

Simplicial arrangements on convex cones

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ABSTRACT – We introduce the notion of a Tits arrangement on a convex open cone as a special case of (infinite) simplicial arrangements. Such an object carries a simplicial structure similar to the geometric representation of Coxeter groups. The standard constructions of subarrangements and restrictions, which are known in the case of finite hyperplane arrangements, work as well in this more general setting.

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CONTENTS

1. Introductory examples	151
2. Hyperplane arrangements, subarrangements and restrictions	154
3. Tits cones and Tits arrangements	164
4. Parabolic subarrangements and restrictions of Tits arrangements	175

APPENDICES

A. The simplicial complex \mathcal{S}	181
B. \mathcal{S} as a gated, numbered chamber complex	184
REFERENCES	190

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Introduction

Let (W, S) be a Coxeter system. There is a canonical real representation of W which is called the *geometric representation* of (W, S) and which is known to be faithful by the work of Tits in the 1960s (a proof can be found in [25]). In the proof he considers a convex cone T in the dual module which is stabilized under the action of W . By means of the standard generating set S he defines an open simplicial cone C which turns out to be a *prefundamental domain* for the action of W on T .

It is now the standard terminology to call the cone T the *Tits cone* and C the (*open*) *fundamental chamber*. The conjugates of the fundamental generators act as linear reflections on the module and its dual. Their reflection hyperplanes yield a simplicial decomposition of T and thus provide a simplicial complex which is the *Coxeter complex* of (W, S) . The geometric representation of (W, S) , the geometry of the Tits cone, and the properties of the Coxeter complex are fundamental tools for the investigation of Coxeter groups.

Coxeter systems play an important role in various branches of mathematics. In combinatorics and geometric group theory they are a rich source of interesting phenomena ([3] and [12]). In representation theory and the theory of algebraic groups the *crystallographic* Coxeter systems occur as *Weyl groups* of several structures. There, they play the role of basic invariants, often called the *type* of the algebraic object under consideration.

In [18] and [8] *Weyl groupoids* and their root systems have been introduced. They emerged as basic invariants of Nichols algebras in earlier work ([17] and [1]) and are natural generalizations of Weyl groups. Weyl groupoids of finite type are those which generalize finite Weyl groups. These have been studied intensively in [7], [11], [10], and [9], where a classification has been obtained. This classification is considerably harder than the classification of the finite Weyl groups. It was observed by the first two authors in 2009 that the final outcome of this classification in rank 3 is intimately related to Grünbaum's list of simplicial arrangements in [16]. There is an obvious explanation of this connection: to each Weyl groupoid one can associate a Tits cone which is the whole space if and only if the Weyl groupoid is of finite type. Based on these considerations, *crystallographic* arrangements have been introduced in [6], where it is shown that their classification is a consequence of the classification of finite Weyl groupoids.

In view of the importance of the Tits cone and the Coxeter complex for the understanding of Coxeter systems it is natural to investigate their analogues in the context of Weyl groupoids. Although it is intuitively clear what has to be done in order to generalize these notions, there are instances where things have

to be modified or some extra argument is needed. Our intention is to provide the basic theory of the Tits cone and the Coxeter complex of a Weyl groupoid. This paper deals with the combinatorial aspects of this project and therefore the crystallographic condition doesn't play a role at all. Hence, here we deal with *Tits arrangements* rather than with Weyl groupoids.

The basic strategy for setting up the framework is to start with a simplicial arrangement \mathcal{A} on a convex open cone T and to investigate the abstract simplicial complex $\mathcal{S}(\mathcal{A}, T)$ associated with it. This is a *gated chamber complex* and has therefore a natural type function. We call a simplicial arrangement on a convex open cone a Tits arrangement if $\mathcal{S}(\mathcal{A}, T)$ is a thin chamber complex because these arrangements are very close to the reflection arrangements of a Coxeter group on a Tits cone. The latter have been used by Tits in his fundamental paper [25] to define the Coxeter complex associated with a Coxeter system and all his ideas and results on the combinatorial side remain valid for thin simplicial arrangements on open convex cones. We elaborate this in detail in Section 3, where we also introduce root-systems associated with Tits-arrangements. There are two standard procedures to obtain from a given simplicial arrangement new simplicial arrangements of lower rank. Those will be considered in Section 4.

We would like to point out that this paper is meant to be a contribution to the foundations of the theory of Weyl groupoids of arbitrary type. The concepts and ideas are at least folklore and several of the results for which we give proofs are well established in the literature. The only exception is probably our systematic use of gated chamber complexes at some places. Our principal goal here is to provide a fairly complete account of the basic theory of the Tits cone of a Weyl groupoid by developing the corresponding notions to the extent that is needed for just this purpose. Therefore, we have decided to include short proofs for standard facts and refer to other sources only when we need a more elaborate result.

We are not able to give a systematic account of the origins of the concepts and ideas which play a role in this paper. Here are some comments based on the best of our knowledge.

1. Simplicial arrangements were first introduced and studied by Melchior [20] and subsequently by Grünbaum [16]. Shortly afterwards, simplicial arrangements attracted attention in the seminal work of Deligne [13]: they are a natural context to study the $K(\pi, 1)$ property of complements of reflection arrangements, since the set of reflection hyperplanes of a finite Coxeter group is a simplicial arrangement. They further appeared as examples or counterexamples to conjectures on arrangements.

2. We do not know where arrangements of hyperplanes on convex cones were considered for the first time. The concept seems most natural and they are mentioned in [23] without further reference. Of course, our definition of a simplicial arrangement on an open convex cone is inspired by the Tits cone of a Coxeter system.
3. The fact that arrangements of hyperplanes provide interesting examples of gated sets in metric spaces appears in [4] for the first time. At least in the simplicial case it was observed much earlier [24].
4. The observation that there is a natural link between root systems and simplicial arrangements is quite natural. We already mentioned that it was our starting point to investigate the Tits cone of a Weyl groupoid. But it also appears in Dyer's work on rootoids [14] and [15]. It is conceivable that the observation was made much earlier by other people and is hidden somewhere in the literature.

The paper is organized as follows.

In Section 1 we will provide some basic examples for our objects of interest. These will be used for later reference, since they are either basic examples or counterexamples for the properties which we introduce. Like in Example 1.2, the geometric representation of a Coxeter group is a prominent sample for simplicial arrangements.

In Section 2 we fix notation and develop the notion of a hyperplane arrangement on an open convex cone in a real vector space V . We introduce the common substructures for these objects, i. e. subarrangements and restrictions. Furthermore we define the chamber graph of an arrangement and show that parabolic subsets of the set of chambers are gated subsets with respect to the canonical metric.

In Section 3 we introduce simplicial arrangements and Tits arrangements. We add additional combinatorial structure to a Tits arrangement by associating to the set of hyperplanes a set of roots, linear forms which define the hyperplanes. With respect to the study of Nichols algebras root systems with additional properties will be interesting, however in this paper we deal with root systems as very general objects. We also associate to a simplicial arrangement a canonical simplicial complex. The main results regarding this complex are proven in the appendix.

In Section 4 we consider subarrangements and restrictions of simplicial arrangements and Tits arrangements. We give criteria when the substructures of a simplicial/Tits arrangement is again a simplicial/Tits arrangement, and in the case of a Tits arrangement we describe canonical root systems.

The appendices provides proofs for the properties of the poset associated to a simplicial hyperplane arrangement which are stated in Section 4. While most of these properties are quite intuitive, a rigid proof can be tedious.

In Appendix A we show that the poset $\mathcal{S}(\mathcal{A}, T)$ associated to a simplicial hyperplane arrangement (\mathcal{A}, T) is indeed a simplicial complex. The results of this section are summarized in Proposition 3.18. In particular, we provide an equivalent definition for simplicial cones, which has an implicit simplicial structure.

Appendix B provides the remaining properties of \mathcal{S} . The first part recalls the definitions of chamber complexes and type functions.

In the second part it is shown that \mathcal{S} is a gated chamber complex with a type function, a collection of the results can be found in Proposition 3.26. We also show that the notions of being spherical and thin, which we introduced for simplicial hyperplane arrangements before, coincide with the classical notions for chamber complexes.

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1. Introductory examples

EXAMPLE 1.1. Consider the following setting. Let

$$\begin{aligned} V &= \mathbb{R}^2, \\ T &= \{(x, y) \mid x, y \in \mathbb{R}_{>0}\}, \\ L &= \{(x, x) \mid x \in \mathbb{R}\}. \end{aligned}$$

Then $T \setminus L$ consists of two connected components

$$K_1 := \{(x, y) \in T \mid x < y\}, K_2 := \{(x, y) \in T \mid x > y\}.$$

Let α_1, α_2 be the dual basis in V^* to $(1, 0), (0, 1)$, then we can write

$$\begin{aligned} K_1 &= \alpha_1^{-1}(\mathbb{R}_{>0}) \cap (-\alpha_1 + \alpha_2)^{-1}(\mathbb{R}_{>0}), \\ K_2 &= \alpha_2^{-1}(\mathbb{R}_{>0}) \cap (\alpha_1 - \alpha_2)^{-1}(\mathbb{R}_{>0}). \end{aligned}$$

We will later define objects which can be written in this way, i. e. as intersections of half spaces, as simplicial cones.

Both K_1 and K_2 are cones with exactly two bounding hyperplanes, for K_1 these are $L = \ker(\alpha_1 - \alpha_2)$ and $\{(0, y) \mid y \in \mathbb{R}\} = \ker(\alpha_1)$, for K_2 these are L and $\{(x, 0) \mid x \in \mathbb{R}\} = \ker(\alpha_2)$.

In this example only one of the bounding hyperplanes, namely L , of K_1, K_2 meets T , while the other meets \bar{T} , but not T itself.

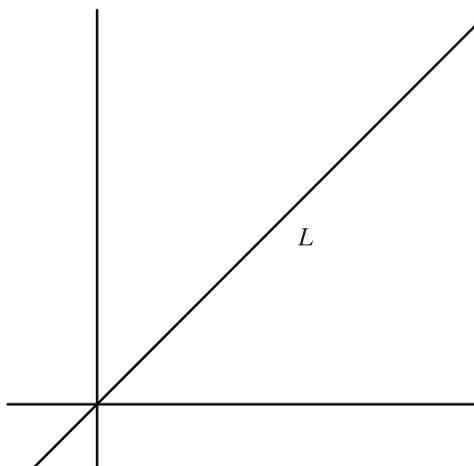


Figure 1. The line setting from Example 1.1.

EXAMPLE 1.2. Let V and T be as before. Define for $n \in \mathbb{N}_{>0}$ the lines

$$L_n = \{(x, y) \mid y - nx = 0\},$$

$$L_n' = \left\{ (x, y) \mid y - \frac{1}{n}x = 0 \right\}.$$

Then the connected components of $T \setminus \bigcup_{n \in \mathbb{N}_{>0}} (L_n \cup L_n')$ are again simplicial cones for suitable linear forms. The number of connected components is not finite in this case, and every component has bounding hyperplanes which meet T (see Figure 2).

EXAMPLE 1.3. Let $V = \mathbb{R}^3$ and

$$T := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 < 0\}.$$

Then T is a convex open cone. Consider the universal Coxeter group W in 3 generators, i. e.

$$W = \langle s, t, u \mid s^2 = t^2 = u^2 = 1 \rangle.$$

The geometric representation (see [19, Chapter 5.3]) of W yields a set of reflection hyperplanes \mathcal{A} , which meet T after choosing a suitable basis.

We obtain the picture in Figure 3 by intersecting T with a hyperplane parallel to $\langle e_1, e_2 \rangle$, which corresponds to the Beltrami–Klein model of hyperbolic 2-space. The connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are again cones which carry a simplicial structure. However, the vertices of the simplices are not contained in T , but in its boundary ∂T .

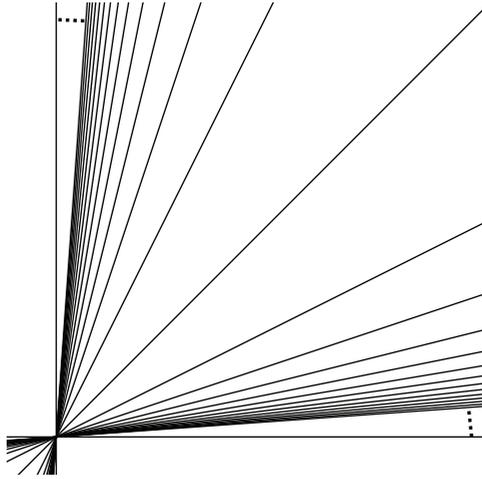


Figure 2. An infinite collection of hyperplanes, tessellating $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$.

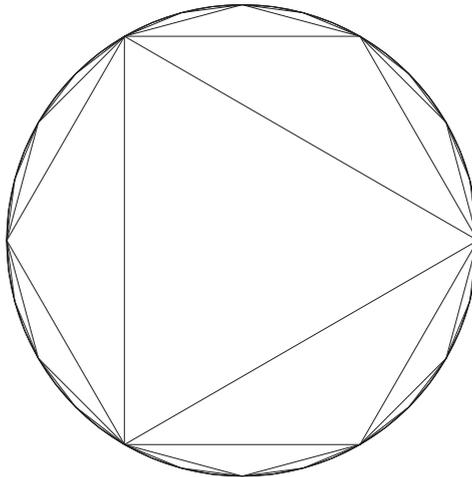


Figure 3. The hyperbolic hyperplane arrangement associated to W .

2. Hyperplane arrangements, subarrangements and restrictions

2.1 – Hyperplane arrangements

Throughout this paper, all topological properties are with respect to the standard topology of \mathbb{R}^r , unless stated otherwise. We also use the standard notation, in particular \bar{X} denotes the closure of $X \subset \mathbb{R}^r$.

DEFINITION 2.1. Let $V = \mathbb{R}^r$. A subset $K \subset V$ is called a *cone*, if $\lambda v \in K$ for all $v \in K$, $0 < \lambda \in \mathbb{R}$. For a subset $X \subset V$ we call

$$\mathbb{R}_{>0}X := \{\lambda x \mid x \in X, \lambda \in \mathbb{R}_{>0}\}$$

the *cone over X*.

Let \mathcal{A} be a set of linear hyperplanes in $V = \mathbb{R}^r$, and T an open convex cone. We say that \mathcal{A} is *locally finite in T*, if for every $x \in T$ there exists a neighborhood $U_x \subset T$ of x , such that $\{H \in \mathcal{A} \mid H \cap U_x \neq \emptyset\}$ is a finite set.

A *hyperplane arrangement (of rank r)* is a pair (\mathcal{A}, T) , where T is a convex open cone in V , and \mathcal{A} is a (possibly infinite) set of linear hyperplanes such that

- (1) $H \cap T \neq \emptyset$ for all $H \in \mathcal{A}$,
- (2) \mathcal{A} is locally finite in T .

If T is unambiguous from the context, we also call the set \mathcal{A} a hyperplane arrangement.

The connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are called *chambers*, denoted with $\mathcal{K}(\mathcal{A}, T)$ or just \mathcal{K} , if (\mathcal{A}, T) is unambiguous.

Let $K \in \mathcal{K}(\mathcal{A}, T)$. Define the *walls of K* as

$$W^K := \{H \leq V \mid H \text{ hyperplane, } \langle H \cap \bar{K} \rangle = H, H \cap K^\circ = \emptyset\}.$$

A hyperplane arrangement is *thin*, if $W^K \subset \mathcal{A}$ for all $K \in \mathcal{K}$.

If $\mathcal{A}' \subset \mathcal{A}$, we call (\mathcal{A}', T) a *subarrangement of (\mathcal{A}, T)* . Subarrangements satisfy the conditions (1) and (2) above and are therefore again hyperplane arrangements.

Let $X \subset \bar{T}$. Then the *localisation at X (in \mathcal{A})* is defined as the subset

$$\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subset H\}.$$

If $X = \{x\}$ is a singleton, we write \mathcal{A}_x instead of $\mathcal{A}_{\{x\}}$. We call the pairs (\mathcal{A}_x, T) the *parabolic subarrangement at x*. A *parabolic subarrangement of (\mathcal{A}, T)* is a subarrangement (\mathcal{A}', T) , such that $\mathcal{A}' = \mathcal{A}_x$ for some $x \in \bar{T}$. Furthermore set $\mathcal{K}_x := \{K \in \mathcal{K} \mid x \in \bar{K}\}$.

The *support* of X (in \mathcal{A}) is defined to be the subspace $\text{supp}_{\mathcal{A}}(X) = \bigcap_{H \in \mathcal{A}_X} H$. Define the *radical* of \mathcal{A} as $\text{Rad}(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H = \text{supp}_{\mathcal{A}}(0)$. We call the arrangement *essential*, if $\text{Rad}(\mathcal{A}) = 0$, and *degenerate* otherwise.

In this paper we call the set

$$\text{sec}_{\mathcal{A}}(X) := \bigcup_{x \in X} \mathcal{A}_x = \{H \in \mathcal{A} \mid H \cap X \neq \emptyset\}$$

the *section* of X (in \mathcal{A}). We will omit \mathcal{A} when there is no danger of confusion.

REMARK 2.2. (1) By construction, the chambers \mathcal{K} are open sets.

(2) In our notation, if (\mathcal{A}, T) is a hyperplane arrangement, \mathcal{A} being locally finite in T is equivalent to: for every point $x \in T$ there exists a neighborhood $U_x \subset T$ of x such that $\text{sec}_{\mathcal{A}}(U_x)$ is finite.

LEMMA 2.3. *Let (\mathcal{A}, T) be a hyperplane arrangement. Then for every point $x \in T$ there exists a neighborhood U_x such that $\mathcal{A}_x = \text{sec}(U_x)$. Furthermore the set $\text{sec}(X)$ is finite for every compact set $X \subset T$.*

PROOF. Let $x \in T$, in particular there exists an open neighborhood U of x such that $\text{sec}(U)$ is finite. By taking the smallest open ε -ball contained in U and centred at x , we can assume $U = U_{\varepsilon}(x)$. Let $H \in \text{sec}(U)$, with $x \notin H$. Let $\delta = d(H, x) > 0$, then $\delta < \varepsilon$, and $U' := U_{\frac{\delta}{2}}(x)$ is an open subset such that $\text{sec}(U') \subseteq \text{sec}(U) \setminus \{H\}$. Since $\text{sec}(U)$ is finite, $\text{sec}(U) \setminus \text{sec}(x)$ is finite. We can therefore repeat this process finitely many times until we find an open ball B such that $\text{sec}(B) = \mathcal{A}_x$.

The second assertion is a consequence of the first. Let X be compact, and for $x \in X$ let U_x denote an open subset such that $\text{sec}(U_x) = \mathcal{A}_x$. Then

$$\text{sec}(X) \subseteq \bigcup_{x \in X} \text{sec}(U_x)$$

as $X \subseteq \bigcup_{x \in X} U_x$. But the U_x are open and X is compact, therefore there exists a finite set $\{x_1, \dots, x_n\} \subset X$, $n \in \mathbb{N}$, such that

$$X \subseteq \bigcup_{i=1}^n U_{x_i}.$$

Consequently, if $H \cap X \neq \emptyset$, then there exists an index i such that $H \cap U_{x_i} \neq \emptyset$ and $H \in \text{sec}(U_{x_i})$. Hence

$$\text{sec}(X) \subseteq \bigcup_{i=1}^n \text{sec}(U_{x_i})$$

and $\text{sec}(X)$ is finite. □

2.2 – Subarrangements

We want to say more about hyperplane arrangements arising as (\mathcal{A}_x, T) for points $x \in \bar{T}$.

DEFINITION 2.4. Let (V, d) be a connected metric space. Define the *segment between* $x, y \in V$ to be

$$\sigma(x, y) := \{z \in V \mid d(x, z) + d(z, y) = d(x, y)\}.$$

When referring to the metric of \mathbb{R}^r , we will use the more common notion of the *interval* between two points x and y :

$$[x, y] := \sigma(x, y) = \{\lambda x + (1 - \lambda)y \in \mathbb{R}^r \mid \lambda \in [0, 1]\},$$

$$(x, y) := \sigma(x, y) \setminus \{x, y\} = \{\lambda x + (1 - \lambda)y \in \mathbb{R}^r \mid \lambda \in (0, 1)\}.$$

The intervals $(x, y]$, $[x, y)$ can be defined analogously.

DEFINITION 2.5. Let $\alpha \in V^*$ be a linear form on V , then

$$\alpha^\perp := \ker \alpha,$$

$$\alpha^+ := \alpha^{-1}(\mathbb{R}_{>0}),$$

$$\alpha^- := \alpha^{-1}(\mathbb{R}_{<0}).$$

With notation as above we find

$$\overline{\alpha^+} = \alpha^{-1}(\mathbb{R}_{\geq 0}) = \alpha^\perp \cup \alpha^+,$$

$$\overline{\alpha^-} = \alpha^{-1}(\mathbb{R}_{\leq 0}) = \alpha^\perp \cup \alpha^-.$$

DEFINITION AND REMARK 2.6. Every hyperplane H separates V into half-spaces. One way to describe the half-spaces uses linear forms.

Choose an arbitrary linear form $\alpha \in V^*$ such that $\alpha^\perp = H$. Then α^+ and α^- ($\overline{\alpha^+}$ and $\overline{\alpha^-}$) are the two *open (closed) half-spaces bounded by* H .

For an arbitrary subset $X \subset T$, if X is contained in one open half space bounded by H , we denote this particular half-space by $D_H(X)$. In this case we write $-D_H(X)$ for the unique half-space not containing X . By definition every chamber K is contained in a unique half space of $H \in \mathcal{A}$, therefore the sets $D_H(K)$ exist for all $H \in \mathcal{A}$, $K \in \mathcal{K}$. Let K, L be chambers, we say that $H \in \mathcal{A}$ *separates* K and L , if $D_H(K) = -D_H(L)$.

We denote by $S(K, L) := \{H \in \mathcal{A} \mid D_H(K) \neq D_H(L)\}$ the set of hyperplanes separating K and L .

LEMMA 2.7. *If $K, L \in \mathcal{K}_x$, $H \in S(K, L)$, then $H \in \mathcal{A}_x$.*

PROOF. Assume $H \notin \mathcal{A}_x$, then the half-space $D_H(\{x\})$ is well defined and unique. As a consequence we have $D_H(K) = D_H(\{x\}) = D_H(L)$, and H does not separate K and L . \square

LEMMA 2.8. *The chambers of the hyperplane arrangement (\mathcal{A}_x, T) correspond to \mathcal{K}_x .*

PROOF. Let \mathcal{K}' denote the chambers of (\mathcal{A}_x, T) . Assume $K_1, K_2 \in \mathcal{K}_x$ are both contained in $K \in \mathcal{K}'$. Then there exists a hyperplane $H \in \mathcal{A}$ such that K_1, K_2 are contained in different half-spaces with respect to H , in contradiction to Lemma 2.7. Since $\mathcal{A}_x \subset \mathcal{A}$, every chamber in \mathcal{K}_x is therefore contained in a unique chamber in \mathcal{K}' .

Likewise, every chamber in \mathcal{K}' contains a chamber in \mathcal{K}_x , which completes the proof. \square

2.3 – Reductions

We show that it is always possible to mod out the radical of an arrangement to obtain an essential one.

DEFINITION 2.9. Assume that (\mathcal{A}, T) is a hyperplane arrangement, set $W = \text{Rad}(\mathcal{A})$ and

$$\begin{aligned} V^{\text{red}} &:= V/W, \\ \pi : V &\longrightarrow V^{\text{red}}, \quad v \longmapsto v + W, \\ T^{\text{red}} &:= \pi(T), \\ \mathcal{A}^{\text{red}} &:= \pi(\mathcal{A}). \end{aligned}$$

LEMMA 2.10. *The set T^{red} is an open convex cone in V^{red} .*

PROOF. The set $\pi(T)$ is open as the image of an open set and π is an open map. Furthermore, if $[y, z]$ is an interval in $\pi(T)$, then there exist $y', z' \in T$ with $y = y' + W$, $z = z' + W$. The interval $[y', z']$ is contained in T as T is convex, and we find $\pi([y', z']) = [y, z]$.

Finally, T^{red} is a cone, since for $y \in T^{\text{red}}$ we find $y' \in T$ with $y = y' + W_x$. Since T is a cone, for every $\lambda > 0$ we have $\lambda y' \in T$, so $\lambda y \in T^{\text{red}}$. \square

We gather basic properties of \mathcal{A}^{red} .

LEMMA 2.11. *The pair $(\mathcal{A}^{\text{red}}, T^{\text{red}})$ is an essential hyperplane arrangement with chambers $\{\pi(K) \mid K \in \mathcal{K}\}$. The chambers are furthermore in one to one correspondence to \mathcal{K} .*

PROOF. The set \mathcal{A}^{red} is a set of hyperplanes in V^{red} , since $W \subset H$ for all $H \in \mathcal{A}$.

Let $H' \in \mathcal{A}^{\text{red}}$ with $H' = \pi(H)$ for $H \in \mathcal{A}$, then $H' \cap T^{\text{red}} \neq \emptyset$, since $H \cap T \neq \emptyset$.

We show that \mathcal{A}^{red} is locally finite in T^{red} . Let $y' \in \pi(T)$, then there exists a $y \in T$ such that $y' = y + W$. Since \mathcal{A} is locally finite in T , there exists a neighborhood $U \subset T$ containing y such that $\{H \in \mathcal{A} \mid H \cap U \neq \emptyset\}$ is finite. As π maps open sets to open sets, $\pi(U)$ is a neighborhood of y' in T^{red} . Let $H' \in \mathcal{A}^{\text{red}}$ and $H \in \mathcal{A}$ with $\pi(H) = H'$ and assume $z \in H' \cap \pi(U)$. Then there exist $z' \in H$ with $\pi(z') = z$ and $z'' \in U$ with $\pi(z'') = z$, thus $z' + w = z''$ for some $w \in W$. Hence $z'' \in H$ as well, and we conclude that $H \cap U \neq \emptyset$.

We have thus established for $H \in \mathcal{A}$

$$H \cap U \neq \emptyset \iff \pi(H) \cap \pi(U) \neq \emptyset.$$

Therefore $\{H \in \mathcal{A}^{\text{red}} \mid H \cap \pi(U)\}$ is finite.

For all $K \in \mathcal{K}$ the set $\pi(K)$ is a connected component of $T^{\text{red}} \setminus \bigcup_{H \in \mathcal{A}^{\text{red}}} H$, in particular, as $K \subset T$, $\pi(K) \subset T^{\text{red}}$, furthermore for a connected component K' of $T^{\text{red}} \setminus \bigcup_{H \in \mathcal{A}^{\text{red}}} H$, $\pi^{-1}(K')$ is a connected component of $T \setminus \bigcup_{H \in \mathcal{A}} H$. This completes the proof. \square

DEFINITION 2.12. Let (\mathcal{A}, T) be a hyperplane arrangement, $x \in \bar{T}$. Then set

$$\begin{aligned} W_x &:= \text{supp}_{\mathcal{A}}(\{x\}), \\ V_x &:= V/W_x, \\ \pi &:= \pi_x: V \longrightarrow V_x, \quad v \longmapsto v + W_x, \\ T_x &:= \pi(T), \\ \mathcal{A}_x^\pi &:= \pi(\mathcal{A}_x). \end{aligned}$$

Note that the set \mathcal{A}_x is finite if $x \in T$.

COROLLARY 2.13. *The pair (\mathcal{A}_x^π, T_x) is an essential hyperplane arrangement with chambers $\{\pi(K) \mid K \in \mathcal{K}_x\}$.*

PROOF. This follows from Lemma 2.8 and Lemma 2.11, since $W_x = \text{Rad}(\mathcal{A}_x)$. \square

EXAMPLE 2.14. The hyperplane arrangement in Example 1.1 is degenerate, as $\text{Rad}(\mathcal{A}) = L$. Reducing this arrangement yields the essential and thin hyperplane arrangement $(\{0\}, \mathbb{R})$.

2.4 – The chamber graph and gated parabolics

We now consider the structure given by the chambers and their adjacency.

DEFINITION 2.15. Let (\mathcal{A}, T) be a hyperplane arrangement. We call two chambers K, L *adjacent*, if $\langle \bar{K} \cap \bar{L} \rangle$ is a hyperplane. If K, L are adjacent and $\langle \bar{K} \cap \bar{L} \rangle = H$, then we also say that K, L are *H-adjacent*.

Define the *chamber graph* $\Gamma = \Gamma(\mathcal{A}, T)$ to be the simplicial graph with vertex set \mathcal{K} , $\{K, L\}$ is an edge if and only if K and L are adjacent.

We call a path in Γ connecting $K, L \in \mathcal{K}$ a *gallery from K to L*. We say a gallery from K to L is *minimal*, if it is of length $d_\Gamma(K, L)$, where d_Γ is the distance in Γ .

REMARK 2.16. Although we consider arbitrary hyperplanes in the previous definition, we will obtain in Lemma 2.18 that if K and L are H -adjacent, H is already contained in \mathcal{A} .

LEMMA 2.17. *Let $K, L \in \mathcal{K}$, $v \in K$, $w \in L$. Then $S(K, L) = \text{sec}([v, w])$, and $|S(K, L)|$ is finite.*

PROOF. As T is convex, the line $[v, w]$ is contained in T , and is compact. Therefore, as \mathcal{A} is locally finite, by Lemma 2.3 $\text{sec}([v, w])$ is finite.

Let $H \in \mathcal{A}$ and assume $H \cap [v, w]$ is empty, then $D_H([v, w])$ is well-defined and $D_H(K) = D_H([v, w]) = D_H(L)$, so H does not separate K and L . We can conclude $S(K, L) \subseteq \text{sec}([v, w])$. On the other hand, let $H \in \text{sec}([v, w])$. Since $v \in K$, $w \in L$, the interval $[v, w]$ is not contained in H and neither v nor w is in H . Assume $D_H(\{v\}) = D_H(\{w\})$, since $D_H(\{v\})$ is convex, we obtain $[v, w] \subset D_H(\{v\})$, contradicting our assumption. Therefore $D_H(K) = D_H(\{v\}) = -D_H(\{w\}) = -D_H(L)$. This yields $\text{sec}([v, w]) \subseteq S(K, L)$. \square

LEMMA 2.18. *Two chambers $K, L \in \mathcal{K}$ are H -adjacent if and only if $S(K, L) = \{H\}$. In this situation L is the unique chamber H -adjacent to K .*

PROOF. Assume first that K, L are H -adjacent. So $H = \langle \bar{K} \cap \bar{L} \rangle$. Note that $H \in W^K \cap W^L$ by definition. Since $H \in W^K$, the set $F := \bar{K} \cap \bar{L}$ is a convex cone spanning H . Take a point x in the interior of $F \subset H$. For a suitable $\varepsilon > 0$ we find $U_\varepsilon(x) \cap H \subset F$ and $\text{sec}(U_\varepsilon(x)) = \{H\}$. The hyperplane H separates $U_\varepsilon(x)$ into two connected components, say U_+ and U_- . Since K is open, $F \subset \bar{K}$, without loss of generality K meets U_+ . Since $\text{sec}(U_\varepsilon(x)) = \{H\}$, indeed we find $U_+ \subset K$. Assuming that L meets U_+ implies that $L = K$, but then $K \cap L$ generates V , contradicting the assumption of K and L being adjacent. Thus $U_- \subset L$, and H separates K and L .

Assume $H' \in \mathcal{A}$ such that H' separates K and L . Then $\bar{K} \subset \overline{(D_{H'}(K))}$ and $\bar{L} \subset \overline{(D_{H'}(L))}$, in particular $\bar{K} \cap \bar{L} \subset H'$. This implies $H' = H$, therefore $S(K, L) = \{H\}$.

For the other implication assume that H is the unique hyperplane separating K and L . In this case, note that K and L are distinct. If H is not in W^K , K and L are on the same side of every wall of K , thus equal, a contradiction. Therefore $H \in W^K \cap W^L$.

Consider $v \in K$ and $w \in L$. The interval $[v, w]$ meets only H by Lemma 2.17. Therefore $[v, w] \cap H = \{p\}$, with $p \in \bar{K} \cap \bar{L}$. Since \mathcal{A} is locally finite, by Lemma 2.3 we find a neighborhood U of p such that $\text{sec}(U) = \{H\}$. Again U is separated by H into two connected components, one contained in K , one in L . Therefore $U \cap H$ is contained in $\bar{K} \cap \bar{L}$ and this set generates H . Thus K and L are adjacent.

For the second statement assume L' is a chamber adjacent to K by H . Then we established $S(K, L') = \{H\}$. If $L \neq L'$, there exists a hyperplane H' separating L and L' , and we can assume $D_{H'}(K) = D_{H'}(L)$ and $D_{H'}(K) = -D_{H'}(L')$. Then H' separates K and L' , which implies $H' = H$. This contradicts the fact that $D_{H'}(K) = D_{H'}(L)$. \square

LEMMA 2.19. *Let $K \in \mathcal{K}$, $H \in \mathcal{A} \cap W^K$. Then there exists a chamber L adjacent to K by H .*

PROOF. Take a point x in the interior of $\bar{K} \cap H$ with respect to H and a neighborhood U of x contained in T with $\text{sec}(U) = \{H\}$. Since $H \in \mathcal{A}$, $-D_H(K) \cap U$ is contained in a chamber L , which is adjacent to K by H . \square

In the case where (\mathcal{A}, T) is not thin, we need to be able to handle hyperplanes which are not in \mathcal{A} but occur as a wall of a chamber.

LEMMA 2.20. *Assume $K \in \mathcal{K}$, $H \in W^K$ and $H \notin \mathcal{A}$. Then $T \cap H = \emptyset$. In particular for all $K, L \in \mathcal{K}$ the set $D_H(K)$ is well defined and $D_H(K) = D_H(L)$ holds.*

PROOF. Assume $T \cap H \neq \emptyset$, then H separates T into two half spaces. As H is a wall of K and $H \notin \mathcal{A}$, we find that K is not a connected component of $T \setminus \bigcup_{H \in \mathcal{A}} H$. Therefore $T \cap H = \emptyset$, which proves our claim. \square

LEMMA 2.21. *Let $K, L \in \mathcal{K}$. Then there exists a minimal gallery of length $|S(K, L)|$ connecting K and L .*

PROOF. By Lemma 2.17 $S(K, L)$ is finite, so let $n = |S(K, L)|$. For $n = 0$, we have $D_H(K) = D_H(L)$ for every $H \in \mathcal{A}$ and all $H \in W^K \cup W^L$ by Lemma 2.20. This implies that K and L are the same connected component. Let $n = 1$, then we obtain the statement from Lemma 2.18.

We now use induction on n , let $S(K, L) = \{H_1, \dots, H_n\}$. We can sort $S(K, L)$ in a way, such that $H_1 \in W^K$. By Lemma 2.19 and Lemma 2.18 there exists a unique chamber K' adjacent to K such that $S(K, K') = \{H_1\}$. We will show $S(K', L) = \{H_2, \dots, H_n\}$, then the statement of the Lemma follows by induction. Since we have $S(K, K') = \{H_1\}$, we have $D_{H_i}(K) = D_{H_i}(K')$ for $i = 2, \dots, n$. Assuming that there exists an additional $H' \in \mathcal{A}$, $H' \neq H_i$ for $i = 1, \dots, n$ with the property that H' separates K' and L , provides a contradiction as H' also separates K and L . By induction this yields a gallery of length n from K to L .

We show that this gallery is also minimal. For this purpose let $\bar{K} = \bar{K}_0, \bar{K}_1, \dots, \bar{K}_m = \bar{L}$ be an arbitrary gallery connecting K, L and let for $i = 1, \dots, m$ the hyperplane $H_i \in \mathcal{A}$ such that $\bar{K}_{i-1} \cap \bar{K}_i \subset H_i$. Note that H_i is unique with that property by Lemma 2.18. Assume $H \in S(K, L)$, then there exists an index $0 \leq i < m$ such that H separates K_i and L but not K_{i+1} and L . So $D_H(K_i) = -D_H(L)$ and $D_H(K_{i+1}) = D_H(L)$, H separates K_i and K_{i+1} . We obtain $H = H_i$, as K_i, K_{i+1} are H_i -adjacent and H_i is unique with that property by Lemma 2.18.

Thus $S(K, L) \subset \{H_1, \dots, H_m\}$ and this yields $|S(K, L)| \leq m$. Above we constructed a gallery of length $|S(K, L)|$, hence the gallery is already minimal. \square

COROLLARY 2.22. *The graph Γ is connected.*

DEFINITION 2.23. Let (M, d) be a connected metric space. Let $x \in M$ and $A \subseteq M$. A point $y \in A$ is called a *gate of x to A* or *the projection of x on A* if $y \in \sigma(x, z)$ for all $z \in A$. The gate of x to A is uniquely determined, if it exists, and we will then denote it by $\text{proj}_A(x)$. The set A is called *gated* if every $x \in M$ has a gate to A .

DEFINITION 2.24. Let $x \in \bar{T}$. Define the graph $\Gamma_x = (\mathcal{K}_x, E_x)$, where $E_x = \{\{K, K'\} \in E \mid K, K' \in \mathcal{K}_x\}$. Then Γ_x is a subgraph of Γ and is called the *parabolic subgraph at x* . A subgraph Γ' of Γ is called *parabolic*, if it is a parabolic subgraph at x for some $x \in \bar{T}$.

REMARK 2.25. The parabolic subgraphs and subarrangements correspond to the residues in the case where \mathcal{A} is a hyperplane arrangement coming from a Coxeter group.

LEMMA 2.26. *Parabolic subgraphs are connected.*

PROOF. The graph Γ_x corresponds to the chamber graph of the hyperplane arrangement (\mathcal{A}_x, T) , and is therefore connected by Corollary 2.22. \square

LEMMA 2.27. *Let $x \in \bar{T}$, $K \in \mathcal{K}$. Then there exists a unique chamber $G \in \mathcal{K}_x$, such that $D_H(K) = D_H(G)$ for all $H \in \mathcal{A}_x$.*

PROOF. We prove existence first. Let $K \in \mathcal{K}$ and consider the intersection

$$S := T \cap \bigcap_{H \in \mathcal{A}_x} D_H(K).$$

Then S is a chamber of (\mathcal{A}_x, T) , since it is a non-empty connected component of T . By Lemma 2.8 S corresponds to a unique chamber G in \mathcal{K}_x , and $D_H(G) = D_H(S) = D_H(K)$ holds.

So let $G \neq G'$ be another chamber in \mathcal{K}_x . Then there exists an element $H \in \mathcal{A}$ such that H separates G and G' . By Lemma 2.7 we obtain $H \in \mathcal{A}_x$, but this implies $D_H(G') \neq D_H(G) = D_H(K)$. Hence G is unique with the property that $D_H(K) = D_H(G)$ for all $H \in \mathcal{A}_x$. \square

PROPOSITION 2.28. *For $x \in \bar{T}$ the set \mathcal{K}_x is a gated subset of (\mathcal{K}, d_Γ) .*

PROOF. Let $x \in \bar{T}$ and Γ_x be a parabolic subgraph. Let $K \in \mathcal{K}$. By Lemma 2.27 there exists a unique chamber $G_K \in \mathcal{K}_x$ such that $D_H(G_K) = D_H(K)$ for all $H \in \mathcal{A}_x$. We will prove that G_K is a gate of K to \mathcal{K}_x . So let $L \in \mathcal{K}$. If $H \in \mathcal{A}$ separates G_K and L , we immediately get by Lemma 2.7 that $H \in \mathcal{A}_x$. On the other hand we get that if $H' \in \mathcal{A}$ separates G_K and K , by construction of G_K we find $H' \notin \mathcal{A}_x$. Now assume $H \in S(K, L) \cap \mathcal{A}_x$, we find that since $H \in \mathcal{A}_x$, H does not separate G_K and K and therefore must separate G_K and L .

Assume on the other hand that $H \in S(K, L) \cap (\mathcal{A} \setminus \mathcal{A}_x)$, then it cannot separate L and G_K , and therefore must separate G_K and K . Summarized this yields

$$\begin{aligned} S(K, L) &= (S(K, L) \cap \mathcal{A}_x) \dot{\cup} (S(K, L) \cap (\mathcal{A} \setminus \mathcal{A}_x)) \\ &= S(K, G_K) \dot{\cup} S(G_K, L). \end{aligned}$$

We thus find for all $L \in \mathcal{K}_x$ that $d_\Gamma(K, L) = d_\Gamma(K, G_K) + d_\Gamma(G_K, L)$. So G_K is indeed a gate for K to Γ_x . \square

2.5 – Restrictions

We will now consider the structure induced on hyperplanes H by the elements in \mathcal{A} .

DEFINITION 2.29. Let (\mathcal{A}, T) be a hyperplane arrangement. Let $H \leq V$ be a hyperplane in V , such that $H \cap T \neq \emptyset$. Set

$$\mathcal{A}^H := \{H' \cap H \leq H \mid H' \in \mathcal{A} \setminus \{H\}, H' \cap H \cap T \neq \emptyset\},$$

this is a set of hyperplanes in H which have non empty intersection with $T \cap H$, if $T \cap H$ is not empty itself.

We also define the connected components of $T \cap H \setminus \bigcup_{H' \in \mathcal{A}^H} H'$ as \mathcal{K}^H .

LEMMA 2.30. If (\mathcal{A}, T) is a hyperplane arrangement of rank r and $H \cap T \neq \emptyset$, then the pair $(\mathcal{A}^H, T \cap H)$ is a hyperplane arrangement of rank $r-1$ with chambers \mathcal{K}^H .

PROOF. By definition for $H' \in \mathcal{A}^H$ we find $H' \cap H \cap T \neq \emptyset$. The set $T \cap H$ is an open convex cone in H , the latter being isomorphic to \mathbb{R}^{r-1} . Let $x \in T \cap H$, U a neighborhood of x such that $\text{sec}(U)$ is finite. Then $\text{sec}_{\mathcal{A}^H}(U \cap H)$ is also finite, hence \mathcal{A}^H is locally finite in $T \cap H$.

By definition \mathcal{K}^H are the chambers of this arrangement. \square

DEFINITION 2.31. If (\mathcal{A}, T) is a simplicial arrangement, then we call $(\mathcal{A}^H, T \cap H)$ for $H \in \mathcal{A}$ the *induced hyperplane arrangement (by (\mathcal{A}, T)) on H* or the *restriction of (\mathcal{A}, T) to H* .

In the case where $H \in \mathcal{A}$, we need some basic relations between chambers in \mathcal{K} and in \mathcal{K}^H .

LEMMA 2.32. Assume $H \in \mathcal{A}$ and let $K \in \mathcal{K}$. If $H \in W^K$, then $\overline{K} \cap H$ is the closure of a chamber in \mathcal{K}^H . For every $K' \in \mathcal{K}^H$ there exists exactly two chambers $K_1, K_2 \in \mathcal{K}$, such that $H \in W^{K_i}$ and $\overline{K_i} \cap H = \overline{K'}$ for $i = 1, 2$.

PROOF. For the first statement, $\overline{K} \cap H$ is contained in the closure of a chamber K' of $(\mathcal{A}^H, T \cap H)$, since its interior with respect to H does not meet any hyperplane. Let $x \in \overline{K'} \setminus (\overline{K} \cap H)$. Then $x \in \overline{L}$ for some chamber $L \in \mathcal{K}$. Let $H' \in S(K, L)$. Assume $x \notin H'$, then $x \in D_{H'}(L)$, while $\overline{K} \cap H$ is either contained in H' or in $-D_{H'}(L) = D_{H'}(L)$. A contradiction to $x \in K'$. Thus we find $x \in H'$. Since $x \notin \overline{K}$, there exists a wall of K separating K and x . A contradiction, hence $\overline{K'} = \overline{K} \cap H$.

For the second statement, take $x \in K'$, then $x \in H$ and $\mathcal{A}_x = \{H\}$. Choose a neighborhood U of x such that $\text{sec}(U) = \{H\}$, which is possible by Lemma 2.3. The hyperplane H separates U into two connected components, which are each contained in chambers K_1, K_2 . By definition K_1 and K_2 are H -adjacent, and by Lemma 2.18 there can not exist another chamber being adjacent to K_1 or K_2 by H . \square

3. Tits cones and Tits arrangements

3.1 – Simplicial cones

In this section we introduce simplicial cones and collect some basic properties. As before, throughout this section let $V = \mathbb{R}^r$.

DEFINITION 3.1. Let B be a basis of V^* , then the *open simplicial cone associated to B* is

$$K^B := \bigcap_{\alpha \in B} \alpha^+.$$

DEFINITION AND REMARK 3.2. Let B be a basis of V^* . We can then define the *closed simplicial cone associated to B* as

$$\overline{K^B} = \bigcap_{\alpha \in B} \overline{\alpha^+}.$$

A set $K \subset V$ is an *open simplicial cone*, if $K = K^B$ for some Basis B of V^* , and likewise a *closed simplicial cone*, if $K = \overline{K^B}$ for some Basis B of V^* .

We say a cone is *simplicial* if it is open simplicial or closed simplicial.

A simplicial cone can also be defined using bases of V . Let C be a basis of V , then the open simplicial cone associated to C is

$$K^C = \left\{ \sum_{v \in C} \lambda_v v \mid \lambda_v > 0 \right\}$$

and the closed simplicial cone associated to C is

$$\overline{K^C} = \left\{ \sum_{v \in C} \lambda_v v \mid \lambda_v \geq 0 \right\}.$$

Both concepts are equivalent, and it is immediate from the definition that if $B \subset V^*$ and $C \subset V$ are bases, then $K^B = K^C$ if and only if B is, up to positive scalar multiples and permutation, dual to C .

A simplicial cone K associated to B carries a natural structure of a simplex, to be precise:

$$\mathcal{S}_K := \left\{ \overline{K} \cap \bigcap_{\alpha \in B'} \alpha^\perp \mid B' \subset B \right\}$$

is a poset with respect to inclusion, which is isomorphic to $\mathcal{P}(B)$ with inverse inclusion. If C is the basis of V dual to B , we find \mathcal{S}_K to be the set of all convex combinations of subsets of C , and \mathcal{S}_K is also isomorphic to $\mathcal{P}(C)$. Moreover, $\{\mathbb{R}_{\geq 0}c \mid c \in C\}$ is the vertex set of the simplex \mathcal{S}_K .

Before we introduce root bases later in this paper, when considering simplicial cones K we will often be interested in bases B of V^* satisfying $K^B = K$ without specifying one. Therefore in the following we will denote with $B_K \subset V^*$ one (arbitrarily chosen) basis of V^* such that $K^{B_K} = K$.

LEMMA 3.3. *An open or closed simplicial cone is convex and has non-empty interior.*

PROOF. Let K be an open simplicial cone and $B_K \subset V^*$ a basis associated to K . Then K is convex and open as an intersection of convex and open subsets. Furthermore let C be dual to B_K , then we obtain that $\sum_{v \in C} v \in K$, hence K is not empty and thus has non-empty interior.

If T is a closed simplicial cone, it contains the closure of an open simplicial cone, and has non-empty interior. \square

REMARK 3.4. There is a definition of *properness* of a cone K as the property of having non-empty interior and being closed, convex, and pointed, the latter meaning that $v, -v \in K \implies v = 0$. Thus all closed simplicial cones are proper. In our context being proper is not of interest, the cones we are dealing with are either convex and open or already simplicial.

3.2 – Root systems and Tits arrangements

In the following let $V = \mathbb{R}^r$ and $T \subseteq V$ be an open convex cone. In this section we define Tits arrangements on T . The interesting cases the reader may think of are $T = \mathbb{R}^r$, or a half-space $T = \alpha^+$ for some $\alpha \in V^*$.

We can now define our main objects of interest.

DEFINITION 3.5. Let $T \subseteq V$ be a convex open cone and \mathcal{A} a set of linear hyperplanes in V . We call a hyperplane arrangement (\mathcal{A}, T) a *simplicial arrangement (of rank r)*, if every $K \in \mathcal{K}(\mathcal{A})$ is an open simplicial cone.

The cone T is the *Tits cone* of the arrangement. A simplicial arrangement is a *Tits arrangement*, if it is thin.

REMARK 3.6. (1) The definition of “thin” requires that all possible walls are already contained in \mathcal{A} . If we do not require this, a bounding hyperplane may arise as a bounding hyperplane of T itself. This happens in Example 1.1, which is not thin. Consider also the case where T itself is an open simplicial cone. Even the empty set then satisfies that the one chamber, which is T itself, is a simplicial cone. However, it has no walls in \mathcal{A} , hence it is not thin.

(2) From the definition it follows that in dimensions 0 and 1, there are very few possibilities for simplicial arrangements and Tits arrangements.

In dimension 0, the empty set is the only possible Tits arrangement.

In the case $V = \mathbb{R}$, we find $T = \mathbb{R}$ or $T = \mathbb{R}_{>0}$ or $T = \mathbb{R}_{<0}$. In the first case, $\{0_{\mathbb{R}}\}$ is a Tits arrangement and in the other two cases, $\{0_{\mathbb{R}}\}$ is a simplicial arrangement, but not thin any more.

(3) Later in this section we introduce the notion of k -spherical arrangements, which is a refinement of being thin.

(4) In the case where $T = \mathbb{R}^r$, the property of \mathcal{A} being locally finite is equivalent to \mathcal{A} being finite, as 0 is contained in every hyperplane. Furthermore $T = \mathbb{R}^r$ is the only case where $0 \in T$, as the cone over every neighborhood of 0 is already \mathbb{R}^r .

EXAMPLE 3.7. (i) Consider Example 1.1, there K_1 and K_2 are simplicial cones, with $W^{K_1} = \{L, \ker(\alpha_1)\}$ and $W^{K_2} = \{L, \ker(\alpha_2)\}$. The set $\{L\}$ is therefore a simplicial arrangement of rank 2 in T , which is finite and hence also locally finite. But $(\{L\}, T)$ is not thin, as

$$T \cap \ker(\alpha_1) = \emptyset = T \cap \ker(\alpha_2).$$

While this particular example can be turned into a thin arrangement by factoring out the radical (cp. Example 2.14). However, one can easily find examples of essential simplicial arrangements which are not thin.

(ii) In Example 1.2, the set $\{L_n, L_n' \mid n \in \mathbb{N}_{>0}\}$ of hyperplanes is not finite, but it is locally finite in T . If $(x, y) \in \mathbb{R}^2$, we cannot find a neighborhood U of (x, y) which meets only finitely many hyperplanes if and only if $x = 0$ or $y = 0$, and hence $(x, y) \notin T$.

A chamber K satisfies one of the following:

- (1) $W^K = \{L_n, L_{n+1}\}$ for some $n \geq 0$, or
- (2) $W^K = \{L_n', L_{n+1}'\}$ for some $n \geq 0$.

Therefore, $(\{L_n, L_n' \mid n \in \mathbb{N}_{>0}\}, T)$ is a Tits arrangement of rank 2.

Likewise, the hyperplane arrangement in Example 1.3 is a Tits arrangement of rank 3 in T . The vertices of the simplicial cones are contained in ∂T , but all walls meet T .

(iii) Figure 4 shows a Tits arrangement coming from the Weyl groupoid of an infinite dimensional rank three Nichols algebra of diagonal type.

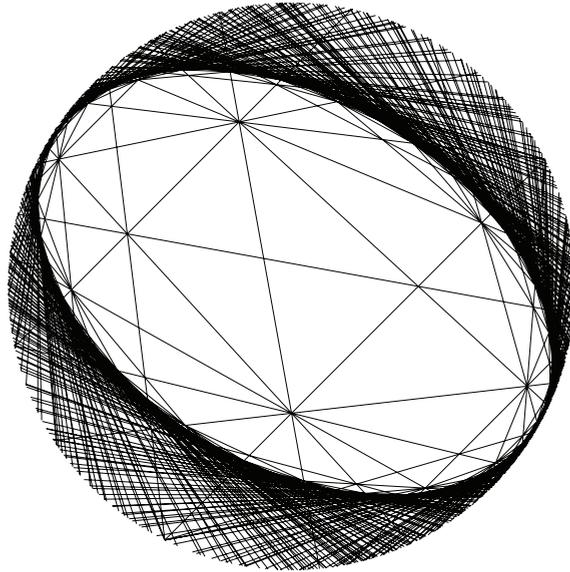


Figure 4. A Tits arrangement of rank three. The area outside of the cone is not entirely black because we display only a finite subset of the hyperplanes.

LEMMA 3.8. *Tits arrangements are essential.*

PROOF. Let $K \in \mathcal{K}$, B a basis of V^* such that $K = K^B$. Since $W^K \subset \mathcal{A}$ and B is a basis, we find $\bigcap_{\alpha \in B} \alpha^\perp = \{0\}$. \square

Some of the “classical” cases are the following choices for T .

DEFINITION 3.9. A simplicial arrangement (\mathcal{A}, T) is *spherical*, if $T = \mathbb{R}^r$. We say (\mathcal{A}, T) is *affine*, if $T = \gamma^\perp$ for some $0 \neq \gamma \in V^*$. For an affine arrangement we call γ the *imaginary root* of the arrangement.

REMARK 3.10. The Tits cone of a Tits arrangement (\mathcal{A}, T) resembles the interior of the Tits cone for Coxeter groups. The geometric representation of an irreducible spherical or affine Coxeter group is a prototype of spherical or affine Tits arrangements.

DEFINITION 3.11. Let $V = \mathbb{R}^r$, a *root system* is a set $R \subset V^*$ such that

- 1) $0 \notin R$,
- 2) $-\alpha \in R$ for all $\alpha \in R$,
- 3) there exists a Tits arrangement (\mathcal{A}, T) such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$.

If R is a root system, (\mathcal{A}, T) as in 3), we say that the Tits arrangement (\mathcal{A}, T) is *associated to R* .

A root system R is *reduced*, if $\alpha \in R$ and $\beta \in \langle \alpha \rangle \cap R$ implies $\beta \in \{\pm\alpha\}$.

Let $(\mathcal{A}, T), (\mathcal{A}', T')$ be Tits arrangements associated to the root systems R and R' respectively. Then (\mathcal{A}, T) and (\mathcal{A}', T') are called *combinatorially equivalent*, if there exists an $g \in \text{GL}(V)$ such that $g\mathcal{A} = \mathcal{A}'$, $g * R = R'$, $g(T) = T'$. Here $*$ denotes the dual action of $\text{GL}(V)$ on V^* , defined by $g * \alpha = \alpha \circ g^{-1}$.

REMARK 3.12. If (\mathcal{A}, T) is associated to R , then so is $(\mathcal{A}, -T)$, but both arrangements are combinatorially equivalent (via $-id_V$). While a given root system R determines \mathcal{A} uniquely, it is not clear whether or not there can exist an arrangement (\mathcal{A}, T') associated to R , which is not equivalent to (\mathcal{A}, T) .

We note some immediate consequences of Definition 3.11.

Given a Tits arrangement (\mathcal{A}, T) , a reduced root system to which they are associated always exists.

LEMMA 3.13. *Let (\mathcal{A}, T) be a Tits arrangement of rank r , \mathbb{S}^{r-1} the unit sphere in \mathbb{R}^r with respect to the standard metric associated to the standard scalar product (\cdot, \cdot) . Let*

$$S = \bigcup_{H \in \mathcal{A}} H^\perp \cap \mathbb{S}^{r-1}.$$

Then $R = \{(s, \cdot) \in V^ \mid s \in S\}$ is a reduced root system for \mathcal{A} .*

Furthermore every reduced root system R' associated to (\mathcal{A}, T) is of the form $R' = \{\lambda_\alpha \alpha \mid \lambda_\alpha = \lambda_{-\alpha} \in \mathbb{R}_{>0}, \alpha \in R\}$, and every such set is a reduced root system associated to (\mathcal{A}, T) .

PROOF. As $H \in \mathcal{A}$ is $r - 1$ -dimensional, $\dim H^\perp = 1$, so $H^\perp \cap \mathbb{S}^{r-1} = \{\pm s\}$ for some vector $s \in V$, with $s^\perp = H$. So R is a root system associated to (\mathcal{A}, T) . Let $\alpha, \lambda\alpha \in R$ for some $\lambda \in \mathbb{R}$, then $\alpha^\perp = (\lambda\alpha)^\perp$. As the map $s \mapsto (s, \cdot)$ is bijective, we find $\lambda \in \{\pm 1\}$. Therefore R is reduced.

For the second statement note that the hyperplane $H \in \mathcal{A}$ determines $s \in H^\perp$ uniquely up to a scalar. So any R' of the given form satisfies $\{\alpha^\perp \mid \alpha \in R'\} = \mathcal{A}$. Furthermore it is reduced, as $\langle \alpha \rangle \cap R' = \{\pm \lambda_\alpha \alpha\}$ for $\alpha \in R$. □

DEFINITION 3.14. Let (\mathcal{A}, T) be a Tits arrangement and R a reduced root system associated to (\mathcal{A}, T) . Let K be a chamber. The *root basis of K* is the set

$$B^K := \{\alpha \in R \mid \alpha^\perp \in W^K, \alpha(x) > 0 \text{ for all } x \in K\}.$$

REMARK 3.15. If R is a root system associated to the Tits arrangement (\mathcal{A}, T) , $K \in \mathcal{K}$, then

$$W^K := \{\alpha^\perp \mid \alpha \in B^K\}.$$

Also, as a simplicial cone $K \subset \mathbb{R}^r$ has exactly r walls, the set B^K is a basis of V^* . Furthermore, $K^{B^K} = K$.

The following lemma is crucial for the theory and motivates the notion of root bases.

LEMMA 3.16. *Let (\mathcal{A}, T) be a Tits arrangement, R a root system associated to (\mathcal{A}, T) and K a chamber. Then $R \subset \pm \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0} \alpha$. In other words, every root is a non-negative or non-positive linear combination of B^K .*

PROOF. The proof works exactly as in the spherical case, see [6, Lemma 2.2]. □

It is useful to identify Tits arrangements which basically only differ by the choice of basis, therefore we make the following definition.

3.3 – The simplicial complex associated to a simplicial arrangement

DEFINITION AND REMARK 3.17. Let (\mathcal{A}, T) be a simplicial arrangement. The set of chambers \mathcal{K} gives rise to a poset

$$\mathcal{S} := \left\{ \bar{K} \cap \bigcap_{H \in \mathcal{A}'} H \mid K \in \mathcal{K}, \mathcal{A}' \subseteq W^K \right\} = \bigcup_{K \in \mathcal{K}} \mathcal{S}_K,$$

with set-wise inclusion giving a poset-structure. Note that we do not require any of these intersections to be in T . By construction they are contained in the closure of T , as every K is an open subset in T .

We will at this point just note that \mathcal{S} is a simplicial complex. This will be elaborated in Appendix A.

PROPOSITION 3.18. *The poset \mathcal{S} is a simplicial complex.*

DEFINITION AND REMARK 3.19. The complex \mathcal{S} is furthermore a chamber complex, which justifies the notion of chambers and is shown in Appendix B.

We have a slight ambiguity of notation at this point, as a chamber in the simplicial complex \mathcal{S} is the closure of a chamber in \mathcal{K} . For the readers convenience, a chamber will always be an element $K \in \mathcal{K}$, while we will refer to a chamber in \mathcal{S} as a *closed chamber*, written with the usual notation \bar{K} . We show in Appendix B that the closed chambers are indeed chambers in a classical sense.

REMARK 3.20. (1) Depending on T , the above mentioned simplicial complex can also be seen as canonically isomorphic to the simplicial decomposition of certain objects, arising by intersection with the respective simplicial cones. In case $T = \mathbb{R}^r$, \mathcal{S} corresponds to a simplicial decomposition of the sphere \mathbb{S}^{n-1} . If $T = \alpha^+$ for $\alpha \in V^*$, we find \mathcal{S} to be a decomposition of the affine space \mathbb{A}^{n-1} , which we identify with the set $\alpha^{-1}(1)$. If T is the light cone and \mathcal{A} is locally spherical as defined in Definition 3.21 below, we can find a corresponding decomposition of \mathbb{H}^{n-1} .

(2) In the literature (see [5, Chapter V, §1]) the simplicial complex associated to a finite or affine simplicial hyperplane arrangement is defined in a slightly different manner. Let V be a Euclidean space, \mathcal{A} a locally finite set of (possibly affine) hyperplanes. Define for $A \in \mathcal{A}$, $v \in V \setminus A$ the set $D_H(v)$ to be the halfspace with respect to H containing v . Set

$$v \sim w \quad \text{if } w \in \bigcap_{v \in H \in \mathcal{A}} H \cap \bigcap_{v \notin H \in \mathcal{A}} D_H(v).$$

Then \sim is an equivalence relation, its classes are called facets. Facets correspond to simplices, and form a poset with respect to the inclusion $F \leq F' \iff F \subseteq \bar{F}'$.

One obtains immediately that every point in V is contained in a unique facet. However, when the space is not the entire euclidean space but a convex open cone, it is desirable to consider some of the points in the boundary, as they contribute to the simplicial structure of \mathcal{S} . Therefore, we prefer our approach before the classical one.

(3) The above approach could also be used to define k -sphericity (see below) for hyperplane arrangements, which are not necessarily simplicial.

Using the simplicial complex \mathcal{S} , it is now possible to refine the notion of being thin for a simplicial arrangement.

DEFINITION 3.21. A simplicial arrangement (\mathcal{A}, T) of rank r is called k -spherical for $k \in \mathbb{N}_0$ if every simplex S of \mathcal{S} , such that $\text{codim}(S) = k$, meets T . We say (\mathcal{A}, T) is *locally spherical* if it is $r - 1$ -spherical.

REMARK 3.22. Immediate from the definition are the following properties.

- (1) A simplicial hyperplane arrangement of rank r is 0-spherical, as $K \in \mathcal{K}$ is constructed as an open subset of T .
- (2) The arrangement (\mathcal{A}, T) is spherical if and only if it is r -spherical.
- (3) The arrangement (\mathcal{A}, T) is thin and therefore a Tits arrangement if and only if it is 1-spherical.
- (4) As \mathcal{S} is a simplicial complex with respect to inclusion, being k -spherical implies being $k - 1$ -spherical for $1 \leq k \leq r$.
- (5) Examples of locally spherical arrangements are all arrangements belonging to affine Weyl groups, where the affine $(r - 1)$ -plane is embedded into a real vector space of dimension r , as well as all arrangements belonging to compact hyperbolic Coxeter groups, where T is the light cone. The notion locally spherical is inspired by exactly this property for groups.
- (6) As a generalization of (5), take a Coxeter system (W, S) of finite rank. Then W is said to be k -spherical if every rank k subset of S generates a finite Coxeter group. The geometric representation of W then yields a hyperplane arrangement which is k -spherical in the way defined above. Therefore, being k -spherical can be seen as a generalization of the respective property of Coxeter groups.

- (7) An equivalent condition for (\mathcal{A}, T) to be k -spherical, which we will use often, is that every $(r - k - 1)$ -simplex meets T . This uses just the fact that simplices of codimension k are exactly $(r - k - 1)$ -simplices.

EXAMPLE 3.23. The arrangement in Example 1.1 yields the simplicial complex \mathcal{S} , with maximal elements K_1, K_2 , vertices $L \cap \bar{T} = \{(x, x) \mid x \geq 0\}, \{(x, 0) \mid x \geq 0\}, \{(0, x) \mid x \geq 0\}$, and the set $\{0\}$ as the minimal element. Thus $(\{L\}, T)$ is 0-spherical, but not 1-spherical, as it is not thin (see also Remark 3.6 and Example 3.7).

The arrangement in Example 1.2 is thin and therefore 1-spherical and locally spherical. The vertices are the rays $L_n \cap \bar{T}, L'_n \cap \bar{T}$, the minimal element, which has codimension 2, is $\{0\}$. Since $0 \notin T$, it is not 2-spherical and in particular not spherical.

In Example 1.3, the arrangement (\mathcal{A}, T) is thin and therefore 1-spherical, but since all vertices are contained in ∂T , it is not 2-spherical and thus not locally spherical. The group W from this example is not compact hyperbolic.

An important observation is the fact that the cone T can be reconstructed from the chambers.

LEMMA 3.24. *For a simplicial arrangement (\mathcal{A}, T) we find*

$$\bar{T} = \overline{\bigcup_{K \in \mathcal{K}} K}.$$

Furthermore (\mathcal{A}, T) is locally spherical if and only if

$$T = \bigcup_{K \in \mathcal{K}} \bar{K}$$

holds.

PROOF. As $\bigcup_{K \in \mathcal{K}} K \subset T$, the inclusion $\overline{\bigcup_{K \in \mathcal{K}} K} \subset \bar{T}$ holds. If $x \in T$, either $x \in K$ for some $K \in \mathcal{K}$, or x is contained in a finite number of hyperplanes, thus in a simplex in \mathcal{S} and also in the closure of a chamber, and therefore $x \in \overline{\bigcup_{K \in \mathcal{K}} K}$.

So let $x \in \bar{T} \setminus T$. Assume further that x is not contained in any simplex in \mathcal{S} , else the statement follows immediately as above. Therefore $\mathcal{A}_x = \emptyset$, and as T is convex, this means x is in the boundary of T . So let $U := U_\delta(x)$ be the open δ -ball with center x , then $U \cap T \neq \emptyset$. As U is open, $U \cap T$ is open again. The set $\text{sec}(U \cap T)$ can not be empty, else $U_x \cap T$ is not contained in a chamber, in contradiction to the construction of chambers. So let \mathcal{K}_0 be the set of chambers

K with $K \cap U \neq \emptyset$. If x is not contained in the closure of $\bigcup_{K \in \mathcal{K}_0} K$, we find an $\delta > \varepsilon > 0$ such that the open ball $U_\delta(x)$ does not intersect any $K \in \mathcal{K}_0$. But $U_\delta(x) \cap T$ must again meet some chambers K , which are then also in \mathcal{K}_0 , a contradiction. So $x \in \overline{\bigcup_{K \in \mathcal{K}} K}$ and equality holds.

For the second statement, if (A, T) is locally spherical, every $x \in T$ is contained in some simplex $F \in \mathcal{S}$, so $T \subset \bigcup_{K \in \mathcal{K}} \bar{K}$ holds. If $x \in \bar{K}$ for some $K \in \mathcal{K}$, x is contained in some simplex $F \subset \bar{K}$. Now A is locally spherical, so F meets T . Boundaries of simplices are simplices, hence the intersection $F \cap (\bar{T} \setminus T)$ is again a simplex in \mathcal{S} , being locally spherical yields that this intersection is empty, which proves the other inclusion.

The other direction of the second statement is immediate from the definition of being $r - 1$ -spherical. \square

REMARK 3.25. For a Tits arrangement (A, T) it is possible to show that we can also describe T as the convex closure of $T_0 := \bigcup_{K \in \mathcal{K}} K$, or alternatively as

$$T = \bigcup_{x, y \in T_0} [x, y].$$

We will require the existence of a type function of \mathcal{S} (for the definition, see Definition B.3), which is given by the following proposition, proven in Appendix B.

PROPOSITION 3.26. *Let (A, T) be a simplicial arrangement. The complex $\mathcal{S} := \mathcal{S}(A, T)$ is a chamber complex of rank r with*

$$\text{Cham}(\mathcal{S}) = \{\bar{K} \mid K \in \mathcal{K}\}.$$

The complex \mathcal{S} is gated and strongly connected. Furthermore there exists a type function $\tau: \mathcal{S} \rightarrow I$ of \mathcal{S} , where $I = \{1, \dots, r\}$. The complex \mathcal{S} is thin if and only if (A, T) is thin, and \mathcal{S} is spherical if and only if (A, T) is spherical.

We will now take a closer look at the relations between the bases of adjacent chambers. Again the proof follows [6, Lemma 2.8] closely.

LEMMA 3.27. *Let (A, T) be a Tits arrangement, R a root system associated to (A, T) , and let $K, L \in \mathcal{K}$ be adjacent chambers. Assume $\bar{K} \cap \bar{L} \subset \alpha_1^\perp$ for $\alpha_1 \in B^K$. If $\beta \in B^L$, then either*

i) $\beta = -\alpha_1$ or

ii) $\beta \in \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0} \alpha$.

holds.

PROOF. By Lemma 3.16 we can assume β is either as in case ii) or $-\beta = \sum_{\alpha \in B^c} \lambda_\alpha \alpha$ with $\lambda_\alpha \in \mathbb{R}_{\geq 0}$. Using that for arbitrary $\varphi, \psi \in V^*$ we have $\overline{\varphi^+} \cap \overline{\psi^+} \subseteq \overline{(\varphi + \psi)^+}$, we get $\overline{K} = \bigcap_{\alpha \in B^K} \overline{\alpha^+} \subseteq \overline{(-\beta)^+}$. We also observe that $\overline{L} \subseteq \overline{\beta^+}$, therefore we get $\overline{K} \cap \overline{L} \subseteq \overline{\beta^+} \cap \overline{(-\beta)^+} = \beta^\perp$.

By choice of α_1 and since the elements in B^K, B^L are reduced by definition, we find $\alpha_1^\perp = \beta^\perp$ and $\beta = -\alpha_1$. \square

DEFINITION AND REMARK 3.28. Assume for $K \in \mathcal{K}$ that B^K is indexed in some way, i.e. $B^K = \{\alpha_1, \dots, \alpha_r\}$. For any set I , define the map $\kappa_I: \mathcal{P}(I) \rightarrow \mathcal{P}(I)$ by $\kappa(I) = I \setminus J$. Set $\kappa := \kappa_{\{1, \dots, r\}}$.

For every simplex $F \subset \overline{K}$ there exists a description of the form $F = \overline{K} \cap \bigcap_{\alpha \in B_F} \alpha^\perp$ for some $B_F \subset B^K$ by Remark 3.2, which gives an index set $J_F = \{i \mid \alpha_i \in B_F\}$.

This gives rise to a type function of \overline{K} in \mathcal{S} , by taking the map $\tau_{\overline{K}}: F \mapsto \kappa(J_F)$. By Theorem B.15 the map $\tau_{\overline{K}}$ yields a unique type function τ of the whole simplicial complex \mathcal{S} . So let $L \in \mathcal{K}$ be another chamber, then the restriction $\tau|_{\overline{L}}$ is a type function of \overline{L} as well. Assume $B^L = \{\beta_1, \dots, \beta_r\}$, this yields a second type function of \overline{L} in the same way we acquired a type function of \overline{K} before,

$$\tau_{\overline{L}}: F \mapsto \kappa(\{i \mid F \subset \beta_i^\perp\}).$$

We now call the indexing of B^L *compatible with B^K* , if $\tau_{\overline{L}} = \tau|_{\overline{L}}$.

Since the type function τ is unique, there is a unique indexing of B^L compatible with B^K .

The existence of the type function and Lemma 3.27 allow us to state the following lemma, which gives another characterization root bases of adjacent chambers.

LEMMA 3.29. Assume that (\mathcal{A}, T) is a Tits arrangement, R a root system associated to (\mathcal{A}, T) . Let $K, L \in \mathcal{K}$ be adjacent chambers and choose a indexing $B^K = \{\alpha_1, \dots, \alpha_r\}$. Let the indexing of $B^L = \{\beta_1, \dots, \beta_r\}$ be compatible with B^K . Assume $\overline{K} \cap \overline{L} \subset \alpha_k^\perp$ for some $1 \leq k \leq r$. Then $\beta_i \in \langle \alpha_i, \alpha_k \rangle$.

PROOF. If $i = k$, this is immediate from Lemma 3.27. So let $i \neq k$ and assume without loss of generality $k = 1$.

Consider the type function $\tau_{\overline{K}}$ of \overline{K} and $\tau_{\overline{L}}$ of \overline{L} as restrictions of the unique type function τ . Being compatible yields that $\tau(\overline{K} \cap \overline{L}) = \tau(\overline{K} \cap \alpha_1^\perp) = \{2, \dots, r\}$.

Then $\tau(\bar{K} \cap \bar{L} \cap \alpha_i^\perp) = \{2, \dots, r\} \setminus \{i\} = \tau(\bar{K} \cap \bar{L} \cap \beta_i^\perp)$ holds and we get $\tau(\bar{K} \cap \alpha_i^\perp \cap \alpha_1^\perp) = \{2, \dots, n\} \setminus \{i\} = \tau(\bar{d} \cap \beta_i^\perp \cap \beta_1^\perp)$. We conclude that $\beta_i^\perp \cap \beta_1^\perp = \beta_i^\perp \cap \beta_1^\perp$, and as $\alpha_1 = -\beta_1$ we find $\beta_i^\perp \cap \alpha_1^\perp = \alpha_i^\perp \cap \alpha_1^\perp$.

Then we find $\langle \beta_i, \alpha_1 \rangle = \langle \alpha_i, \alpha_1 \rangle$, so β_i is a linear combination of α_1 and α_i , which proves our claim. □

4. Parabolic subarrangements and restrictions of Tits arrangements

4.1 – Parabolic subarrangements of Tits arrangements

We will collect more properties of parabolic subarrangements of Tits arrangements. In particular we will indicate suitable root systems associated to these subarrangements. Remember from Corollary 2.13 that (\mathcal{A}_x^π, T_x) is a hyperplane arrangement.

DEFINITION 4.1. Assume (\mathcal{A}, T) is a Tits arrangement and R a root system associated to (\mathcal{A}, T) . Let $x \in \bar{T}$, define $R_x := \{\alpha \in R \mid \alpha^\perp \in \mathcal{A}_x\}$. For $K \in \mathcal{K}_x$ set $B_x^K := B^K \cap R_x$.

LEMMA 4.2. *With notation as above, let $K \in \mathcal{K}_x$, then $\langle B_x^K \rangle = \langle R_x \rangle$. In particular, B_x^K is a basis of $\langle R_x \rangle$.*

PROOF. The inclusion $\langle B_x^K \rangle \subset \langle R_x \rangle$ holds due to $B_x^K \subset R_x$.

Now the space $(B_x^K)^\perp = \{v \in V \mid \alpha(v) = 0 \ \forall \alpha \in B_x^K\}$ is a subspace of V and $(B_x^K)^\perp \cap \bar{K}$ is a face of \bar{K} containing x . Assume F is a face of $(B_x^K)^\perp \cap \bar{K}$ containing x . Then F is also a face of \bar{K} and has the structure $F = \bar{K} \cap \bigcap_{i=1}^m \alpha_i^\perp$ for some $\alpha_1, \dots, \alpha_m \in B^K$. Then by definition $x \in \alpha_i^\perp$ for $i = 1, \dots, m$, and $(B_x^K)^\perp \cap \bar{K} \subset F$, so $(B_x^K)^\perp \cap \bar{K}$ is a minimal face of \bar{K} containing x .

The inclusion $\langle B_x^K \rangle \subset \langle R_x \rangle$ implies $(R_x)^\perp \subset (B_x^K)^\perp$, therefore the set $(R_x)^\perp \cap \bar{K}$ is also a face of \bar{K} , furthermore it contains x by definition. We can conclude $(B_x^K)^\perp \subset (R_x)^\perp$, which yields the equality.

The second claim follows since $B_x^K := B^K \cap R_x$ and the elements in B^K are linearly independent. □

REMARK 4.3. The linear form $\alpha \in R_x$ defines a map $V_x \rightarrow \mathbb{R}$ by $v + W \mapsto \alpha(v)$, which is well defined since $W \in \ker \alpha$. We will identify this map with α and think of R_x as a subset of V_x^* .

Most of the time we are interested only in R_x and its combinatorial properties, therefore it does not matter whether we consider these in V_x^* or in V^* . But formally the transition to T_x is necessary to obtain a root system in the strict sense.

We will consider V_x as a topological space with respect to the topology in V/W , which comes from the standard topology on \mathbb{R}^r .

PROPOSITION 4.4. *Assume (A, T) is a Tits arrangement, R a root system associated to (A, T) and $x \in \bar{T}$. The hyperplane arrangement (\mathcal{A}_x^π, T_x) is simplicial. The chambers of this arrangement correspond to \mathcal{K}_x .*

Moreover, if $x \in T$, then (\mathcal{A}_x^π, T_x) is spherical. If $x \notin T$, $\dim V_x = m$, and (A, T) is k -spherical for some $k \in \mathbb{N}$, then (\mathcal{A}_x^π, T_x) is $\min(m, k)$ -spherical.

In particular (\mathcal{A}_x^π, T_x) is a Tits arrangement and R_x a root system associated to (\mathcal{A}_x^π, T_x) .

PROOF. Assume $\dim V_x = m$, so $\dim \langle R_x \rangle = m$ and $\dim W^\perp = r - m$.

Let $K \in \mathcal{K}_x$, $\pi: V \rightarrow V_x$ denote the standard epimorphism. Let \mathcal{K}' be the connected components of $T_x \setminus \bigcup_{\alpha \in R_x} \alpha^\perp$. The set $\pi(B_x^K)$ is a basis for V_x by Lemma 4.2 and we find $\pi(K) \subset T_x \cap \bigcap_{\alpha \in B_x^K} \pi^*(\alpha)^\perp$ by definition. Denote this intersection by K' .

First we show that $K' \in \mathcal{K}'$. Assume there exists $\beta \in R_x$ such that there exist $y', z' \in K'$ with $\beta(y') > 0$, $\beta(z') < 0$. Then we find $y, z \in T$ with the properties: $\beta(y) > 0$, $\beta(z) < 0$, $\alpha(y) > 0$, $\alpha(z) > 0$ for all $\alpha \in B_x^K$. For any $0 < \lambda < 1$ and $\alpha \in B_x^K$ we find $\alpha(y - \lambda(y - x)) = \alpha(y)(1 - \lambda) > 0$ and $\alpha(z - \lambda(z - x)) < 0$ as $\alpha(x) = 0$. So for $0 < \lambda_y, \lambda_z < 1$ the points $y - \lambda_y(y - x)$, $z - \lambda_z(z - x)$ still satisfy the above inequalities.

Now let $\alpha \in B^K \setminus B_x^K$, then $\alpha(x) > 0$. So choosing $0 < \lambda_y, \lambda_z < 1$ large enough we find that the points $y_1 := y - \lambda_y(y - x)$, $z_1 := z - \lambda_z(z - x)$ are close enough to x to satisfy $\beta(y_1) > 0$, $\beta(z_1) < 0$ and $\alpha(y_1) > 0 < \alpha(z_1)$ for all $\alpha \in B^K$. So $y_1, z_1 \in K$, in contradiction to the simplicial structure of \mathcal{S} .

By its definition, K' is a simplicial cone, the arrangement (\mathcal{A}_x^π, T_x) therefore is simplicial. It remains to determine for which k the pair (\mathcal{A}_x^π, T_x) is k -spherical.

In the case where $x \in T$, $\pi(T) = V_x$ and \mathcal{A}_x^π is spherical. Construct a simplex F' in the following way.

Since K is a simplicial cone, there exists an closed simplex S with the property $\bar{K} = \mathbb{R}_{>0}S \cup \{0\}$ (for details, see Remark A.1). Let F_x denote the minimal face of S such that $x \in \mathbb{R}_{>0}F_x$, and let $V(F_x)$ be the vertex set of F_x . Then the vertices $V(S) \setminus V(F_x)$ span a face of S , denote this face by F' . The vertices $\pi(V(F'))$ are linearly independent: Assume $\sum_{v \in V(F')} \lambda_v \pi(v) = 0$, then $\sum_{v \in V(F')} \lambda_v v \in W^\perp$. Furthermore note that W^\perp is spanned by $V(F_x)$. So we get a linear combination of the form $\sum_{v \in V(F')} \lambda_v v = \sum_{v \in V(F_x)} \lambda_v v$. Linear independence of $V(S)$ yields $\lambda_v = 0$ for all $v \in V(S)$.

This also gives us $|V(F')| \leq m$. Assuming inequality, we find more than $r - m$ vertices in W^\perp , a contradiction to $\dim W^\perp = r - m$. So $\pi(F')$ spans indeed an $(m - 1)$ -simplex in V_x .

We show $\overline{K'} = \mathbb{R}_{>0}\pi(F') \cup \{0\}$. In general for every $v \in V(\overline{K'})$ there exists a unique $\alpha \in B^K$ such that $\alpha(v) > 0$ by Remark 3.2, all other $\alpha \neq \beta \in B^c$ satisfy $\beta(v) = 0$. Since for $v \in V(F')$ the vector v is not contained in any proper face of \overline{K} containing x , we find a unique $\alpha \in B_x^K$ such that $\alpha(v) > 0$.

Let $y \in \mathbb{R}_{>0}\pi(F')$, then $y = \sum_{v \in V(F')} \lambda_v \pi(v)$ with $\lambda_v \geq 0$ for all $v \in V(F')$. Then $\alpha(v) \geq 0$ for all $v \in F'$, $\alpha \in B^{K,x}$ yields $\alpha(\pi(v)) \geq 0$. Therefore $y \in \overline{K'}$.

So assume $0 \neq y \notin \mathbb{R}_{>0}\pi(F')$, then $y = \sum_{v \in V(F')} \lambda_v \pi(v)$ and there exists a $w \in \pi(V(F'))$ such that $\lambda_w < 0$. Now there exists a unique $\alpha \in B_x^K$ such that $\alpha(w) > 0$, so we find $\alpha(\sum_{v \in V(F')} \lambda_v v) < 0$. Therefore $\alpha(y) < 0$ holds, and therefore $y \notin \overline{K'}$. We can conclude $\overline{K'} = \mathbb{R}_{>0}\pi(F') \cup \{0\}$.

Now assume $x \notin T$ and \mathcal{A} is k -spherical. Let F' be as above, then $\underline{F'}$ is isomorphic as a simplicial complex to the closed chamber $\mathbb{R}_{>0}\pi(F') \cup \{0\}$ and all simplices contained therein. A face of F' meets T if and only if a face of $\mathbb{R}_{>0}\pi(F') \cup \{0\}$ meets T_x . Now F' is an $(m - 1)$ -simplex, as it is spanned by m vertices, likewise F_x is an $(r - m - 1)$ -simplex.

Let $F_1 \subset F'$ be a face of F' , and assume F_1 is an l -simplex. Then $V(F_1) \cup V(F_x)$ generate an $(r - m + l)$ -simplex F_2 of \mathcal{S} . As \mathcal{A} is k -spherical, F_2 therefore meets T if $r - k - 1 \leq r - m + l$ is satisfied, or equivalently $m - k - 1 \leq l$. Under this condition also $\pi_x(F_2) = \pi_x(F_1)$ meets T_x , we can conclude that \mathcal{A}_x^π is k -spherical. Since \mathcal{A}_x^π will not be more than m -spherical, since it is an arrangement of rank m , we find that \mathcal{A}_x^π is $\min(k, m)$ -spherical.

So as (\mathcal{A}, T) is thin, (\mathcal{A}_x^π, T_x) is a Tits arrangement with root system R_x . Since chambers $K \in \mathcal{K}_x$ and in $K' \in \mathcal{K}'$ are uniquely determined by the sets B_x^K and $B^{K'}$ we find that π induces a bijection between K_x and K' . \square

REMARK 4.5. (1) The Tits arrangement (\mathcal{A}_x^π, T_x) is not so much dependent on the point x as on the subspace spanned by x , as one gets the same arrangement for every λx , $0 < \lambda \in \mathbb{R}$. This hints to the fact that one can see a simplicial arrangement as a simplicial complex in projective space. However, we will not elaborate this further.

(2) It is worth mentioning that the chambers \mathcal{K}_x and the underlying simplicial complex correspond to the star of the smallest simplex in \mathcal{S} containing x .

(3) It is possible to show that for a simplicial hyperplane arrangement (\mathcal{A}, T) , the arrangement (\mathcal{A}_x^π, T_x) is also simplicial. However, the existence of a root system simplifies the proof.

(4) When $x \notin T$, (\mathcal{A}_x^π, T_x) may become k' -spherical for some $k' > \min(k, m)$. The reason for this is that \mathcal{A} not being k' -spherical does not imply that every $(r - k' - 1)$ -simplex contained in a chamber in \mathcal{K}_x does not meet T .

The above statements make sense if $R_x = \emptyset$, this occurs if and only if either $x \in T$ is in the interior of a chamber, or $x \notin T$ does not meet any hyperplane $H \in \mathcal{A}$. However, in this case we have $\langle B_x^K \rangle = \{0\}$ and the induced arrangement is the empty arrangement. This is not a problem, since in this case $W^\perp = V$ and $V_x = \{0\}$, but this case is somewhat trivial. Therefore the requirement $R_x \neq \emptyset$ is quite natural to make, and we will assume this from now on.

Another trivial case is $x = \{0\}$, in which case $R_x = R$, $\mathcal{A}_x = \mathcal{A}$ and $T_x = T$.

We can also give an exact criterion to when \mathcal{A}_x^π is a spherical arrangement.

COROLLARY 4.6. *Assume (\mathcal{A}, T) is a Tits arrangement of rank $r \geq 2$ and R a root system associated to (\mathcal{A}, T) . Let $x \in \bar{T}$. Then \mathcal{A}_x is finite if and only if $x \in T$.*

In particular, a simplicial arrangement is finite if and only if it is spherical.

PROOF. If $x \in T$, \mathcal{A}_x, R_x are finite by Proposition 4.4. So assume \mathcal{A}_x, R_x are finite and let $x \in \bar{T}$. Then also \mathcal{K}_x is a finite set, so let $K \in \mathcal{K}_x$ and by Lemma 2.26 we find $K' \in \mathcal{K}_x$ such that $d(K, K')$ is maximal. Any minimal gallery between chambers in \mathcal{K}_x is already in \mathcal{K}_x , since for $H \in \mathcal{A}$ with $D_H(K) = -D_H(K')$ we obtain $H \in \mathcal{A}_x$ by Lemma 2.7.

Let $\dim V_x = m$, then the m adjacent chambers K^1, \dots, K^m of K' exist, since \mathcal{A} is thin, and are all closer to K than to K' . Let K^i be H_i -adjacent to K , we can conclude $D_{H_i}(K) = -D_{H_i}(K')$ for $i = 1, \dots, m$. Let U be an open ball with center x , and let $U' = U \cap K', U'' = U \cap K$, then U' and U'' are open as well and contained in T . Take $y' \in U'$, so $y' = x + (y' - x) \in U'$ and $y'' := x - (y' - x)$ is in U'' . We find $x \in \sigma(y', y'') \subset T$, which shows our first statement.

The last statement is obtained by taking $x = 0_V$. □

We finish this section with two observations.

LEMMA 4.7. *The root system R is reduced if and only if R_x is reduced for every $x \in T$.*

PROOF. This follows immediately since R_x is constructed as a subset of R and $R = \bigcup_{x \in T} R_x$, note that for $\alpha \in R_x$ we find $\alpha^\perp \cap T \neq \emptyset$. □

LEMMA 4.8. *Let (\mathcal{A}, T) be a Tits arrangement and R a root system associated to (\mathcal{A}, T) . Let $x \in \bar{T}$ with $R_x \neq \emptyset$. Let $K, L \in \mathcal{K}_x$ be α_1^\perp -adjacent, and $B^K = \{\alpha_1, \dots, \alpha_n\}$, $B^L = \{\beta_1, \dots, \beta_n\}$ indexed compatible with B^K . Then $B_x^K \rightarrow B_x^L, \alpha_i \mapsto \beta_i$ is a bijection.*

PROOF. We know by Lemma 4.2 that $\langle B_x^K \rangle = \langle B_x^L \rangle$. Since B_x^K, B_x^L consist of linear independent vectors, we get $|B_x^K| = |B_x^L|$.

It remains to show that the map is well defined. For $\alpha_i \in B_x^K$ we find $x \in \beta_i^\perp$. We have $x \in \alpha_1^\perp$ and $x \in \alpha_i^\perp$. Now $\beta_i = \lambda_1 \alpha_1 + \lambda_i \alpha_i$ for some $\lambda_1, \lambda_i \in \mathbb{Z}$ by Lemma 3.29, so $\beta_i(x) = \lambda_1 \alpha_1(x) + \lambda_i \alpha_i(x) = 0$ and we are done. \square

4.2 – Restrictions of simplicial arrangements

In this section, we will discuss how a Tits arrangement (\mathcal{A}, T) induces a Tits arrangement on hyperplanes in \mathcal{A} . In the classical theory of hyperplane arrangements this is also called the restriction of an arrangement [cp. [22]].

DEFINITION 4.9. Let (\mathcal{A}, T) be a Tits arrangement, R be a root system associated to (\mathcal{A}, T) and let H be a hyperplane. Define

$$\pi_H^*: V^* \longrightarrow H^*, \quad \alpha \longmapsto \alpha|_H,$$

and set

$$R^H := \pi_H^*(R) \setminus (\{0\} \cup \{\alpha \in \pi_H^*(R) \mid \alpha^\perp \cap H \cap T = \emptyset\}).$$

We can also define the connected components of $H \setminus \bigcup_{H' \in \mathcal{A}^H} H'$ as \mathcal{K}^H .

REMARK 4.10. In the case $r = 0$ there exists no hyperplane which is not in the arrangement. In the case $r = 1$ the set \mathcal{A}^H is just the point $\{0\}$ or empty. Our statements will remain true in these cases, but most of the time they will be empty.

In particular we will examine the case where $H \in \mathcal{A}$, since otherwise we will not necessarily see an induced simplicial complex.

An interesting special case occurs for affine arrangements with imaginary root γ and $H = \gamma^\perp$, since in many cases this yields as R^H a root system associated to a spherical Tits arrangement.

LEMMA 4.11. *With notation as above, we find $\mathcal{A}^H = \{\alpha^\perp \cap H \mid \pi_H^*(\alpha) \in R^H\}$.*

PROOF. This follows immediately from the definition. \square

COROLLARY 4.12. *The elements $K' \in \mathcal{K}^H$ are simplicial cones in $H \cap T$.*

PROOF. If $r = 0$, the statement is empty, so let $r \geq 1$. Lemma 2.32 yields that for $K' \in \mathcal{K}_H$ the set $\overline{K'}$ is actually an maximal face of some chamber $K \in \mathcal{K}$. This is a simplicial cone by Remark 3.2. \square

The following observation is immediate from a geometric point of view, but necessary to point out.

LEMMA 4.13. *Let $H \leq V$ be an arbitrary hyperplane, and let $\alpha \in R$. Then we find $\pi_H^*(\alpha)^\perp = \alpha^\perp \cap H$, $\pi_H^*(\alpha)^+ = \alpha^+ \cap H$ and $\pi_H^*(\alpha)^- = \alpha^- \cap H$.*

PROOF. First note that $\pi_H^*(\alpha)(x) = \alpha(x)$ by definition of π_H^* . The equalities follow immediately by considering the cases $\alpha(x) = 0$ or $\alpha(x) > 0$. \square

The above lemma immediately yields the following statement.

COROLLARY 4.14. $\mathcal{A}^H = \{\alpha^\perp \leq H \mid \alpha \in R^H\}$.

LEMMA 4.15. *Let (\mathcal{A}, T) be a Tits arrangement, R a root system associated to (\mathcal{A}, T) , $K \in \mathcal{K}$, $H \in W^K$. Let $B := \pi_H^*(B^K) \setminus \{0\}$. Then*

- i) $H \cap \overline{K} = \overline{K'}$ for a unique $K' \in \mathcal{K}^H$,
- ii) $\langle \overline{K'} \cap \alpha^\perp \rangle = \alpha^\perp$ and $\alpha^\perp \cap K' = \emptyset$ for $\alpha \in B$,
- iii) $K' = \{x \in H \mid \alpha(x) > 0 \text{ for all } \alpha \in B\}$.

PROOF. Part i) is clear by the definition of \mathcal{K} and \mathcal{K}^H , since $H \cap \overline{K}$ is a unique maximal face of \overline{K} .

For the second statement assume $H = \beta^\perp$ for $\beta \in B^K$. We use that $\overline{K'}$ is a maximal face of \overline{K} . The maximal faces of $\overline{K'}$ are exactly the sets of the form $\overline{K} \cap H \cap \alpha^\perp = \overline{K'} \cap \alpha^\perp$ for $\alpha \in B^K \setminus \{\beta\}$, by Lemma 4.13 we obtain that the faces can also be written as $\overline{K'} \cap \alpha^\perp$ for $\alpha \in B$. As the maximal face $\overline{K'} \cap \alpha^\perp$ spans a hyperplane in H contained in α^\perp , we conclude $\langle \overline{K'} \cap \alpha^\perp \rangle = \alpha^\perp$ for $\alpha \in B$ and ii) holds.

Assertion iii) is a direct consequence of Lemma 4.13. \square

PROPOSITION 4.16. *Let (\mathcal{A}, T) be a k -spherical Tits arrangement for $k \geq 1$. Let R be a root system associated to (\mathcal{A}, T) and $H \in \mathcal{A}$. Then $(\mathcal{A}^H, T \cap H)$ is a $(k - 1)$ -spherical simplicial arrangement. If $(\mathcal{A}^H, T \cap H)$ is thin, it is a Tits arrangement associated to R^H .*

PROOF. The set R^H does not contain 0 by definition, and if $\alpha \in R^H$, we find $\alpha' \in R$ with $\alpha = \pi_H^*(\alpha')$, so $-\alpha' \in R$ and $-\alpha \in R^H$.

By Lemma 4.13 we know that $\mathcal{A}^H = \{\alpha^\perp \leq H \mid \alpha \in R^H\}$ and by definition we have $\alpha^\perp \cap H \cap T \neq \emptyset$. Furthermore we know that the connected components in \mathcal{K}^H are simplicial cones by Corollary 4.12.

Let $K' \in \mathcal{K}^H$ and $K \in \mathcal{K}$ such that K' is a face of K . Let $\beta \in B^K$ such that $\beta^\perp = H$. By ii) in Lemma 4.15 we find $W^{K'} = \{\pi_H^*(\alpha)^\perp \mid \alpha \in B^K \setminus \{\beta\}\}$, and with iii) in Lemma 4.15 we $B^{K'}$ is of the form $\{\lambda_\alpha \alpha \in R^H \mid \alpha \in \pi_H^*(B^K) \setminus \{0\}\}$, where $\lambda_\alpha \in \mathbb{R}_{>0}$. Hence \mathcal{A}^H is a simplicial arrangement in $T \cap H$, and $\mathcal{A}^H = \{\alpha^\perp \mid \alpha \in R^H\}$ holds. So if \mathcal{A}^H is thin, R^H is a root system for \mathcal{A}^H .

Now assume F is an m -simplex in the simplicial complex \mathcal{S}^H associated to the simplicial arrangement $(\mathcal{A}^H, T \cap H)$. Since this is a subset of \mathcal{S} , F meets $T \cap H$ if and only if F meets T , and therefore $(\mathcal{A}^H, T \cap H)$ is $k - 1$ -spherical if (\mathcal{A}, T) is k -spherical. \square

COROLLARY 4.17. *Assume that in the setting of Proposition 4.16 the pair $(\mathcal{A}^H, T \cap H)$ is a Tits arrangement and R^H a root system associated to $(\mathcal{A}^H, T \cap H)$. Let $K \in \mathcal{K}$ such that $H \in W^H$. Then the set $\pi_H^*(B^K) \setminus \{0\}$ is a basis of H^* .*

PROOF. By Proposition 4.16 the set $B^{K'}$ is a basis of H^* , by Lemma 4.15 iii) we find that the elements in $\pi_H^*(B^K) \setminus \{0\}$ are non zero scalar multiples of $B^{K'}$. \square

Appendices

A. The simplicial complex \mathcal{S}

We freely use the standard notions of simplicial complexes and refer in particular to [12, Appendix A2] and [24]. Notice that in this article, a chamber is a maximal simplex and a morphism of simplicial complexes is rank preserving on the set of simplices.

Fix a simplicial arrangement (\mathcal{A}, T) . We will show that \mathcal{S} is actually a simplicial complex and furthermore a chamber complex with set of chambers \mathcal{K} . We already showed in Remark 3.2 that the simplicial structure on a closed chamber \bar{K} is induced from the simplicial structure of \bar{S} , where S is an open simplex such that $K = \mathbb{R}_{>0}S$.

Recall the notion of the convex hull of a set.

DEFINITION AND REMARK A.1. Let (V, d) be a connected metric space. For an arbitrary subset $X \subset V$ the convex hull of X is the smallest convex set $Y \subset V$, such that $X \subset Y$. For a different approach, recall that the segment between $x, y \in M$ is

$$\sigma(x, y) := \{z \in M \mid d(x, z) + d(z, y) = d(x, y)\}.$$

This can be used to define

$$H(X) := \bigcup_{x, y \in X} \sigma(x, y).$$

We can then recursively define

$$H^{(0)}(X) := X,$$

$$H^{(n)}(X) := H(H^{(n-1)}(X)) \text{ for } 1 \leq n \in \mathbb{N}.$$

Then the convex hull of X is the set $\bigcup_{n \in \mathbb{N}} H^{(n)}(X)$.

For $V = \mathbb{R}^r$ and a linearly independent set $X \subset \mathbb{R}^r$ the convex hull of X can be more easily described as

$$\left\{ \sum_{x \in X} \lambda_x x \mid 0 \leq \lambda_x \leq 1 \text{ for all } x \in X, \sum_{x \in X} \lambda_x = 1 \right\}.$$

In this setting, we will refer to the set

$$\left\{ \sum_{x \in X} \lambda_x x \mid 0 < \lambda_x < 1 \text{ for all } x \in X, \sum_{x \in X} \lambda_x = 1 \right\}$$

as the *open convex hull* of X .

If $X \subset \mathbb{R}^r$ is linearly independent, then its convex hull S carries the structure of a simplex of rank $|X| - 1$. The vertex set is $V(S) = X$ and the simplicial structure coincides with the poset $(\mathcal{P}(X), \subseteq)$.

Moreover, if C is a basis of \mathbb{R}^r , and S is the open convex hull of C , the cone

$$\mathbb{R}_{>0}S := \{\lambda v \mid \lambda \in \mathbb{R}_{>0}, v \in S\}$$

coincides with K^C as defined in Remark 3.2. If F is a face of S , there exists a subset $C_F \subset C$ such that F is the convex hull of C_F . Then $\overline{\mathbb{R}_{>0}F}$ is a face \bar{K} in the simplicial complex \bar{K} . Therefore $\mathcal{P}(C)$, S , and $\overline{K^C}$ are all isomorphic simplices, via the isomorphisms

$$\begin{aligned} \underline{S} &\longrightarrow \bar{K}, & F &\longmapsto \overline{\mathbb{R}_{>0}F}, \\ \underline{S} &\longrightarrow \mathcal{P}(C), & F &\longmapsto C_F. \end{aligned}$$

LEMMA A.2. Let $C = \{v_1, \dots, v_r\}$, $C' = \{v'_1, \dots, v'_r\}$ be bases of V such that $K^C = K^{C'}$. Then, up to permutation, $\alpha_i = \lambda_i \alpha'_i$ for some $\lambda_i \in \mathbb{R}_{>0}$ for all $1 \leq i \leq r$.

The same holds for two bases B, B' of V^* such that $K^B = K^{B'}$.

PROOF. Let S be the open convex hull of C . Then $\mathbb{R}_{>0}S = K^C = K^{C'}$. One can choose positive scalars $\mu_i \in \mathbb{R}_{>0}$ such that $w_i = \mu_i v'_i \in \bar{S}$. Then $w_i = \sum_{k=1}^r \kappa_{ik} v_k$ with $\sum_{k=1}^r \kappa_{ik} = 1$, $0 \leq \kappa_{ik} \leq 1$. Let $C'' := \{w_1, \dots, w_r\}$, we find $K^C = K^{C''}$. Hence $v_i = \sum_{j=1}^r v_{ij} w_j$ with $\sum_{j=1}^r v_{ij} = 1$ and $0 \leq v_{ij} \leq 1$.

Therefore the matrix $M_{C''}^{C'}(\text{id})$ describing the base change is non-negative, and the same holds for its inverse $M_{C'}^{C''}(\text{id})$. It is well known (see Theorem 4.6 in Chapter 3 of [2] for example) that the inverse of a non-negative matrix is non-negative if and only if the matrix is monomial. Adding the fact that the sum of every column adds up to 1, we get that $M_{C''}^{C'}(\text{id})$ and $M_{C'}^{C''}(\text{id})$ are already permutation matrices.

The statement for $B, B' \subset V^*$ just follows from Remark 3.2 by considering the dual basis. □

REMARK A.3. For the following statement recall from 3.2 that for a basis B of V^* the cone $K^B \subset V$ is given by

$$K^B = \bigcap_{\alpha \in B} \alpha^+.$$

For a basis C of V the cone K^C is given by

$$K^C = \left\{ \sum_{v \in C} \lambda_v v \mid 0 < \lambda_v \right\}.$$

COROLLARY A.4. Let $K = K^C = K^B$ for a basis C of V and a basis B of V^* . Let $\beta \in V^*$. Then $\beta(v) \geq 0$ for all $v \in C$ if and only if $\beta \in \sum_{\alpha \in B} \mathbb{R}_{\geq 0} \alpha$. Likewise $\beta(v) \leq 0$ for all $v \in C$ if and only if $\beta \in -\sum_{\alpha \in B} \mathbb{R}_{\geq 0} \alpha$.

PROOF. Let $\beta = \sum_{\alpha \in B} \lambda_\alpha \alpha$. By Remark 3.2 we find that B is dual to C up to positive scalar multiples. Denote with $\alpha_v \in B$ the dual to $v \in C$. So $\beta = \sum_{v \in C} \lambda_v \alpha_v$. Applying this to C yields $\beta(v) = \lambda_v$ for all $v \in C$. This immediately yields both equivalences. □

LEMMA A.5. Let $F \in \mathcal{S}$. For every $H \in \mathcal{A}$, either $F \subset H$ or F is contained in a unique closed half-space of H , denoted by $\underline{D}_H(F)$. Furthermore $F \cap H \in \mathcal{S}$, and if $F \in \bar{K}$ for some $K \in \mathcal{K}$, then $F \cap H \in \bar{K}$.

PROOF. In case $F \subset H$, there is nothing to show, so assume $F \not\subset H$. Then the first statement is an immediate consequence of Corollary A.4 and the fact that the elements K are defined as connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$. Let $\alpha \in V^*$ such that $H = \alpha^\perp$ and $D_H(F) = \alpha^+$.

Let F be the convex hull of the vertices $\mathbb{R}_{>0}v_1, \dots, \mathbb{R}_{>0}v_k$, where $k \geq 1$ as we assume $F \not\subset H$. For the same reason we can assume $v_i, \dots, v_k \notin H$ for some $i < k$, and without loss of generality we can assume $v_1, \dots, v_{j-1} \in H$. Then $F \cap H$ is the convex hull of $\mathbb{R}_{>0}v_1, \dots, \mathbb{R}_{>0}v_{i-1}$. By Remark 3.2 \bar{K} is a simplicial complex, hence $F \cap H$ is a simplex in \mathcal{S} contained in \bar{K} . \square

Proof of Proposition 3.18. Let $F \in \mathcal{S}$, then $F = \bar{K} \cap \bigcap_{H \in \mathcal{A}_1} H$ for some $K \in \mathcal{K}$, $\mathcal{A}_1 \subset W^K$. In particular \underline{F} carries the structure of a simplex, since by Remark 3.2 \bar{K} is a simplex.

So let $F' \in \mathcal{S}$, we have to show $F \cap F' \in \mathcal{S}$. The intersection $F \cap F'$ is not empty, as it contains 0_V .

Assume $F' = \bar{K}' \cap \bigcap_{H \in \mathcal{A}_2} H$, for $K' \in \mathcal{K}$, $\mathcal{A}_2 \subset W^{K'}$. In the case $K = K'$ there is nothing to show, as \bar{K} is a simplicial complex by Remark 3.2. So from now on let $K \neq K'$.

Consider the case that $\mathcal{A}_2 = \emptyset$, we find $F' = \bar{K}'$. The set \bar{K}' can be written as $\bar{K}' = \bigcap_{H \in W^{K'}} \overline{D_H(K')}$ by Remark 3.2. By Lemma A.5 the intersection $F \cap \overline{D_H(K')}$ for $H \in W^{K'}$ is either F , in case that $D_H(K')$ contains F , or equals $F \cap H$. In both cases, it is again a simplex in \bar{K} . We can conclude that $F \cap \bar{K}' \in \mathcal{S}$ for every $F \in \mathcal{S}$.

Now let $\mathcal{A}_2 \neq \emptyset$, then $F \cap \bigcap_{H \in \mathcal{A}_2} H$ is a simplex in \bar{K} by Lemma A.5.

But $F \cap F'$ can be written as $F \cap \bigcap_{H \in \mathcal{A}_2} H \cap \bar{K}'$, using the previous part of the proof shows our claim.

B. \mathcal{S} as a gated, numbered chamber complex

In this section we recall definitions and basic facts regarding chamber complexes and type functions following [21] and [24].

DEFINITION B.1. Let Δ be a simplicial complex. We call Δ a *chamber complex* if it satisfies.

- 1) Every $A \in \Delta$ is contained in a chamber.

2) For two chambers $C, C' \in \Delta$ there is a sequence

$$C = C_0, \quad C_1, \quad \dots, \quad C_k = D$$

such that

$$\text{codim}_{C_{i-1}}(C_{i-1} \cap C_i) = \text{codim}_{C_i}(C_{i-1} \cap C_i) \leq 1$$

for $1 \leq i \leq k$.

We call a sequence as in 2) a *gallery* (from C to D) and k its *length*.

The first property is always satisfied if $\text{rk}(\Delta)$ is finite. In this situation it is easy to see (cp. [21, 1.3, p.15]) that every chamber of a chamber complex has the same rank. A consequence from 2) is that every element in a gallery is again a chamber. If C, D are chambers, note that for $\text{codim}_C(C \cap D) = \text{codim}_D(C \cap D) = 0$ implies $C = D$. Thus in a gallery the same chamber can occur repeatedly.

DEFINITION B.2. Let Δ be a chamber complex. For two chambers $C, D \in \Delta$ and $A \in \Delta$ with $A \leq C$ and $A \leq D$ we have $\text{codim}_C(A) = \text{codim}_D(A)$ (cp. [24, 1.3]). This allows us to define the *corank* of A as $\text{corank}(A) = \text{codim}_C(A)$ for any chamber C containing A . We call C and D *adjacent*, if $\text{corank}(C \cap D) = 1$.

For $A \in \Delta$ with $\text{corank}(A) = 1$, we call $\text{Cham}(\text{St}(A))$ a *panel* of Δ .

The complex Δ is meagre (resp. thin, firm, thick), if every panel contains at most two (exactly two, at least two, at least three) chambers.

A chamber complex Δ is *strongly connected*, if $\text{St}(A)$ is a chamber complex for every $A \in \Delta$.

DEFINITION B.3. Let Δ be a chamber complex and I be an index set. A *type function* of Δ is a morphism of chamber complexes $\tau: \Delta \rightarrow \mathcal{P}(I)$.

A *weak type function* of Δ is a family of type functions

$$(\tau_C: \underline{C} \rightarrow \mathcal{P}(I))_{C \in \text{Cham}(\Delta)}$$

which is compatible in the sense that $\tau_C|_{\underline{C} \cap \underline{D}} = \tau_D|_{\underline{C} \cap \underline{D}}$ for adjacent chambers C and D .

REMARK B.4. A type function τ with index set I induces a weak type function $(\tau|_{\underline{C}}: \underline{C} \rightarrow \mathcal{P}(I))_{C \in \text{Cham}(\Delta)}$. Conversely, we show in Lemma B.14 that a weak type function $(\tau_C: \underline{C} \rightarrow \mathcal{P}(I))_{C \in \text{Cham}(\Delta)}$ on a strongly connected chamber complex gives rise to a type function τ such that $\tau|_{\underline{C}} = \tau_C$.

The following lemma is immediate from the definitions.

LEMMA B.5. *Let Δ be a chamber complex, K, K' chambers. Define $d_\Delta(K, K')$ to be the length of a minimal gallery from K to K' . Then $(\text{Cham}(\Delta), d_\Delta)$ is a connected metric space.*

We recall the following proposition.

PROPOSITION B.6 (see [21, 1.5.3]). *Let Δ be a chamber complex, such that all sets $\text{Cham}(\text{St}(x))$ are gated for $F \in \Delta$ with $\text{codim}_\Delta(F) \in \{1, 2\}$. Let C be a chamber and τ be type function of \underline{C} , there exists a unique weak type function $(\tau_D)_{D \in \text{Cham}(\Delta)}$ such that $\tau_C = \tau$.*

>From now on, let $V = \mathbb{R}^r$ and (\mathcal{A}, T) be a simplicial arrangement, with the respective simplicial complex \mathcal{S} .

The set T itself is a metric space as a convex open subset of \mathbb{R}^r . We will for the rest of the chapter denote this metric as d_T , and the more metric on the chambers as $d_{\mathcal{S}}$.

Remember that \mathcal{A} is locally finite in T , which means that if we take a compact (with respect to d_T) subset $X \subseteq T$, the set $\text{sec}(X)$ is finite.

At this point we can justify the notion of the chambers \mathcal{K} .

LEMMA B.7. $\text{Cham}(\mathcal{S}) = \{\bar{K} \mid K \in \mathcal{K}\}$.

PROOF. By definition every maximal element in \mathcal{S} is of the form \bar{K} for some $K \in \mathcal{K}$. Assume $\bar{K} \subset \bar{K}'$ for $K, K' \in \mathcal{K}$, then \bar{K} is contained in some $H \in \mathcal{A}$ by Remark 3.2, which contradicts to the definition of \mathcal{K} . This proves $\text{Cham}(\mathcal{S}) = \{\bar{K} \mid K \in \mathcal{K}\}$. □

To prove that \mathcal{S} is already a chamber complex, we need a bit more information about the distance between two chambers, depending on the number of hyperplanes separating them.

LEMMA B.8. *Let $\bar{K} \in \text{Cham}(\mathcal{S})$, then $\bar{K} = \bigcap_{H \in W^K} D_H(K)$.*

PROOF. This is a direct consequence of the fact that $\bar{K} = \{x \in T \mid \alpha(x) \geq 0 \text{ for all } \alpha \in B_K\}$ and $W^K = \{r^\perp \mid r \in B_K\}$ for some basis $B_K \subset V^*$. For $\alpha \in B_K$ with $\alpha^\perp = H \in W^K$ we find $D_H(K) = \{v \in T \mid \alpha(v) \geq 0\}$. □

LEMMA B.9. *Two chambers are adjacent in the chamber graph if and only if they are adjacent in the chamber complex. Hence minimal galleries in Γ correspond to minimal galleries in $\text{Cham}(\mathcal{S})$.*

DEFINITION AND REMARK B.10. Two closed chambers \bar{K}, \bar{L} are adjacent, if $\text{codim}_{\bar{K}}(\bar{K} \cap \bar{L}) = 1 = \text{codim}_{\bar{L}}(\bar{K} \cap \bar{L})$. With respect to Lemma 2.18 we find that K, L are adjacent if and only if (K, L) is a unique minimal gallery from K to L .

DEFINITION B.11. For a simplex $F \in \mathcal{S}$ we set

$$\mathcal{K}_F := \{K \in \mathcal{K} \mid F \subset \bar{K}\}$$

and

$$\mathcal{A}_F := \{H \in \mathcal{A} \mid F \subset H\}.$$

With this notation $\text{Cham}(\text{St}(F)) = \{\bar{K} \mid K \in \mathcal{K}_F\}$ holds.

PROPOSITION B.12. *The simplicial complex \mathcal{S} is a strongly connected chamber complex.*

PROOF. The complex \mathcal{S} is a chamber complex, since every simplex is contained in a chamber and two chambers $K, L \in \mathcal{K}$ are connected by a gallery of length $|S(K, L)|$ by Lemma 2.21 and $|S(K, L)|$ is finite by Lemma 2.17. For two elements $K, L \in \mathcal{K}$ we can therefore define the distance $d_{\mathcal{S}}(K, L)$ as the length of a minimal gallery connecting K, L .

Let F be a simplex in \mathcal{S} , and consider the simplicial complex $\text{St}(F)$ with chambers \mathcal{K}_F . Let $K, L \in \mathcal{K}_F$ and assume $d_{\mathcal{S}}(K, L) = n = |S(K, L)| \geq 1$, so $K \neq L$. The fact that $K, L \in \mathcal{K}_F$ implies $F \in \bar{K} \cap \bar{L}$.

We need to show that there exists a gallery in $\text{St}(F)$ from K to L , which we do by induction on $d_{\mathcal{S}}(K, L)$. For $d_{\mathcal{S}}(K, L) = 1$ we have that K, L are adjacent. So let $d_{\mathcal{S}}(K, L) = n$ and assume $K' \in \mathcal{K}$ with the properties that K, K' are adjacent, $K \cap K' \subset H_1$ and $S(K, L) = \{H_1, \dots, H_n\}$. Then $H_1 \in \mathcal{A}_F$ by Lemma 2.7, and $F \in \bar{K} \cap H_1$ implies $F \subset \bar{K} \cap \bar{K}'$. In particular we get $\bar{K}' \in \mathcal{K}_F$ and by induction there exists a gallery from K' to L in \mathcal{K}_F , so we are done. \square

With respect to Lemma 2.28, the following Lemma yields that the sets $\text{Cham}(\text{St}(F))$ are gated for all $F \in \mathcal{S}$.

LEMMA B.13. *For every $F \in \mathcal{S}$, there exists an $x \in \bar{T}$ such that F is the minimal simplex in \mathcal{S} which contains x .*

PROOF. Let $F \in \mathcal{S}$, then there exists $K \in \mathcal{K}$ and $\mathcal{A}' \subset W^K$ such that $F = \bar{K} \cap \bigcap_{H \in \mathcal{A}'} H$. In particular, $F \subset \bigcap_{H \in \mathcal{A}'} H$. A point x as above exists, since F has nonempty interior with respect to $\bigcap_{H \in \mathcal{A}'} H$. \square

We now make use of an abstract result for type functions of chamber complexes. The following lemma is mentioned in [21] as an easy consequence, but we elaborate this result.

LEMMA B.14. *Let Δ be a strongly connected chamber complex. If Δ is weakly numbered, it is already numbered. In particular, if $(\tau_K)_{K \in \text{Cham}(\Delta)}$ is a weak type function, there exists a type function τ such that $\tau|_{\underline{K}} = \tau_K$ for all $K \in \text{Cham}(\Delta)$.*

PROOF. Let $\mathcal{C} = \text{Cham}(\Delta)$ and $(\tau_K)_{K \in \mathcal{C}}$ be a weak type function, so $\tau_K|_{\underline{K} \cap \underline{L}} = \tau_L|_{\underline{K} \cap \underline{L}}$ for all $K, L \in \mathcal{C}$.

Assume $F \in \Delta$, and let K, L be chambers with $F \in \underline{K}, \underline{L}$. Since Δ is strongly connected, $St(F)$ is connected, and we find $K, L \in St(F)$. Thus we find a gallery $K = K_0, K_1, \dots, K_{k-1}, K_m = L$ from K to L with all $K_i \in St(F)$, so in particular $F \in \underline{K}_i$ for all $0 \leq i \leq m$. Also K_{i-1} and K_i are adjacent for $1 \leq i \leq m$, therefore $\tau_{K_{i-1}}(F) = \tau_{K_i}(F)$ and inductively we obtain $\tau_K(F) = \tau_L(F)$.

This allows us to define $\tau(F) := \tau_K(F)$ for every simplex F and every chamber K containing F . By definition τ coincides with τ_K on all simplices F' contained in K , in particular $\tau|_{\underline{K}}$ is a type function and thus a morphism of chamber complexes.

Finally, τ itself is a morphism, since every simplex F is contained in a chamber K , and $\tau|_{\underline{K}}$ is a morphism. □

With respect to Theorem B.6, a direct consequence is the following theorem.

THEOREM B.15. *The complex \mathcal{S} has a type function. In particular, if we have a type function τ_K of a closed chamber \bar{K} , this extends uniquely to a type function of \mathcal{S} .*

REMARK B.16. The construction of the weak type function is actually quite simple. Begin with a chamber K and consider a type function τ of \bar{K} . Let L be adjacent to K such that $F = \bar{K} \cap \bar{L}$. Then set $\tau_L|_F = \tau|_F$. Let $i \in I$ be the unique index such that $i \notin \tau(F)$, then τ maps the vertex not contained in F to i , so τ_L must map the vertex v in \bar{L} not contained in F to i as well. So as every simplex $S \neq \emptyset$ is either contained in F or contains v , if S is contained in F then $\tau_L(S)$ is already defined, if it contains v set $\tau_d(S) = \tau(S \cap F) \cup \{i\}$. One can check that τ_L is a morphism of chamber complexes, and furthermore τ_L is the only possible type function of \bar{L} satisfying $\tau_L|_F = \tau|_F$.

In this way we can inductively construct type functions for all chambers with arbitrary distance to K . This construction works always, however being well defined arises as a problem. Given a chamber L with $d_{\mathcal{S}}(K, L) = n \geq 2$, there

may be two chambers K_1, K_2 with $d_{\mathcal{K}}(K, K_1) = d_{\mathcal{K}}(K, K_2) = n - 1$ and K_1, K_2 adjacent to L . Then L has induced type functions from K_1 as well as from K_2 . Now Theorem B.15 yields that these two induced type functions coincide, and thus the method gives us a weak type function of \mathcal{S} .

Since we introduced the notion of being spherical for arrangements, we recall the respective notion for chamber complexes.

DEFINITION B.17. We say that a complex Δ is *spherical* if it contains a pair of chambers K, K' such that $\text{proj}_P(K') \neq K$ for all panels P containing K . The chamber K' with such a property is called *opposite* to K .

Some properties of spherical complexes are the following, which can be found in [21].

LEMMA B.18. *Let Δ be a chamber complex.*

- i) *If Δ is firm, it is spherical if and only if it has finite diameter.*
- ii) *If Δ is meager and spherical, K, K' opposite chambers. Then $\text{Cham}(\Delta) = \sigma(K, K')$.*

To complete the proof of Proposition 3.26, we also need the following observation.

LEMMA B.19. *The simplicial complex \mathcal{S} is thin (resp. spherical) if and only if the simplicial arrangement (\mathcal{A}, T) is thin (resp. spherical).*

PROOF. THIN. Since $V \setminus H$ has two connected components, the complex \mathcal{S} is meager. It is thin if and only if for every chamber C and every wall $H \in W^C$ there exists a chamber C_H which is H -adjacent to C . In this case $\bar{C} \cap \bar{C}_H \subset T$, since T is convex. Then $\bar{C} \cap \bar{C}_H \subset H$, therefore H meets T and is contained in \mathcal{A} .

SPHERICAL. Assume (\mathcal{A}, T) is spherical, then $T = V$ by definition and \mathcal{A} is finite. Hence also \mathcal{K} is finite and \mathcal{S} is thin, and therefore spherical by Lemma B.18.

Let \mathcal{S} be spherical. By definition we find two opposite chambers C and C' . As \mathcal{S} is meager by construction, we also know $\mathcal{K} = \sigma(C, C')$. In particular $\mathcal{S}(C, C') = \mathcal{A}$, and \mathcal{K} is finite as well as \mathcal{A} .

Assume C has no i -adjacent chamber for an $i \in I$ and let P be the i -panel containing C , then $\text{proj}_P(D) = C$ for all $D \in \mathcal{K}$, a contradiction. Thus \mathcal{S} is also thin. By our previous argument therefore (\mathcal{A}, T) is thin.

Let $x \in \partial T$. Since T can not be written as a finite union of hyperplanes, we can assume that $x \notin H$ for all $H \in \mathcal{A}$. Thus take a neighborhood U of x and consider the chambers intersecting U . By taking a smaller U we can also assume that only a single chamber D intersects U . Thus there exists a wall of D not meeting T , a contradiction to \mathcal{A} being thin. Hence ∂T is empty and $T = V$ holds. \square

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