixed point or ψ' is a power of φ and so ψ' admits 7 as a fixed point. In any case, we reach a contradiction. Thus we have $k_{C_3}(T_3) = \frac{\mu_{[6]}(S_6)}{\phi(6)}$ and so $\tilde{c}_0(S_7)_{T_3} = \frac{\mu_{[1,6]}(S_7)}{\mu_{[6]}(S_6)} = 7$. Since there are no other types left in S_7 , we conclude that $c_0(\mathcal{T}(S_7)) = 3$ and $\tilde{c}_0(S_7) = 128$. Moreover $c_0(\mathcal{O}(S_7)) = 2$ with the two components of $\mathcal{O}_0(S_7)$ having as vertex sets $\{7\}$ and $\{2, 3, 4, 5, 6, 10\}$. \square

We are ready to show that, for $n \geq 8$, the main role is played by the component $\tilde{\Delta}_n$ defined in Lemma 7.4.

PROPOSITION 7.7. *For $n \geq 8$, all the vertices of $\tilde{\mathcal{P}}_0(S_n)$ apart from those of prime order $p \geq n - 1$ are contained in $\tilde{\Delta}_n$.*

PROOF. Let $n \geq 8$. We start showing that each $[\psi] \in [S_n]_0$ having even order is a vertex of $\tilde{\Delta}_n$. Let $o(\psi) = 2k$, for k a positive integer. Since $o(\psi^k) = 2$, ψ^k is the product of $s \geq 1$ transpositions. If $s = 1$, then we have $\psi^k = (a\ b)$ for suitable $a, b \in N$ and, by Lemma 7.4, the path $[\psi], [(a\ b)]$ has its end vertex in $\tilde{\Delta}_n$. If $s = 2$, then $\psi^k = (a\ b)(c\ d)$, for suitable $a, b, c, d \in N$. Since $n \geq 8$, there exist distinct $e, f, g \in N \setminus \{a, b, c, d\}$ and we have the path

$$[\psi], \quad [\psi^k], \quad [(a\ b)(c\ d)(e\ f\ g)], \quad [(e\ f\ g)], \quad [(a\ b)(e\ f\ g)], \quad [(a\ b)]$$

with an end vertex belonging to $\tilde{\Delta}_n$. Finally if $s \geq 3$, then we have $\psi^k = (a\ b)(c\ d)(e\ f)\sigma$, for suitable $a, b, c, d, e, f \in N$ and $\sigma \in S_n$, with $\sigma^2 = id$. Let $\varphi = (a\ c\ e\ b\ d\ f)\sigma$, so that $\varphi^3 = \psi^k$. Since $n \geq 8$, then there exist distinct $g, h \in N \setminus \{a, b, c, d, e, f\}$ and we have the path

$$[\psi], \quad [\psi^k], \quad [\varphi], \quad [(a\ e\ d)(c\ b\ f)], \quad [(a\ e\ d)(c\ b\ f)(g\ h)], \quad [(g\ h)]$$

with an end vertex belonging to $\tilde{\Delta}_n$.

Next let $o(\psi) = p$, where p is an odd prime such that $p \leq n - 2$. If $|M_\psi| \leq n - 2$, pick $a, b \in \{1, 2, \dots, n\} \setminus M_\psi$ and consider the path $[\psi], [\psi(a b)], [(a b)]$. If $|M_\psi| \geq n - 1$, observe that, since $p \leq n - 2$, ψ is the product of $s \geq 2$ cycles of length p , say $\psi = (a_1 a_2 \dots a_p)(b_1 b_2 \dots b_p)\sigma$, where $\sigma = id$ or σ is the product of $s - 2$ cycles of length p . Let $\varphi = (a_1 b_1 a_2 b_2 \dots a_p b_p)\sigma^{\frac{(p+1)}{2}}$. Since $o(\varphi)$ is even, by what shown above, we get $[\varphi] \in V_{\tilde{\Delta}_n}$. Moreover we have $\varphi^2 = \psi$ and thus $[\psi] \in V_{\tilde{\Delta}_n}$.

Finally let $o(\psi) = upq$, where $p, q \geq 3$ are distinct prime numbers and u is an odd positive integer. Then $o(\psi^u) = pq$ and in the split of ψ^u into disjoint cycles, there exists either a cycle of length pq or two cycles of length p and q . In the first case $pq \leq n$ gives $p \leq n/q \leq n/3 < n - 2$. In the second case we have $p + q \leq n$, which gives $p \leq n - q < n - 2$. Thus $o(\psi^{uq}) = p < n - 2$ and, by the previous case, we obtain $[\psi^{uq}] \in V_{\tilde{\Delta}_n}$ so that also $[\psi] \in V_{\tilde{\Delta}_n}$. \square

PROOF OF THEOREM B (ii) AND THEOREM D. Let $n \geq 8$ be fixed and recall that $c_0(S_n) = \tilde{c}_0(S_n)$. First of all, observe that $P \cap O_0(S_n) = \{p \in P : p \leq n\}$. Therefore we can reformulate Proposition 7.7 by saying that all the vertices in $[S_n]_0$ of order not belonging to the set $B(n) = P \cap \{n, n - 1\}$ are in $\tilde{\Delta}_n$. In particular, $V_{\tilde{o}(\tilde{\Delta}_n)} \supseteq O_0(S_n) \setminus B(n)$. Let Σ_n be the unique component of $\mathcal{O}_0(S_n)$ such that $V_{\Sigma_n} \supseteq V_{\tilde{o}(\tilde{\Delta}_n)}$.

If $n \notin P \cup (P + 1)$, then $B(n) = \emptyset$ and thus all the vertices of $\tilde{\mathcal{P}}_0(S_n)$ are in $\tilde{\Delta}_n$. In this case $\tilde{\mathcal{P}}_0(S_n) = \tilde{\Delta}_n$ is connected and, by [2, Proposition 3.2], both its quotients $\mathcal{T}_0(S_n)$ and $\mathcal{O}_0(S_n)$ are connected. Note that the completeness of \tilde{t} and \tilde{o} implies $\tilde{t}(\tilde{\Delta}_n) = \mathcal{T}_0(S_n)$ and $\tilde{o}(\tilde{\Delta}_n) = \mathcal{O}_0(S_n) = \Sigma_n$.

Next let $n \in P \cup (P + 1)$, say $n = p$ or $n = p + 1$ for some $p \in P$, necessarily odd. Then $B(n) = \{p\}$. We need to understand only the components of $\tilde{\mathcal{P}}_0(S_n)$ and $\mathcal{T}_0(S_n)$ containing vertices of order p , and decide whether or not $p \in V_{\Sigma_n}$. By Lemma 7.3, p is isolated in $\mathcal{O}_0(S_p)$ and the unique type T of order p is isolated in $\mathcal{T}_0(S_n)$. Moreover each component of $\tilde{\mathcal{P}}_0(S_n)$ admissible for T is an isolated vertex and their number is known. The values for $c_0(S_n)$ as displayed in Table 2 and the fact that $c_0(\mathcal{T}(S_n)) = 2$ immediately follows. Now note that Σ_n cannot reduce to the isolated vertex p , because Σ_n contains at least the vertex $2 \in O_0(S_n) \setminus \{p\}$. Thus $c_0(\mathcal{O}_0(S_n)) = 2$ and the two components of $\mathcal{O}_0(S_n)$ have as vertex sets $\{p\}$ and $V_{\Sigma_n} = O_0(S_n) \setminus \{p\} = V_{\tilde{o}(\tilde{\Delta}_n)}$. Since we have shown that $\tilde{\Delta}_n$ is the only possible component of $\tilde{\mathcal{P}}_0(S_n)$ not reduced to an isolated vertex and \tilde{o} is complete, applying [2, Proposition 5.2] we get $\Sigma_n = \tilde{o}(\tilde{\Delta}_n)$.

So far, for every $n \geq 8$, we have shown that $\tilde{\Delta}_n$ is the only possible component of $\tilde{\mathcal{P}}_0(S_n)$ not reduced to an isolated vertex; $\tilde{t}(\tilde{\Delta}_n)$ is the only possible component

of $\mathcal{T}_0(S_n)$ not reduced to an isolated vertex; $\tilde{o}(\tilde{\Delta}_n)$ is the only possible component of $\mathcal{O}_0(S_n)$ not reduced to an isolated vertex. For the graphs $\mathcal{P}_0(S_n)$, $\tilde{\mathcal{P}}_0(S_n)$, $\mathcal{T}_0(S_n)$ and $\mathcal{O}_0(S_n)$, we use now the term main component for, respectively, the component Δ_n such that $\pi(\Delta_n) = \tilde{\Delta}_n$ defined in Corollary 7.6, $\tilde{\Delta}_n$, $\tilde{t}(\tilde{\Delta}_n)$ and $\tilde{o}(\tilde{\Delta}_n)$. Then, by Corollary 7.6, no main component is complete. We show finally that every component C of $\mathcal{P}_0(S_n)$, with $C \neq \Delta_n$, is a complete graph on $p - 1$ vertices. Let $\psi \in V_C$. Then $[\psi] \notin V_{\tilde{\Delta}_n}$ and thus $[\psi]$ is isolated in $\tilde{\mathcal{P}}_0(S_n)$. Hence, to conclude, we invoke Lemma 7.2 (iii). \square

COROLLARY 7.8. *The following are equivalent:*

- (i) every component C of $\tilde{\mathcal{P}}_0(S_n)$ is isomorphic to the component of $\mathcal{T}_0(S_n)$ induced on $\mathfrak{T}(C)$;
- (ii) $2 \leq n \leq 5$.

PROOF. (ii) \implies (i) For $2 \leq n \leq 5$, the case-by-case proof of Theorem B (i) shows directly the required isomorphism.

(i) \implies (ii) For $n \geq 6$, we show that the component $\tilde{\Delta}_n$ is not isomorphic to the component of $\mathcal{T}_0(S_n)$ induced on $\mathfrak{T}(\tilde{\Delta}_n)$. Namely $\tilde{\Delta}_n$ contains all the vertices of type $T = [1^{n-2}, 2]$ and since $k_{\tilde{\mathcal{P}}_0(S_n)}(T) = \frac{n(n-1)}{2} > 1$, Corollary 6.7 (iv) applies. \square

PROOF OF COROLLARY C. A check on Tables 1 and 2 of Theorem B. \square

COROLLARY 7.9. *Apart from the trivial case $n = 2$, the minimum $n \in \mathbb{N}$ such that $\mathcal{P}(S_n)$ is 2-connected is $n = 9$. There exists infinitely many $n \in \mathbb{N}$ such that $\mathcal{P}(S_n)$ is 2-connected.*

PROOF. Let $n = k^2$, for some $k \geq 3$. Then $n \geq 8$, $n \notin P$ and $n - 1 = k^2 - 1 = (k - 1)(k + 1)$ is not a prime. Thus, by Theorem B, $\mathcal{P}(S_n)$ is 2-connected. In particular $c_0(S_9) = 1$. Moreover by Tables 1 and 2, we have $c_0(S_n) > 1$ for all $3 \leq n \leq 8$. \square

Acknowledgements. The authors wish to thank Gena Hahn and Silvio Dolfi for some suggestions on a preliminary version of the paper.

Daniela Bubboloni is partially supported by GNSAGA of INdAM. Mohammad A. Iranmanesh and Seyed M. Shaker are financially supported by Yadz University Research Council.

REFERENCES

- [1] J. ABAWAJY – A. KELAREV – M. CHOWDHURY, *Power graphs: a survey*, Electron. J. Graph Theory Appl. (EJGTA) **1** (2013), no. 2, pp. 125–147.
- [2] D. BUBBOLONI, *Graph homomorphisms and components of quotient graphs*, Rend. Sem. Mat. Univ. Padova **138** (2017), pp. 39–60.
- [3] D. BUBBOLONI – M. A. IRANMANESH – S. M. SHAKER, *On some graphs associated with the finite alternating groups*, Comm. Algebra **45** (2017), no. 12, pp. 5355–5373.
- [4] P. J. CAMERON, *The power graph of a finite group II*, J. Group Theory **13** (2010), pp. 779–783.
- [5] P. J. CAMERON – S. GHOSH, *The power graph of a finite group*, Discrete Mathematics **311** (2011), pp. 1220–1222.
- [6] I. CHAKRABARTY – S. GHOSH – M. K. SEN, *Undirected power graphs of semigroups*, Semigroup Form **78** (2009), pp. 410–426.
- [7] B. CURTIN – G. R. POURGHOLI – H. YOUSEFI-AZARI, *On the punctured power graphs of finite groups*, Australasian J. Combinatorics **62** (2015), pp. 1–7.
- [8] R. DIESTEL, *Graph theory*, 4th ed., Graduate Texts in Mathematics, 173, Springer-Verlag, Heidelberg, 2010.
- [9] A. DOOSTABADI – M. FARROKHI D. G., *On the connectivity of proper power graphs of finite groups*, Comm. Algebra **43** (2015), pp. 4305–4319.
- [10] A. V. KELAREV – S. J. QUINN, *A combinatorial property and power graphs of groups*, Contrib. General Algebra **12** (2000), pp. 229–235.
- [11] U. KNAUER, *Algebraic graph theory*, Morphisms, monoids and matrices, De Gruyter Studies in Mathematics, 41, Walter de Gruyter & Co., Berlin, 2011.
- [12] U. KNAUER, *Endomorphism spectra of graphs*, Discrete Mathematics **109** (1992), pp. 45–57.
- [13] M. MIRZARGAR – A. R. ASHRAFI – M. J. NADJAFI-ARANI, *On the power graph of a finite group*, Filomat **26** (2012), no. 6, pp. 1201–1208.
- [14] A. R. MOGHADDAMFAR – S. RAHBARIYAN – W. J. SHI, *Certain properties of the power graph associated with a finite group*, J. Algebra Appl. **13** (2014), no. 7, 1450040, 18 pp.
- [15] G. R. POURGHOLI – H. YOUSEFI-AZARI – A. R. ASHRAFI, *The undirected power graph of a finite group*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 4, pp. 1517–1525.
- [16] S. M. SHAKER – M. A. IRANMANESH, *On groups with specified quotient power graphs*, Int. J. Group Theory **5** (2016), no. 3, pp. 49–60.