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On groups with the same character degrees as almost simple groups with socle Mathieu groups

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Abstract – Let G be a finite group and $\mathsf{cd}(G)$ denote the set of complex irreducible character degrees of G. In this paper, we prove that if G is a finite group and H is an almost simple group whose socle is a Mathieu group such that $\mathsf{cd}(G) = \mathsf{cd}(H)$, then there exists an abelian subgroup A of G such that G/A is isomorphic to H. In view of Huppert's conjecture (2000), we also provide some examples to show that G is not necessarily a direct product of A and H, and hence we cannot extend this conjecture to almost simple groups.

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1. Introduction

Let G be a finite group, and let $\mathsf{Irr}(G)$ be the set of all complex irreducible character degrees of G. Denote by $\mathsf{cd}(G)$ the set of character degrees of G, that is to say, $\mathsf{cd}(G) = \{\chi(1) | \chi \in \mathsf{Irr}(G)\}$. It is well-known that the complex group algebra $\mathbb{C}G$ determines the character degrees of G and their multiplicities.

There is growing interest in a question regarding the structure of G which can be recovered from the character degree set of G with or without multiplicity.

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It is well-known that $\mathsf{cd}(G)$ does not completely determine the structure of G in general. For example, the non-isomorphic groups D_8 and Q_8 not only have the same set of character degrees, but also share the same character table. The character degree set cannot be used to distinguish between solvable and nilpotent groups. For example, if G is either Q_8 or S_3, then $\mathsf{cd}(G) = \{1, 2\}$. Recently, Navarro [15] showed that the character degree set alone cannot determine the solvability of the group. Indeed, he constructed a finite perfect group H and a finite solvable group G such that $\mathsf{cd}(G) = \mathsf{cd}(H)$. It is also discovered by Navarro and Rizo [16] that there exists a finite perfect group and a finite nilpotent group with the same character degree set. Notice that in both examples, these finite perfect groups are not nonabelian simple. It remains open whether the complex group algebra can determine the solvability of the group or not (see Brauer's Problem 2 [5]).

However, the situation for simple groups and related groups is rather different. It has been proved recently that all *quasisimple* groups are uniquely determined up to isomorphism by their complex group algebras [3, 17, 18] in which a finite group G is quasisimple if G is perfect and $G/Z(G)$ is a nonabelian simple group. In the late 1990s, Huppert [11] posed a conjecture which asserts that every nonabelian simple group is essentially characterized by the set of its character degrees.

Conjecture 1.1 (Huppert). *Let G be a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.*

This conjecture has been verified for the alternating groups, many of the simple groups of Lie type [12, 19, 21, 22] and for all sporadic simple groups [1, 2, 10, 20]. Note that this conjecture does not extend to solvable groups, for example, Q_8 and D_8. We moreover cannot extend Huppert's conjecture to *almost simple groups*, that is to say, finite groups H such that $H_0 \leqslant H \leqslant \mathsf{Aut}(H_0)$ with H_0 nonabelian simple groups. In fact, there are four groups G of order 672 with the same character degrees as $\mathsf{Aut}(L_2(7)) = L_2(7) : \mathbb{Z}_2$. These groups are $L_2(7) : \mathbb{Z}_4$, $SL_2(7) : \mathbb{Z}_2$ (split) $SL_2(7).\mathbb{Z}_2$ (non split) and $\mathbb{Z}_2 \times (L_2(7) : \mathbb{Z}_2)$. We further observe that $G' = L_2(7)$ in all these cases. Although it is unfortunate that we cannot establish Huppert's conjecture for almost simple groups, we can prove the following result for finite groups whose character degrees are the same as those of almost simple groups associated to the Mathieu groups:

Theorem 1.2. *Let G be a finite group, and let H be an almost simple group whose socle H_0 is one of the Mathieu groups. If $\mathsf{cd}(G) = \mathsf{cd}(H)$, then $G' \cong H_0$ and $G/Z(G)$ is isomorphic to H.*

In order to prove Theorem 1.2, we establish the following steps which Huppert introduced in [11]. Let H be an almost simple group with socle H_0, and let G be a group with the same character degrees as H. Then we show that

Step 1. $G' = G''$;

Step 2. if G'/M is a chief factor of G, then G'/M is isomorphic to H_0;

Step 3. if $\theta \in \mathsf{Irr}(M)$ with $\theta(1) = 1$, then $I_{G'}(\theta) = G'$ and so $M = M'$;

Step 4. $M = 1$ and $G' \cong H_0$;

Step 5. $G/Z(G)$ is isomorphic to H.

2. Preliminaries

In this section, we present some useful results to prove Theorem 1.2. We first establish some definitions and notation.

Throughout this paper all groups are finite. Recall that a group H is said to be an almost simple group with socle H_0 if $H_0 \leqslant H \leqslant \mathsf{Aut}(H_0)$, where H_0 is a nonabelian simple group. For a positive integer n, $\pi(n)$ denotes the set of all prime divisors of n. If G is a group, we will write $\pi(G)$ instead of $\pi(|G|)$. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the *inertia group* of θ in G is denoted by $I_G(\theta)$ and is defined by $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i \chi_i$, where each χ_i is an irreducible character of G and e_i is a nonnegative integer, then those χ_i with $e_i > 0$ are called the irreducible constituents of χ. The set of all irreducible constituents of θ^G is denoted by $\mathsf{Irr}(G|\theta)$. All further notation and definitions are standard and could be found in [9, 13].

LEMMA 2.1 ([9, Theorems 19.5 and 21.3]). *Suppose $N \trianglelefteq G$ and $\chi \in \mathsf{Irr}(G)$.*

(a) *If $\chi_N = \theta_1 + \theta_2 + \cdots + \theta_k$ with $\theta_i \in \mathsf{Irr}(N)$, then k divides $|G/N|$. In particular, if $\chi(1)$ is prime to $|G/N|$, then $\chi_N \in \mathsf{Irr}(N)$.*

(b) *(Gallagher's Theorem) If $\chi_N \in \mathsf{Irr}(N)$, then $\chi\psi \in \mathsf{Irr}(G)$ for all $\psi \in \mathsf{Irr}(G/N)$.*

LEMMA 2.2 ([9, Theorems 19.6 and 21.2]). *Suppose $N \trianglelefteq G$ and $\theta \in \mathsf{Irr}(N)$. Let $I = I_G(\theta)$.*

(a) *If $\theta^I = \sum_{i=1}^{k} \phi_i$ with $\phi_i \in \mathsf{Irr}(I)$, then $\phi_i^G \in \mathsf{Irr}(G)$. In particular, $\phi_i(1)|G : I| \in \mathsf{cd}(G)$.*

(b) *If θ extends to $\psi \in \mathsf{Irr}(I)$, then $(\psi\tau)^G \in \mathsf{Irr}(G)$ for all $\tau \in \mathsf{Irr}(I/N)$. In particular, $\theta(1)\tau(1)|G : I| \in \mathsf{cd}(G)$.*

(c) *If $\rho \in \mathsf{Irr}(I)$ such that $\rho_N = e\theta$, then $\rho = \theta_0\tau_0$, where θ_0 is a character of an irreducible projective representation of I of degree $\theta(1)$ and τ_0 is a character of an irreducible projective representation of I/N of degree e.*

Lemma 2.3 ([20, Lemma 3]). *Let G/N be a solvable factor group of G minimal with respect to being nonabelian. Then two cases can occur.*

(a) *G/N is an r-group for some prime r. Hence there exists $\psi \in \mathsf{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. If $\chi \in \mathsf{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi\tau \in \mathsf{Irr}(G)$ for all $\tau \in \mathsf{Irr}(G/N)$.*

(b) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Then $f = |G : F| \in \mathsf{cd}(G)$ and $|F/N| = r^a$ for some prime r, and a is the smallest integer such that $r^a \equiv 1 \mod f$. If $\psi \in \mathsf{Irr}(F)$, then either $f\psi(1) \in \mathsf{cd}(G)$ or r^a divides $\psi(1)^2$. In the latter case, r divides $\psi(1)$.*

(1) *If no proper multiple of f is in $\mathsf{cd}(G)$, then $\chi(1)$ divides f for all $\chi \in \mathsf{Irr}(G)$ such that $r \nmid \chi(1)$, and if $\chi \in \mathsf{Irr}(G)$ such that $\chi(1) \nmid f$, then $r^a \mid \chi(1)^2$.*

(2) *If $\chi \in \mathsf{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathsf{cd}(G)$, then either f divides $\chi(1)$ or r^a divides $\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G, then $f = \chi(1)$ or $r^a \mid \chi(1)^2$.*

Lemma 2.4. *Let G be a finite group.*

(a) *If G is a nonabelian simple group, then there exists a nontrivial irreducible character φ of G that extends to $\mathsf{Aut}(G)$.*

(b) *If N is a minimal normal subgroup of G so that $N \cong S^k$, where S is a nonabelian simple group, and $\varphi \in \mathsf{Irr}(S)$ extends to $\mathsf{Aut}(S)$, then $\varphi^k \in \mathsf{Irr}(N)$ extends to G.*

Proof. Part (a) follows from [4, Theorems 2–4]. To prove part (b) see [4, Lemma 5]. □

Lemma 2.5 ([11, Lemma 6]). *Suppose that $M \trianglelefteq G' = G''$ and for every $\lambda \in \mathsf{Irr}(M)$ with $\lambda(1) = 1$, $\lambda^g = \lambda$ for all $g \in G'$. Then $M' = [M, G']$ and $|M/M'|$ divides the order of the Schur multiplier of G'/M.*

Lemma 2.6 ([14, Theorem D]). *Let N be a normal subgroup of a finite group G and let $\varphi \in \mathsf{Irr}(N)$ be G-invariant. Assume that $\chi(1)/\varphi(1)$ is odd, for all $\chi(1) \in \mathsf{Irr}(G|\varphi)$. Then G/N is solvable.*

3. Degree properties of almost simple groups with socle sporadic

In this section, we determine all finite simple groups whose irreducible character degrees divide some irreducible character degrees of almost simple groups whose socles are the Mathieu groups M_{12} and M_{22}.

PROPOSITION 3.1. *Let H be an almost simple group whose socle is M_{12} or M_{22}. Suppose that S is a finite nonabelian simple group whose irreducible character degrees divide some degrees of H.*

(a) *If $H_0 = M_{12}$, then S is isomorphic to A_5, A_6, $L_2(11)$, M_{11} or M_{12}.*

(b) *If $H_0 = M_{22}$, then S is isomorphic to A_5, A_6, A_7, $L_2(7)$, $L_2(8)$ or M_{22}.*

PROOF. Suppose that H is an almost simple group with socle a sporadic simple group $H_0 = M_{12}$ or M_{22}. Suppose also that S is a simple group whose degrees divide some degrees of H. Then the prime divisors of the degrees of S are exactly those primes dividing $|S|$. Therefore, $\pi(S) \subseteq \pi(H)$. All such possible simple groups S are listed in [23], and we only need to check if the degrees of S divide some degrees of H by [7, 8].

Suppose first $H_0 = M_{12}$. Then $\pi(S) \subseteq \{2, 3, 5, 11\}$, and so by [23], S is isomorphic to one of the simple groups A_5, A_6, $L_2(11)$, $U_5(2)$, $S_4(3)$, M_{11} and M_{12}. Note that $U_5(2)$ and $S_4(3)$ have degrees 220 and 81, respectively. Therefore, S is isomorphic to A_5, A_6, $L_2(11)$, M_{11} or M_{12}.

Suppose now $H_0 = M_{22}$. Then $\pi(S) \subseteq \{2, 3, 5, 7, 11\}$. Now we apply [23]. Then S is isomorphic to one of the simple groups in $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where

$$
\begin{aligned}
\mathcal{A}_1 &:= \{A_5, A_6, A_7, L_2(7), L_2(8), M_{22}\},\\
\mathcal{A}_2 &:= \{A_8, L_3(4), L_2(49), U_3(3), S_4(7), M_{11}\},\\
\mathcal{A}_3 &:= \{A_9, A_{10}, A_{11}, A_{12}, L_2(11), U_3(5), U_4(3), U_5(2), U_6(2),\\
&\qquad S_4(3), S_6(2), O_8^+(2), M_{12}, McL, HS, J_2\}.
\end{aligned}
$$

If $S \in \mathcal{A}_2$, then S has a degree divisible by 25, 27, 44, 49 or 64, which is a contradiction. If $S \in \mathcal{A}_3$, then S has a degree which is divisible by 12, which is also a contradiction. Therefore $S \in \mathcal{A}_1$ as claimed. □

4. Groups with socle M_{12}

In this section, we prove Theorem 1.2 for almost simple group H whose socle is $H_0 := M_{12}$. By [10], Theorem 1.2 is proved when $H = H_0 = M_{12}$. Therefore, we only need to deal with the case where $H := M_{12} : 2$. The following result follows from [7, pp. 31-33] and a straightforward calculation.

Lemma 4.1. *Let $H_0 := M_{12}$ and $H := M_{12} : 2$. If K is a maximal subgroup of H_0 whose index in H_0 divides some degrees $\chi(1)$ of H, then one of the following occurs*:

(i) $K \cong M_{11}$ *and* $\chi(1)/|H_0 : K|$ *divides* $2 \cdot 5$ *or* $2^2 \cdot 3$;

(ii) $K \cong M_{10} : 2$ *and* $\chi(1)/|H_0 : K| = 1$;

(iii) $K \cong L_2(11)$ *and* $\chi(1)/|H_0 : K| = 1$.

4.1 – *Proof of Theorem 1.2 for $M_{12} : 2$*

As noted before, by [10], we may assume that $H := M_{12} : 2$. We further assume that G is a finite group with $\mathsf{cd}(G) = \mathsf{cd}(M_{12} : 2)$. The proof of Theorem 1.2 follows from the following lemmas.

Lemma 4.2. $G' = G''$.

Proof. Assume the contrary. Then there is a normal subgroup N of G, where N is maximal such that G/N is a nonabelian solvable group. Now we apply Lemma 2.3 and we have one of the following cases.

(a) G/N is a r-group with r prime. In this case, G/N has an irreducible character ψ of degree $r^b > 1$, and so does G. Since $M_{12} : 2$ has an irreducible character of degree 32, we conclude that $r = 2$. Let now $\chi \in \mathsf{Irr}(G)$ with $\chi(1) = 99$. Then Lemma 2.1(a) implies that $\chi_N \in \mathsf{Irr}(N)$, and so by Lemma 2.1(b), G has an irreducible character of degree $99\psi(1)$, which is a contradiction.

(b) G/N is a Frobenius group with kernel F/N. Then $|G : F| \in \mathsf{cd}(G)$ divides $r^a - 1$, where $|F/N| = r^a$. Let $|G : F| \in \{2\cdot 11, 5\cdot 11\}$ and $\chi(1) = 2^4\cdot 3^2$. Then Lemma 2.3(b) implies that r^a divides $\chi(1)^2 = 2^8 \cdot 3^4$, and it is impossible as $|G : F|$ does not divide $r^a - 1$, for every divisor r^a of $2^8 \cdot 3^4$. Let now $|G : F| \notin \{2 \cdot 11, 5 \cdot 11\}$. Then no proper multiple of $|G : F|$ is in $\mathsf{cd}(G)$. If $r = 2$, then by Lemma 2.3(b.1), both $3^2 \cdot 11$ and $3^2 \cdot 5$ must divide $|G : F|$, which is a contradiction. If $r = 3$, then by Lemma 2.3(b.1), $|G : F|$ is divisible by 2^5 and $2^4 \cdot 11$, which is a contradiction. Similarly, if $r \neq 2, 3$, then 2^5 and $2^4 \cdot 3^2$ divide $|G : F|$, which is a contradiction. □

Lemma 4.3. *Let G'/M be a chief factor of G. Then $G'/M \cong M_{12}$.*

Proof. Suppose $G'/M \cong S^k$, where S is a nonabelian simple group for some positive integer k. Since S is a finite nonabelian simple group whose irreducible character degrees divide some degrees of $M_{12} : 2$, by Proposition 3.1(a), S is isomorphic to one of the groups A_5, A_6, M_{11}, M_{12} or $L_2(11)$. If S is isomorphic to one of the groups A_5, A_6, M_{11}, M_{12}, $L_2(11)$, then $k = 1$ as G has no degree divisible by 5^2. Let now $L := G/M$ and $C := C_L(S)$. Then $T := SC$, and since $S \cap C = 1$, we have that

$$S \cong \frac{S}{S \cap C} \cong \frac{SC}{C} = \frac{T}{C} \trianglelefteq \frac{L}{C} \lesssim \mathsf{Aut}(S).$$

Therefore, L/T may be viewed as a subgroup of $\mathsf{Out}(S)$. Assume now that $\psi \in \mathsf{Irr}(S)$ and $\chi \in \mathsf{Irr}(L)$ is an irreducible constituent of ψ^L. Since $T \cong S \times C$, $\psi \in \mathsf{Irr}(T)$, and since $T \trianglelefteq L$, Lemma 2.1(a) implies that $t = \chi(1)/\psi(1)$ divides $|L/T|$. Note that $|L/T|$ divides $|\mathsf{Out}(S)|$. Then t divides $|\mathsf{Out}(S)|$. Therefore, G must have an irreducible character of degree $t\psi(1)$ where t divides $|\mathsf{Out}(S)|$. Applying this argument to the case where S is isomorphic to A_5 or A_6 when $\psi(1) = 5$, we conclude that G has a character of degree at most 20, which is a contradiction. Similarly, in the case where S is isomorphic to M_{11} or $L_2(11)$, G has a character of degree at most 20, which is a contradiction. Therefore G'/M is isomorphic to M_{12}. □

Lemma 4.4. *Let $\theta \in \mathsf{Irr}(M)$ with $\theta(1) = 1$. Then $I_{G'}(\theta) = G'$ and $M = M'$.*

Proof. Suppose $I = I_{G'}(\theta) < G'$. By Lemma 2.2, we have $\theta^I = \sum_{i=1}^k \phi_i$ where $\phi_i \in \mathsf{Irr}(I)$ for $i = 1, 2, \ldots, k$. Let U/M be a maximal subgroup of $G'/M \cong M_{12}$ containing I/M and set $t := |U : I|$. It follows from Lemma 2.2(a) that $\phi_i(1)|G' : I| \in \mathsf{cd}(G')$, and so $t\phi_i(1)|G' : U|$ divides some degrees of G. Then $|G' : U|$ must divide some character degrees of G, and hence by Lemma 4.1 one of the following holds.

(i) Suppose $U/M \cong M_{11}$. Then $t\phi_i(1)$ divides $2 \cdot 5$ or $2^2 \cdot 3$. If $t = 1$, then $I/M \cong M_{11}$. Since M_{11} has trivial Schur multiplier, it follows that θ extends to $\theta_0 \in \mathsf{Irr}(I)$, and so by Lemma 2.2(b) $(\theta_0\tau)^{G'} \in \mathsf{Irr}(G')$, for all $\tau \in \mathsf{Irr}(I/M)$. For $\tau(1) = 55 \in \mathsf{cd}(M_{11})$, it turns out that $12 \cdot 55 \cdot \theta_0(1)$ divide some degrees of G, which is a contradiction. Therefore, $t \neq 1$, and hence the index of a maximal subgroup of $U/M \cong M_{11}$ containing I/M must divide $2 \cdot 5$ or $2^2 \cdot 3$. This implies that $t\phi_i(1)$ divides $2^2 \cdot 3$ and $I/M \cong L_2(11)$. In particular, $\phi_i(1) = 1$. Thus θ extends to a ϕ_i, and so by Lemma 2.2(b), $144\tau(1) \in \mathsf{cd}(G')$, for all $\tau \in \mathsf{Irr}(I/M)$. This leads us to a contradiction by taking $\tau(1) = 10 \in \mathsf{cd}(L_2(11))$.

(ii) Suppose $U/M \cong M_{10} : 2$. In this case $t = 1$, or equivalently, $I/M = U/M \cong M_{10} : 2$. Moreover, $\phi_i(1) = 1$, for all i. Then θ extends to $\phi_i \in \mathsf{Irr}(I)$, and so by Lemma 2.2(b), $66\tau(1)$ divides some degrees of G, for $\tau(1) = 10$, which is a contradiction.

(iii) Suppose $U/M \cong L_2(11)$ and $t = \phi_i(1) = 1$, for all i. Then $I/M \cong L_2(11)$, and so θ extends to $\phi_i \in \mathsf{Irr}(I)$. Thus $144\tau(1) \in \mathsf{Irr}(G')$, for all $\tau \in \mathsf{Irr}(I/M)$. This is impossible by taking $\tau(1) = 10$.

Therefore, $I_{G'}(\theta) = G'$. By Lemma 2.5, we have that $|M/M'|$ divides the order of Schur multiplier of $G'/M \cong M_{12}$ which is 2. If $|M/M'| = 2$, then G'/M' is isomorphic to $2 \cdot M_{12}$ which has a character of degree 32 [7, p. 33]. Therefore M_{12} must have a degree divisible by 32, which is a contradiction. Hence $|M/M'| = 1$, or equivalently, $M = M'$. □

Lemma 4.5. *The subgroup M is trivial, and hence $G' \cong M_{12}$.*

Proof. By Lemmas 4.3 and 4.4, we have that $G'/M \cong M_{12}$ and $M = M'$. Suppose that M is nonabelian, and let $N \leqslant M$ be a normal subgroup of G' such that M/N is a chief factor of G'. Then $M/N \cong S^k$, for some nonabelian simple group S. It follows from Lemma 2.4 that S possesses a nontrivial irreducible character φ such that $\varphi^k \in \mathsf{Irr}(M/N)$ extends to G'/N. By Lemma 2.1(b), we must have $\varphi(1)^k\tau(1) \in \mathsf{cd}(G'/N) \subseteq \mathsf{cd}(G')$, for all $\tau \in \mathsf{Irr}(G'/M)$. Now we can choose $\tau \in G'/M$ such that $\tau(1)$ is the largest degree of M_{12}, and since φ is nontrivial, $\varphi(1)^k\tau(1)$ divides no degree of G, which is a contradiction. Therefore, M is abelian, and since $M = M'$, we conclude that $M = 1$. □

Lemma 4.6. *The group $G/Z(G)$ is isomorphic to $M_{12} : 2$.*

Proof. Set $A := C_G(G')$. Since $G' \cap A = 1$ and $G'A \cong G' \times A$, it follows that $G' \cong G'A/A \trianglelefteq G/A \leqslant \mathsf{Aut}(G')$. By Lemma 4.5, we have $G' \cong M_{12}$, and so we conclude that G/A is isomorphic to M_{12} or $M_{12} : 2$. In the case where G/A is isomorphic to M_{12}, we must have $G \cong A \times M_{12}$. This is impossible as $32 \in \mathsf{cd}(G)$ but M_{12} has no character of degree 32. Therefore, G/A is isomorphic to $M_{12} : 2$. Since $[G, A] \leqslant G' \cap A = 1$, we conclude that $A = Z(G)$ as claimed. □

5. Groups with socle M_{22}

In this section, we prove Theorem 1.2 for almost simple group H whose socle is $H_0 := M_{22}$. Note that Theorem 1.2 is proved for $H = H_0 = M_{22}$, see [10]. Therefore, we only need to focus on case where $H := M_{22} : 2$. The following result follows from [7, pp. 39–41] and a straightforward calculation.

Lemma 5.1. *Let $H_0 := M_{22}$ and $H := M_{22} : 2$. If K is a maximal subgroup of H_0 whose index in H_0 divides some degrees $\chi(1)$ of H, then one of the following occurs*:

(i) $K \cong L_3(4)$ *and* $\chi(1)/|H_0 : K|$ *divides* 7;

(ii) $K \cong 2^4 : S_5$ *and* $\chi(1)/|H_0 : K| = 1$;

(iii) $K \cong 2^4 : A_6$ *and* $\chi(1)/|H_0 : K|$ *divides* 2, 3 *or* 5.

5.1 – *Proof of Theorem* 1.2 *for* $M_{22} : 2$

Theorem 1.2 is true for the Mathieu group M_{22} by [10]. It remains to assume that $H := M_{22} : 2$. In what follows assume that G is a finite group with $\mathrm{cd}(G) = \mathrm{cd}(M_{22} : 2)$. The proof follows from the following lemmas.

Lemma 5.2. $G' = G''$.

Proof. Assume the contrary. Then there is a normal subgroup N of G where N is a maximal such that G/N is a nonabelian solvable group. Since $M_{22} : 2$ does not have a irreducible character of prime power degree, by Lemma 2.3, G/N is a Frobenius group with kernel F/N of order r^a. If $|G : F| \in \{3 \cdot 7, 5 \cdot 11\}$, then Lemma 2.2(b) implies that r^a divides $\chi^2(1) = 2^2 \cdot 7^2 \cdot 11^2$, and it is impossible as $|G : F|$ does not divide $r^a - 1$. If $|G : F| \notin \{3 \cdot 7, 5 \cdot 11\}$, then no proper multiple of $|G : F|$ is in $\mathrm{cd}(G)$. By the same argument as in Lemma 4.2, as G has irreducible characters of degrees $5 \cdot 7 \cdot 11$, $2^4 \cdot 5 \cdot 7$, $3 \cdot 7 \cdot 11$, $2 \cdot 7 \cdot 11$, $2 \cdot 3 \cdot 5 \cdot 7$ and $2^4 \cdot 5 \cdot 7$, we conclude by Lemma 2.3(b.1) that $r \neq 3$, 5, and 11. Again applying Lemma 2.3(b.1), $3^2 \cdot 11$ and $5 \cdot 11$ divide $|G : F|$, which is impossible. □

Lemma 5.3. *Let G'/M be a chief factor of G. Then $G'/M \cong M_{22}$.*

Proof. Suppose $G'/M \cong S^k$, where S is a nonabelian simple group for some positive integer k. Since G has no degree divisible by 25 and 49, by Proposition 3.1(b), the group $G'/M \cong S$ is isomorphic to A_5, A_6, A_7, $L_2(7)$, $L_2(8)$ or M_{22}. We now apply the same argument as in Lemma 4.3. Assume that S is isomorphic to one of the simple groups as in the first column of Table 1. Then by [7], G'/M has a character ψ of degree as in the third column of the same Table 1. If χ is an irreducible constituent of $\psi^{G/M}$, then $\chi(1) = t\psi(1)$, where t divides $|\mathrm{Out}(S)|$. Consequently, G has a character of degree at most d as in the forth column of Table 1, which is a contradiction.

Table 1. The triples (S, ψ, d) in Lemma 5.3

S	$\lvert\mathrm{Out}(S)\rvert$	$\psi(1)$	d
A_5	2	5	10
A_6	4	5	20
A_7, $L_2(7)$	2	6	12
$L_2(8)$	3	8	24

Therefore $G'/M \cong M_{22}$. $\square$

Lemma 5.4. *If $\theta \in \mathrm{Irr}(M)$, then $I_{G'}(\theta) = G'$ and $M = M'$.*

Proof. Suppose $I := I_{G'}(\theta) < G'$. By Lemma 2.2 we have $\theta^I = \sum_{i=1}^{k} \phi_i$ where $\phi_i \in \mathrm{Irr}(I)$ for $i = 1, 2, \dots, k$. Let U/M be a maximal subgroup of $G'/M \cong M_{22}$ containing I/M and set $t := |U : I|$. It follows from Lemma 2.2(a) that $\phi_i(1)|G' : I| \in \mathrm{cd}(G')$, and so $t\phi_i(1)|G' : U|$ divides some degrees of G. Then $|G' : U|$ must divide some character degrees of G, and hence by Lemma 5.1 one of the following holds.

(i) Suppose $U/M \cong L_3(4)$. Then, for each i, $t\phi_i(1)$ divides 7. As $U/M \cong L_3(4)$ does not have any subgroup of index 7, by [7, p. 23], so $t = 1$ and $I/M \cong U/M \cong L_3(4)$ and $\phi_i(1)$ divides 7. If $\phi_i(1) = 1$, then θ extends to ϕ_i, and so by Lemma 2.2(b), $(\phi_i\tau)^{G'} \in \mathrm{Irr}(G')$, for all $\tau \in \mathrm{Irr}(I/M)$. If $\tau(1) = 64 \in \mathrm{cd}(L_3(4))$, then $22\tau(1) = 2^7 \cdot 11$ divides some degrees of G, which is a contradiction. Hence $\phi_i(1) = 7$, for all i. Then $\phi_{i_M} = e_i\theta$, where $e_i \neq 1$ is the degree of a projective representation of $I/M \cong L_3(4)$, and it is impossible by [7, p. 24].

(ii) Suppose $U/M \cong 2^4 : S_5$. Then $I/M \cong 2^4 : S_5$ and $\phi_i(1) = 1$, for all i. Thus θ extends to ϕ_i in I. It follows from Lemma 2.2(b) that $\tau(1)|G' : I|$ divides some character degrees of G, for all $\tau \in I/M$. This is impossible by taking $\tau(1) = 4$.

(iii) Suppose $U/M \cong 2^4 : A_6$. In this case, $t\phi_i(1)$ divides 2, 3 or 5. It follows from [6] that U/M has no maximal subgroup of index 2, 3 and 5, and this implies that $I = U$. Therefore, $I/M \cong 2^4 : A_6$, that is to say, I/M has an abelian subgroup A/M of order 2^4 such that $I/A \cong A_6$.

Let now $\lambda \in \mathrm{Irr}(A|\theta)$, and write $\lambda^I = \sum f_i\mu_i$. Since $A \trianglelefteq I$, the degree $\mu_i(1)$ divides 2, 3 or 5, for all i. Since the index of a maximal subgroup of $I/A \cong A_6$ is at least 6, λ is I-invariant, and so $\mu_{i_A} = f_i\lambda$, for all i. If $f_i = 1$ for some i, then

λ extends to $\lambda_0 \in \mathsf{Irr}(I)$, and so by Lemma 2.2(b), $\lambda_0\tau$ is an irreducible character of λ^I, for all $\tau \in \mathsf{Irr}(I/A)$, and so $\lambda_0(1)\tau(1) = \tau(1)$ divides 2, 3, or 5. This is impossible as we could take $\tau(1) = 8 \in \mathsf{cd}(A_6)$. Therefore, $f_i > 1$, for all i. Moreover, we know from Lemma 2.2(c) that each f_i is the degree of a nontrivial proper irreducible projective representation of A_6, by [7, p. 5], we observe that $f_i \in \{3, 5\}$. This shows that $\mu(1)/\lambda(1)$ is odd, for all $\mu \in \mathsf{Irr}(I|A)$, and so by Lemma 2.6, the group $I/A \cong A_6$ is solvable, which is a contradiction.

This shows $I_{G'}(\theta) = G'$. By Lemma 2.5, we have that $|M/M'|$ divides the order of Schur multiplier of $G'/M \cong M_{22}$ which is 12. If $|M/M'| \neq 1$, then G has an irreducible character of degree divisible by one the degrees in the second row of Table 2, which is a contradiction. Therefore, $M = M'$. $\square$

Table 2. Some character degrees of G'/M' in Lemma 5.4

G'/M'	$2 \cdot M_{22}$	$3 \cdot M_{22}$	$4 \cdot M_{22}$	$6 \cdot M_{22}$	$12 \cdot M_{22}$
Degree	440	384	440	384	384

Lemma 5.5. *The subgroup M is trivial, and hence $G' \cong M_{22}$.*

Proof. Assume that M is nonabelian, and let M/N be a chief factor of G'. Then $M/N \cong S^k$ for some nonabelian simple group S. We now apply Lemmas 2.4 and 2.1(b) and find a nontrivial irreducible character φ such that $\varphi(1)^k\tau(1) \in \mathsf{cd}(G'/N) \subseteq \mathsf{cd}(G')$, for all $\tau \in \mathsf{Irr}(G'/M)$, this is impossible if $\tau(1)$ is the largest degree of M_{22}. Therefore, M is abelian, and hence we are done. $\square$

Lemma 5.6. *The group $G/Z(G)$ is isomorphic to $M_{22} : 2$.*

Proof. Set $A := C_G(G')$. By the same argument as in Lemma 4.6, we conclude that $A = Z(G)$ and $G/Z(G)$ is isomorphic to M_{22} or $M_{22} : 2$, and since $560 \in \mathsf{cd}(G)$ but M_{22} has no character of degree 560, the group $G/Z(G)$ is isomorphic to $M_{22} : 2$. $\square$

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