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Positive solutions of singular semilinear elliptic problems in NTA-cones

Lassaad Gharbi (∗) – Mohamed Amine Ben Boubaker (∗∗)

Abstract – We study the existence, the uniqueness and the asymptotic behavior of positive solutions of the nonlinear equation

$$\Delta v + f(., v) = 0,$$

in an NTA-cone in $\mathbb{R}^n (n \geq 3)$, when a positive Borel measurable function $f(.,.)$ is continuous and non-increasing with respect to the second variable and satisfies a certain condition related to a Kato class.

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1. Introduction

We work in the Euclidean space $\mathbb{R}^n$, where $n \geq 3$. By Γ, we denote an NTA-cone of vertex 0 (see [14] for the definition), and by $G_\Gamma(x, y)$, the Green function for the Laplacian in Γ. We write $\delta_\Gamma(z)$ for the distance from z in Γ to the Euclidean boundary $\partial\Gamma$ of Γ. By $B(x, r)$ we denote the open ball of centre x and radius r. We write $B(r) = B(0, r)$ for simplicity. Let z_0 be a fixed point in Γ. As proved in [1] and [14], for each $\zeta \in \partial\Gamma \cup \{\infty\}$ there exists exactly one Martin Kernel $K_\Gamma(., \zeta)$ on Γ, that is, a positive harmonic function on Γ vanishing continuously

(∗) *Indirizzo dell'A.*: Institut Préparatoire aux Etudes d'Ingénieur de Nabeul, Université de Carthage, Campus Universitaire Merazka, B.P. 62, 8058 Nabeul, Tunisie
E-mail: lassaad.gharbi@gmail.com

(∗∗) *Indirizzo dell'A.*: (a) LR11ES11 Analyse Mathématiques et Applications, Faculté des Sciences de Tunis, Université de Tunis El Manar, 2092 El Manar, Tunis, Tunisie
(b) Institut Préparatoire aux Etudes d'Ingénieur de Nabeul, Université de Carthage, Campus Universitaire Merazka, B.P. 62, 8058 Nabeul, Tunisie
E-mail: MohamedAmine.BenBoubaker@ipein.rnu.tn

on $\partial\Gamma \cup \{\infty\} \setminus \{\zeta\}$ and satisfying $K_\Gamma(z_0, \zeta) = 1$. Moreover, from [[14], p. 472], there exist a nonnegative constant α and a positive bounded continuous function ω on $\Gamma \cap S(0,1)$ such that

$$K_\Gamma(x,0) = |x|^{2-n-\alpha}\,\omega\Big(\frac{x}{|x|}\Big) \quad \text{and} \quad K_\Gamma(x,\infty) = |x|^{\alpha}\,\omega\Big(\frac{x}{|x|}\Big), \tag{1.1}$$

where $S(0,1)$ is the unit sphere in $\mathbb{R}^n$. In this paper, we study the existence, the uniqueness and the asymptotic behavior of positive solutions of the nonlinear elliptic problem

$$\begin{cases} \Delta v + f(.,v) = 0 & \text{in } \Gamma, \\ v > 0 & \text{in } \Gamma, \\ v = 0 & \text{on } \partial\Gamma. \end{cases} \tag{1.2}$$

Note that the existence results of problem (1.2) have been extensively studied for the special nonlinearity $f(x,t) = p(x)\,q(t)$, for both bounded and unbounded domain D in $\mathbb{R}^n$ $(n \geq 2)$ with smooth compact boundary (see for example [4, 6, 8, 9, 12, 15, 16, 17]). In [12] Edelson studied (1.2) in $\mathbb{R}^2$, when $f(x,t) = p(x)t^{-\gamma}$, $0 < \gamma < 1$. He proved the existence of an entire positive solution with the growth $\ln|.|$ near infinity. In [9] Crandall, Rabinowitz and Tatar studied (1.2) on a bounded open domain, where they proved existence of solutions, and continuity properties of the solutions if $f(x,t)$ does not depend on x, by using the method of sub- and supersolutions. Lazer and MacKenna [17] also dealt the problem (1.2), when $f(x,t) = p(x)t^{-\gamma}$, $\gamma > 0$ on a bounded open domain, with p a continuous function, proving existence and regularity results at the boundary for the solutions. In [15], Lair and Shaker proved the result of [17] in $\mathbb{R}^n$ $(n \geq 3)$. These results were generalized later by Lair and Shaker in [16]. They studied (1.2) on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ when $f(x,t) = p(x)\,q(t)$, q is a positive non-increasing and differentiable function on $]0,+\infty[$ which is integrable near 0. They proved that the problem (1.2) has a unique weak positive solution $v \in H_0^1(\Omega)$, provided that q is a nontrivial, nonnegative $L^2(\Omega)$ function. In [4] Boccardo and Orsina studied (1.2) when $f(x,t) = p(x)t^{-\gamma}$, $\gamma > 0$ on a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$, with p is a nonnegative function. They proved existence, regularity and non existence results which depends on the summability of p in some Lebesgue spaces, and on the value of γ. Recently in [6] Canino and Sciunzi prove the uniqueness of the solution for the problem studied by Boccardo and Orsina in [4]. Yet recently in [8] Carmona and Martinez-Aparicio studied (1.2) when $f(x,t) = p(x)\,t^{-\gamma(x)}$ on an open bounded set of $\mathbb{R}^n$ $(n \geq 2)$, with $\gamma(x)$ is a positive continuous function and p is a positive function that belongs to a certain Lebesgue space. Inspired by [4], they proved existence results for the problem (1.2). In [20] X. Wang, P. Zhao and Li Zhang studied (1.2), when $f(x,t) = \lambda t^{\beta} + p(x)t^{-\gamma}$ in a bounded smooth domain Ω, with $1 < \beta$, $0 < \gamma < 1$ and $p \in \mathcal{C}_0^{\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$. They proved that (1.2) has at least two positive solutions. Thus, our first aim in this paper is to extend the results of [12], [15], [16], and [20] to a more general problem on NTA-cones in $\mathbb{R}^n (n \geq 3)$. In [13], K. Hirata studied (1.2) in uniform cones in $\mathbb{R}^n$ $(n \geq 3)$, where $f(x,t)$ is a Borel measurable function on $\Gamma \times]0, +\infty[$, continuous with respect to the second variable such that

$$|f(x,t)| \leq t\,\psi(x,t), \quad \text{for all } (x,t) \in \Gamma \times]0, +\infty[\,,$$

where ψ is a nonnegative Borel measurable function satisfying, for each $x \in \Gamma$, $\psi(x,.)$ is nondecreasing in $]0, +\infty[$ and $\lim_{t \longrightarrow 0^+} \psi(x,t) = 0$. By applying sharp estimates for the Green function, he proved the existence of infinitely many continuous solutions with growth as the Martin kernel at infinity. On the other hand, Zhang and Zhao [21] studied (1.2) in a bounded Lipschitz domain containing the origin. They showed the existence of singular solutions with the growth $|\cdot|^{2-n}$ near the origin. The existence of bounded solutions in an unbounded domain with a compact Lipschitz boundary was investigated in [22]. In [5] Brezis and Kamin study the sublinear elliptic equation:

$$\Delta u + p(x)(u(x))^{\gamma} = 0 \quad \text{in } \mathbb{R}^n,$$

with $0 < \gamma < 1$ and p is a nonnegative measurable function satisfying some appropriate conditions. They proved the existence and the uniqueness of positive solution. In [3] M. A. Ben Boubaker combine a singular term and a sublinear term in the nonlinearity and studied (1.2), where $f(x,t) = \varphi(x,t) + \psi(x,t)$, φ and ψ are required to satisfy some appropriate hypothesis related to the Kato class $K(\Gamma)$ which are different to those considered to study the problem (1.2) in this paper. Bachar, Mâagli, and Mâatoug [2] studied (1.2) in $D = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 > 0\}$ and showed the existence of solutions with the growth x_2 near infinity. Their discussion was based on the explicit expression of the Green function. Thus our second aim is to extend their result to NTA- cones Γ in $\mathbb{R}^n (n \geq 3)$, by applying the sharp estimates for the Green function established by K. Hirata in [13]. In particular, we show the existence and uniqueness of solutions with the same growth as the Martin kernel at infinity. Our tools are based essentially on same results established by K. Hirata in [13].

Definition 1.1 (Kato class, see[13]). We say that a Borel measurable function φ in Γ belongs to the extended Kato class $K(\Gamma)$ if φ satisfies the following conditions:

$$\lim_{r\longrightarrow 0}\left(\sup_{x\in\Gamma}\int_{\Gamma\cap B(x,r)}\frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\,G_\Gamma(x,y)\,|\varphi(y)|\,dy\right)=0, \tag{1.3}$$

$$\lim_{R\longrightarrow +\infty}\left(\sup_{x\in\Gamma}\int_{\Gamma\setminus B(R)}\frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\,G_\Gamma(x,y)\,|\varphi(y)|\,dy\right)=0. \tag{1.4}$$

Example 1.2 (see[13]). Suppose that $0\le\alpha<1$ and $\omega(z)\simeq\delta_\Gamma(z)$, and let

$$J(y)=(1+|y|)^{\alpha p-q}\ |y|^{p(1-\alpha)}\ \delta_\Gamma(y)^{-p}.$$

Then $J\in K(\Gamma)$ if and only if $p<2<q$.

We impose the following conditions on f:

(H$_1$) $f\colon\Gamma\times\,]0,+\infty[\,\longrightarrow\,]0,+\infty[$ is a measurable, continuous and non-increasing function with respect to the second variable.

(H$_2$) for all $c>0$, $f(.,c)\in K(\Gamma)$;

(H$_3$) for all $c>0$, $V(f(.,c))>0$, where $V=(-\Delta)^{-1}$ is the potential kernel associated to Δ.

The following notations will be adopted.

i) We denote $f\simeq g$ if there exists $A\ge 1$ such that for all $x\in\Gamma$, $(1/A)g(x)\le f(x)\le A\,g(x)$.

ii) $B(\Gamma)$ is the set of Borel measurable functions in Γ and $B^+(\Gamma)$ is the set of non negative one.

iii) $C(\Gamma)$ denotes the space of all bounded continuous functions in Γ endowed with the uniform norm $\|\,.\,\|_\infty$.

iv) $C_0(\overline{\Gamma}\cup\{\infty\})=\{v\in C(\overline{\Gamma}\cup\{\infty\})\colon \lim_{x\longrightarrow\partial\Gamma}v(x)=\lim_{x\longrightarrow\infty}v(x)=0\}$.

v) By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line.

We define the potential kernel V on $B^+(\Gamma)$ by

$$V\varphi(x)=\int_\Gamma G_\Gamma(x,y)\,\varphi(y)\,dy.$$

We note that, for any $\varphi\in B^+(\Gamma)$ such that $\varphi\in L^1_{\mathrm{loc}}(\Gamma)$ and $V\varphi\in L^1_{\mathrm{loc}}(\Gamma)$, we have in the distributional sense

$$\Delta(V\varphi)=-\varphi\quad\text{in }\Gamma. \tag{1.5}$$

We point out that for any $\varphi \in B^+(\Gamma)$ such that $V\varphi \not\equiv +\infty$, we have $V\varphi \in L^1_{\mathrm{loc}}(\Gamma)$, (see [7], p 51). Let us recall that V satisfies the complete maximum principle, i.e. for each $\varphi \in B^+(\Gamma)$ and a nonnegative superharmonic function u on Γ such that $V\varphi \leq u$ in $\{\varphi > 0\}$ we have $V\varphi \leq u$ in Γ, (cf. [18], Theorem 3.6, p 175). Our main results are the following.

THEOREM 1.3. *Assume* (H$_1$)–(H$_3$). *Then for* $b > 0$, *the problem* (1.2) *has at least one positive solution* v_b *continuous on* $\overline{\Gamma}$ *and satisfying for all* x *in* Γ,

$$\begin{aligned} &b\, K_\Gamma(x,\infty) \\ &\quad \leq v_b(x) \\ &\quad \leq bK_\Gamma(x,\infty) + \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, f\,(y, b\, K_\Gamma(y,\infty))\, dy\right), \end{aligned}$$

where $\sigma = \inf_{\theta>0}(\theta + \|Vf(.,\theta)\|_\infty)$. *In particular* $\lim_{t\longrightarrow+\infty} \frac{v_b(tx)}{K_\Gamma(tx,\infty)} = b$.

THEOREM 1.4. *Assume* (H$_1$)–(H$_3$). *Then the problem* (1.2) *has a unique positive solution* $v \in C_0(\overline{\Gamma} \cup \{\infty\})$, *satisfying, for all* x *in* Γ,

$$A\frac{K_\Gamma(x,\infty)}{(|x|+1)^{n+2\alpha-2}} \leq v(x) \leq \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, f(y, A\frac{K_\Gamma(y,\infty)}{(|y|+1)^{n+2\alpha-2}})\, dy\right),$$

where σ *is as in Theorem* 1.3.

This paper consists of 4 sections devoted to the following topics. In Section 2, we recall some results, established by K. Hirata in [13], that will be necessary throughout this paper. In Section 3, we prove Theorem 1.3. In Section 4 we prove Theorem 1.4. Finally we give an interesting example.

2. Preliminaries

COROLLARY 2.1. *For* $x, y \in \Gamma$ *with* $2\,|y| \leq |x|$,

$$G_\Gamma(x,y) \simeq |x|^{2-n-2\alpha}\; K_\Gamma(x,\infty)\, K_\Gamma(y,\infty) = K_\Gamma(x,0)\, K_\Gamma(y,\infty),$$

where the constant of comparison depends only on Γ.

THEOREM 2.2 (3-G inequalities). *There exists a constant* A *depending only on* Γ *such that for* $x, y, z \in \Gamma$,

$$\frac{G_\Gamma(x,y)\, G_\Gamma(y,z)}{G_\Gamma(x,z)} \leq A\Big(\frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\, G_\Gamma(x,y) + \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(y,z)\Big). \tag{2.1}$$

Lemma 2.3. *Let $r > 0$ and $R > 0$. Then there exists a constant A depending only on r, R and Γ such that for $x, y \in \Gamma \cap B(R)$ with $|x-y| \geq r$,*

$$G_\Gamma(x,y) \leq A\, K_\Gamma(x,\infty)\, K(y,\infty). \tag{2.2}$$

Lemma 2.4. *There exists a constant $A > 0$ depending only on Γ such that for all $x, y \in \Gamma$*

$$\begin{aligned} K_\Gamma(x,\infty)\, K_\Gamma(y,\infty) &\leq A\, \max(|x|, |y|)^{n-2+2\alpha}\, G_\Gamma(x,y) \\ &\leq A((1+|x|)\,(1+|y|))^{n-2+2\alpha}\, G_\Gamma(x,y). \end{aligned}$$

Proposition 2.5. *If $\varphi \in K(\Gamma)$, then*

$$\|\varphi\|_H = \sup_{x\in\Gamma} \int_\Gamma \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\, G_\Gamma(x,y)\, |\varphi(y)|\, dy < +\infty.$$

Moreover, for each $R > 0$,

$$\int_{\Gamma\cap B(R)} K_\Gamma(y,\infty)^2\, |\varphi(y)|\, dy < +\infty.$$

Corollary 2.6. *Let $\varphi \in K(\Gamma)$. Then, for each $R > 0$,*

$$\int_{\Gamma\cap B(R)} K_\Gamma(y,\infty)\, |\varphi(y)|\, dy < +\infty. \tag{2.3}$$

Lemma 2.7. *Let $\varphi \in K(\Gamma)$. Then, for each $x_0 \in \overline{\Gamma}$,*

$$\lim_{r\longrightarrow 0} \int_{\Gamma\cap B(x_0,r)} K_\Gamma(y,\infty)^2\, |\varphi(y)|\, dy = 0.$$

3. Proof of Theorem 1.3

Proposition 3.1. *Let $\varphi \in K(\Gamma)$ and h be a positive superharmonic function in Γ.*

a) *We have*

$$\lim_{r\longrightarrow 0} \sup_{x\in\Gamma} \left(\frac{1}{h(x)} \int_{B(x_0,r)\cap\Gamma} G_\Gamma(x,y)\, h(y)\, |\varphi(y)|\, dy \right) = 0 \tag{3.1}$$

for all $x_0 \in \overline{\Gamma}$, and

$$\lim_{M\longrightarrow +\infty} \sup_{x\in\Gamma} \left(\frac{1}{h(x)} \int_{\Gamma\setminus B(M)} G_\Gamma(x,y)\, h(y)\, |\varphi(y)|\, dy \right) = 0. \tag{3.2}$$

b) *For all $x \in \Gamma$ and A as in Theorem 2.2,*

$$\int_\Gamma G_\Gamma(x,y)\, h(y)\, |\varphi(y)|\, dy \le 2\, A\, \|\varphi\|_H\, h(x). \tag{3.3}$$

Proof. Let h be a positive superharmonic function in Γ. Then by [[15], Theorem 2.1, p. 164], there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of positive measurable functions in Γ such that

$$h(y) = \sup_{n\in\mathbb{N}} \int_\Gamma G_\Gamma(y,z)\, f_n(z)\, dz.$$

Hence we need to verify (3.1), (3.2), and (3.3) only for $h(y) = G_\Gamma(y,z)$, uniformly for $z \in \Gamma$.

a) Let $r > 0$. By using Theorem 2.2, we get

$$\begin{aligned}\frac{1}{G_\Gamma(x,z)} &\int_{B(x_0,r)\cap\Gamma} G_\Gamma(x,y)\, G_\Gamma(y,z)\, |\varphi(y)|\, dy \\ &\le 2\, A \sup_{z\in\Gamma} \int_{B(x_0,r)\cap\Gamma} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy.\end{aligned}$$

Let $\varepsilon > 0$. Since $\varphi \in K(\Gamma)$, there exist positive numbers r_1 and R_1 such that

$$\sup_{z\in\Gamma} \int_{\Gamma\cap B(z,r_1)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy \le \varepsilon$$

and

$$\sup_{z\in\Gamma} \int_{\Gamma\setminus B(R_1)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy \le \varepsilon.$$

Let $r > 0$ and $z \in \Gamma$. Then we have by Lemma 2.3

$$\begin{aligned}&\int_{\Gamma\cap B(x_0,r)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy \\ &\quad\le 2\varepsilon + \int_{\Gamma\cap B(x_0,r)\cap B(R_1)\setminus B(z,r_1)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy \\ &\quad\le 2\varepsilon + A \int_{\Gamma\cap B(x_0,r)} K_\Gamma(y,\infty)^2\, |\varphi(y)|\, dy.\end{aligned}$$

Hence (3.1) follows from Lemma 2.7. On the other hand, we have

$$\begin{aligned}\frac{1}{G_\Gamma(x,z)} &\int_{\Gamma\setminus B(M)} G_\Gamma(x,y)\, G_\Gamma(y,z)\, |\varphi(y)|\, dy \\ &\le 2\, A \sup_{z\in\Gamma} \int_{\Gamma\setminus B(M)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\, |\varphi(y)|\, dy\end{aligned}$$

and from (1.4) it converges to zero as $M \longrightarrow +\infty$. This gives (3.2).

b) By using Theorem 2.2, we obtain

$$\frac{1}{G_\Gamma(x,z)}\int_\Gamma G_\Gamma(x,y)\,G_\Gamma(y,z)\,|\varphi(y)|\,dy \le 2\,A\,\|\varphi\|_H. \qquad \square$$

Corollary 3.2. *Let $\varphi \in K(\Gamma)$. Then we have*

$$\sup_{x\in\Gamma}\int_\Gamma G_\Gamma(x,y)\,|\varphi(y)|\,dy < +\infty, \tag{3.4}$$

$$\int_\Gamma \frac{K_\Gamma(y,\infty)}{(1+|y|)^{n+2\alpha-2}}\,|\varphi(y)|\,dy < +\infty. \tag{3.5}$$

Proof. Inequality (3.4) is a consequence of (3.3) with $h = 1$ in Γ and Proposition 2.5. Let $x_0 \in \Gamma$. Then by Lemma 2.4 and (3.4) we get

$$\begin{aligned}
&\int_\Gamma \frac{K_\Gamma(y,\infty)}{(|y|+1)^{n+2\alpha-2}}\,|\varphi(y)|\,dy \\
&\le A\,\frac{(|x_0|+1)^{n+2\alpha-2}}{K_\Gamma(x_0,\infty)}\,\sup_{x\in\Gamma}\int_\Gamma G_\Gamma(x,y)\,|\varphi(y)|\,dy \\
&< +\infty. \qquad \square
\end{aligned}$$

Proposition 3.3. *Let $\varphi \in K(\Gamma)$. Then, the function $V\varphi$ is in $C_0(\overline{\Gamma}\cup\{\infty\})$.*

Proof. Let $x_0 \in \overline{\Gamma}$, $\delta > 0$ and $x, x' \in \Gamma \cap B(x_0, \frac{\delta}{2})$. Then

$$\begin{aligned}
&|V\varphi(x) - V\varphi(x')| \\
&\le 2\sup_{x\in\Gamma}\int_{\Gamma\setminus B(\delta^{-1})} G_\Gamma(x,y)\,|\varphi(y)|\,dy \\
&\quad + 2\sup_{x\in\Gamma}\int_{\Gamma\cap B(x_0,\delta)} G_\Gamma(x,y)\,|\varphi(y)|\,dy \\
&\quad + \int_{\Gamma\cap B(\delta^{-1})\setminus B(x_0,\delta)} |G_\Gamma(x,y) - G_\Gamma(x',y)|\,|\varphi(y)|\,dy.
\end{aligned}$$

By (3.1) and (3.2), the first two quantities of the right hand side are bounded by ε whenever δ is sufficiently small. For δ sufficiently small, $G_\Gamma(.,y)$ can be extended continuously to $B(x_0,\frac{\delta}{2}) \cap \overline{\Gamma}$ whenever $y \in \Gamma \setminus B(x_0,\delta)$. Moreover, by (2.2) and (1.1), there exists $A > 0$ such that

$$G_\Gamma(x,y) \le A\,K_\Gamma(y,\infty),$$

for all $(x, y) \in (B(x_0, \frac{\delta}{2}) \cap \overline{\Gamma}) \times (\Gamma \cap B(\delta^{-1}) \setminus B(x_0, \delta))$. Then by (2.3) and Lebesgue's theorem, we have

$$\int_{(\Gamma \cap B(\delta^{-1})) \setminus B(x_0,\delta)} |G_\Gamma(x, y) - G_\Gamma(x', y)|\ |\varphi(y)|\ dy \xrightarrow[|x-x'|\to 0]{} 0.$$

Hence, $V\varphi$ is continuous in $\overline{\Gamma}$ and $\lim_{x \longrightarrow x_0} V\varphi(x) = 0$ when $x_0 \in \partial\Gamma$ because $G_\Gamma(.,)$ vanishes continuously on $\partial\Gamma$.
Now we will show that $\lim_{|x| \longrightarrow +\infty} V\varphi(x) = 0$. Let $M > 0$. Then

$$|V\varphi(x)| \le \int_{\Gamma \cap B(M)} G_\Gamma(x, y)\ |\varphi(y)|\ dy + \int_{\Gamma \setminus B(M)} G_\Gamma(x, y)\ |\varphi(y)|\ dy.$$

By (3.2), the second term of the right hand side is bounded by ε uniformly for x, whenever M is sufficiently large. Using (1.1) and Corollary 2.1 , we get

$$G_\Gamma(x, y) \le A\ \frac{K_\Gamma(y, \infty)}{|x|^{n-2+\alpha}}, \quad \text{for } x \in \Gamma \setminus B(2M) \text{ and } y \in \Gamma \cap B(M).$$

It follows from Corollary 2.6 that $\lim_{|x| \longrightarrow +\infty} |V\varphi(x)| = 0$. $\square$

THEOREM 3.4. *Assume* (H_1)-(H_3). *Let θ and b be strictly positive numbers. Then the Problem*

$$P_{\theta,b} = \begin{cases} \Delta v + f(., v) = 0 & \text{in } \Gamma, \\ v > 0 & \text{in } \Gamma, \\ v = \theta & \text{on } \partial\Gamma, \end{cases}$$

has at least one positive solution $v_{\theta,b} \in C(\overline{\Gamma})$ satisfying

$$\lim_{x \longrightarrow \infty} \frac{v_{\theta,b}(x)}{\theta + b\ K_\Gamma(x, \infty)} = 1,$$

in particular for each fixed $z \in \Gamma$

$$\lim_{t \longrightarrow +\infty} \frac{v_{\theta,b}(tz)}{K_\Gamma(tz, \infty)} = b.$$

The proof is based on the Schauder fixed point argument. In the sequel, we suppose that Γ is an NTA-cone in $\mathbb{R}^n$ with $n \ge 3$ and f satisfies (H_1)–(H_3). Let $\theta > 0$. It follows from (H_2) and Proposition 3.3 that $Vf(., \theta) \in C_0(\overline{\Gamma} \cup \{\infty\})$. So in the sequel, we denote $F_\theta = \{w \in C(\overline{\Gamma} \cup \{\infty\}) \colon \theta \le w \le \beta = \theta + \|Vf(., \theta)\|_\infty\}$. For $w \in F_\theta$, we define

$$Tw(x) = \theta + \frac{\theta}{\theta + b\ K_\Gamma(x, \infty)} \int_\Gamma G_\Gamma(x, y)\ f\Big(y, \frac{\theta + b\ K_\Gamma(y, \infty)}{\theta}\ w(y)\Big)\ dy,$$

for all $x \in \Gamma$.

Lemma 3.5. *The class* $\{Tw : w \in F_\theta\}$ *is equicontinuous in* $\overline{\Gamma} \cup \{\infty\}$. *Moreover we have* $\{Tw : w \in F_\theta\} \subset C(\overline{\Gamma} \cup \{\infty\})$.

Proof. As in the proof of Proposition 3.3, we show that $\{Tw : w \in F_\theta\}$ is equicontinuous in $(\overline{\Gamma} \cup \{\infty\})$. In particular, for all $w \in F_\theta$, $Tw \in C(\overline{\Gamma} \cup \{\infty\})$. □

Lemma 3.6. $T(F_\theta) \subset F_\theta$. *Moreover,* $T(F_\theta)$ *is relatively compact in* $C(\overline{\Gamma} \cup \{\infty\})$.

Proof. Let $w \in F_\theta$, by (H_1)

$$f\Big(y, \frac{\theta + b\,K_\Gamma(y,\infty)}{\theta}\,w(y)\Big) \le f(y,\theta), \quad \text{for all } y \in \Gamma. \tag{3.6}$$

Then for $w \in F_\theta$

$$\theta \le Tw(x) \le \beta, \quad \text{for all } x \in \Gamma.$$

By using Lemma 3.5, we deduce that $T(F_\theta) \subset F_\theta$. Since $\{Tw : w \in F_\theta\}$ is uniformly bounded in $\overline{\Gamma} \cup \{\infty\}$, it follows by Ascoli's theorem, that $T(F_\theta)$ is relatively compact in $C(\overline{\Gamma} \cup \{\infty\})$. □

Lemma 3.7. T *is continuous in* F_θ.

Proof. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in F_θ which converges uniformly to $w \in F_\theta$ in $\overline{\Gamma} \cup \{\infty\}$. Then we have

$$\begin{aligned}
&|Tw_n(x) - Tw(x)| \\
&\quad \le \frac{\theta}{\theta + bK_\Gamma(x,\infty)} \int_\Gamma G_\Gamma(x,y)\,\Big| f\Big(y, \frac{\theta + b\,K_\Gamma(y,\infty)}{\theta}\,w_n(y)\Big) \\
&\qquad\qquad - f\Big(y, \frac{\theta + b\,K_\Gamma(y,\infty)}{\theta}\,w(y)\Big)\Big|\,dy.
\end{aligned}$$

It follows from (H_1), (H_2), (3.6), (3.4), and Lebesgue's convergence theorem that for each $x \in \Gamma$,

$$\lim_{n\longrightarrow+\infty} Tw_n(x) = Tw(x).$$

Hence, Tw_n converges pointwisely to Tw in Γ as $n \longrightarrow +\infty$. Since $T(F_\theta)$ is relatively compact in $C(\overline{\Gamma} \cup \{\infty\})$, the pointwise convergence implies the uniform convergence. Thus $\lim_{n\longrightarrow+\infty} \|Tw_n - Tw\|_\infty = 0$. Hence, T is continuous on F_θ. □

Proof of Theorem 3.4. Let $\theta > 0$. Note that F_θ is a nonempty bounded closed convex set in $C(\overline{\Gamma} \cup \{\infty\})$. Since T is a compact mapping from F_θ into it self, it follows from the Schauder fixed point theorem that there exists $w_\theta \in F_\theta$ such that $w_\theta = T(w_\theta)$, that is for all $x \in \Gamma$

$$w_\theta(x) = \theta + \frac{\theta}{\theta + b\, K_\Gamma(x,\infty)} \int_\Gamma G_\Gamma(x,y)\, f\Big(y, \frac{\theta + b\, K_\Gamma(y,\infty)}{\theta}\, w_\theta(y)\Big)\, dy.$$

Put

$$v_{\theta,b}(x) = \frac{\theta + b\, K_\Gamma(x,\infty)}{\theta}\, w_\theta(x),$$

for $x \in \Gamma$. Then for all $x \in \Gamma$

$$v_{\theta,b}(x) = \theta + b\, K_\Gamma(x,\infty) + \int_\Gamma G_\Gamma(x,y)\, f(y, v_{\theta,b}(y))\, dy. \tag{3.7}$$

By using (H_1), we get for all $y \in \Gamma$

$$f(y, v_{\theta,b}(y)) \leq f(y,\theta).$$

It follows from (H_2) and Proposition 2.5 that the function $y \longmapsto f(y, v_{\theta,b}(y))$ belongs to $L^1_{\mathrm{loc}}(\Gamma)$ and by Proposition 3.3, we get that $Vf(., v_{\theta,b}) \in C_0(\overline{\Gamma} \cup \{\infty\}) \subset L^1_{\mathrm{loc}}(\Gamma)$. Hence,

$$\Delta v_{\theta,b} + f(., v_{\theta,b}) = 0 \quad \text{in } \Gamma \text{ (in the sense of distribution).}$$

Moreover,

$$\lim_{x \longrightarrow \infty} \frac{v_{\theta,b}(x)}{\theta + bK_\Gamma(x,\infty)} = 1 + \lim_{x \longrightarrow \infty} \frac{Vf(., v_{\theta,b})(x)}{\theta + b\, K_\Gamma(x,\infty)} = 1$$

and $\lim_{x \longrightarrow \partial\Gamma} v_{\theta,b}(x) = \theta$. Thus, $v_{\theta,b} \in C(\overline{\Gamma})$ and $v_{\theta,b}$ is a positive continuous solution of the problem $(P_{\theta,b})$. □

Proposition 3.8. *Let $\theta_1, \theta_2, b_1, b_2$ be real numbers such that $0 \leq \theta_1 \leq \theta_2$ and $0 \leq b_1 \leq b_2$. Then,*

$$0 \leq v_{\theta_2,b_2}(x) - v_{\theta_1,b_1}(x) \leq \theta_2 - \theta_1 + (b_2 - b_1)\, K_\Gamma(x,\infty), \quad \textit{for all } x \in \Gamma.$$

Proof. Let h be the function defined on Γ by

$$h(x) = \begin{cases} \dfrac{f(x, v_{\theta_1,b_1}(x)) - f(x, v_{\theta_2,b_2}(x))}{v_{\theta_2,b_2}(x) - v_{\theta_1,b_1}(x)} & \text{if } v_{\theta_2,b_2}(x) \neq v_{\theta_1,b_1}(x), \\ 0 & \text{if } v_{\theta_2,b_2}(x) = v_{\theta_1,b_1}(x). \end{cases}$$

Let us put $g(x) = v_{\theta_2,b_2}(x) - v_{\theta_1,b_1}(x)$ and denote by

$$g^+ = \max(g,0), \quad g^- = \max(-g,0).$$

It is easy to see that $g \in B(\Gamma)$, $h \in B^+(\Gamma)$ and

$$g + V(h\,g) = \theta_2 - \theta_1 + (b_2 - b_1)\, K_\Gamma(x,\infty).$$

Since

$$\begin{aligned} V(h\,|g|) &\le V(f(.,v_{\theta_1,b_1})) + V(f(.,v_{\theta_2,b_2})) \\ &\le v_{\theta_1,b_1} + v_{\theta_2,b_2} \\ &< \infty, \end{aligned}$$

we have

$$g^+(x) + V(hg^+)(x) = \theta_2 - \theta_1 + (b_2 - b_1)\, K_\Gamma(x,\infty) + g^- + V(hg^-)(x),$$

for all $x \in \Gamma$. As a consequence, we get

$$V(hg^+)(x) \le (\theta_2 - \theta_1) + (b_2 - b_1)\, K_\Gamma(x,\infty) + V(hg^-)(x)$$

on the set $\{x \in \Gamma : g^+(x) > 0\}$. As $\theta_2 - \theta_1 + (b_2 - b_1)\, K_\Gamma(x,\infty) + V(h\,g^-)$ is a non negative superharmonic function in Γ, then the complete maximum principle implies that

$$V(hg^+) \le (\theta_2 - \theta_1) + (b_2 - b_1)\, K_\Gamma(x,\infty) + V(hg^-) \quad \text{in } \Gamma,$$

that is

$$Vhg(x) \le ((\theta_2 - \theta_1) + (b_2 - b_1)\, K_\Gamma(x,\infty)) = g + V(hg)(x), \quad \text{for all } x \in \Gamma.$$

This implies that

$$0 \le g(x) \le (\theta_2 - \theta_1) + (b_2 - b_1)\, K_\Gamma(x,\infty), \quad \text{for all } x \in \Gamma. \qquad \square$$

Proof of Theorem 1.3. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, converging decreasingly to zero. Then, for each $n \in \mathbb{N}$, the problem $(P_{\theta_n,b})$ has a continuous solution $v_{\theta_n,b}$ satisfying for each $x \in \Gamma$

$$v_{\theta_n,b}(x) = \theta_n + bK_\Gamma(x,\infty) + \int_\Gamma G_\Gamma(x,y)\, f(y, v_{\theta_n,b}(y))\, dy.$$

By Proposition 3.8, we deduce that the sequence $(v_{\theta_n,b})_{n\in\mathbb{N}}$ decreases to a function v_b. On the other hand, for all $x \in \Gamma$

$$\begin{aligned} v_{\theta_n,b}(x) - \theta_n &= bK_\Gamma(x,\infty) + \int_\Gamma G_\Gamma(x,y)\, f(y, v_{\theta_n,b}(y))\, dy \\ &\geq bK_\Gamma(x,\infty) \\ &> 0. \end{aligned}$$

It follows from (H$_1$) that the sequence $(v_{\theta_n,b} - \theta_n)_{n\in\mathbb{N}}$ increases to v_b and so $v_b > 0$ in Γ. Finally,

$$v_b = \inf_{n\in\mathbb{N}} v_{\theta_n,b} = \sup_{n\in\mathbb{N}} (v_{\theta_n,b} - \theta_n)$$

is a positive continuous function in $\overline{\Gamma}$. By using (H$_1$) and the monotone convergence theorem, we get

$$v_b(x) = b\, K_\Gamma(x,\infty) + \int_\Gamma G_\Gamma(x,y)\, f(y, v_b(y))\, dy, \quad \text{for all } x \in \Gamma. \tag{3.8}$$

This implies that $Vf(., v_b) \in L^1_{\mathrm{loc}}(\Gamma)$. Moreover, by using Proposition 2.5 and (H$_2$) we obtain that $f(., v_b) \in L^1_{\mathrm{loc}}(\Gamma)$. Since $K_\Gamma(., \infty)$ is harmonic in Γ, we see, by (3.8), that v_b is a distributional solution to

$$\Delta v_b + f(., v_b) = 0 \quad \text{in } \Gamma.$$

Thus $v_b \in C(\overline{\Gamma})$ and v_b is a positive solution of the problem (1.2).
Now, let

$$\sigma = \inf_{\theta>0} (\theta + \|Vf(., \theta)\|_\infty).$$

By (H$_3$) and (H$_1$), we see that $\sigma > 0$ and by (3.7), we get

$$\theta + b\, K_\Gamma(x,\infty) \leq v_{\theta,b}(x) \leq \beta + b\, K_\Gamma(x,\infty), \quad \text{for all } x \in \Gamma.$$

This implies that for all $x \in \Gamma$

$$b\, K_\Gamma(x,\infty) \leq v_b(x) \leq b\, K_\Gamma(x,\infty) + \sigma.$$

By (H$_1$) and (3.8), we obtain

$$b\, K_\Gamma(x,\infty) \leq v_b(x) \leq b\, K_\Gamma(x,\infty) + \int_\Gamma G_\Gamma(x,y)\, f(y, b\, K_\Gamma(y,\infty))\, dy,$$

for all $x \in \Gamma$. Finally, we deduce that

$$\begin{aligned} b\, K_\Gamma(x,\infty) &\leq v_b(x) \\ &\leq b\, K_\Gamma(x,\infty) + \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, f(y, b\, K_\Gamma(y,\infty))\, dy\right). \end{aligned}$$ □

4. Proof of Theorem 1.4

Proposition 4.1 (uniqueness). *Assume that (H_1) is satisfied. Then, the problem (1.2) has at most one positive solution in $C_0(\overline{\Gamma} \cup \{\infty\})$.*

Proof (see [11]). Assume that there exist two positive solutions u, v of (1.2) with $u \neq v$. Suppose that there exists $x_0 \in \Gamma$ such that $v(x_0) > u(x_0)$. Put $w = v - u \in C_0(\Gamma)$. Then, we have

$$\Delta w + f(., v) - f(., u) = 0, \quad \text{in } \Gamma.$$

Let $U = \{x \in \Gamma : w(x) > 0\}$. Then U is an open nonempty set. Since, the function f satisfies (H_1), we deduce that

$$\begin{cases} \Delta w \geq 0, & \text{in } U, \\ w = 0, & \text{on } \partial U. \end{cases}$$

Hence by the maximum principle (see [10], pp. 465–466), we get $w \leq 0$ in U. This is a contradiction to the definition of U. □

Proof of Theorem 1.4. Let $\theta > 0$ and $(b_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging decreasingly to zero. Then for each $n \in \mathbb{N}$ the problem (P_{θ, b_n}) has a continuous solution v_{θ, b_n} satisfying

(4.1)
$$v_{\theta, b_n}(x) = \theta + b_n K_\Gamma(x, \infty) + \int_\Gamma G_\Gamma(x, y) f(y, v_{\theta, b_n}(y))\, dy, \quad \text{for all } x \in \Gamma.$$

Moreover, for each $n \in \mathbb{N}$, the problem (1.2), has a continuous solution v_{b_n} satisfying

$$v_{b_n}(x) = b_n K_\Gamma(x, \infty) + \int_\Gamma G_\Gamma(x, y) f(y, v_{b_n}(y))\, dy, \quad \text{for all } x \in \Gamma. \tag{4.2}$$

By Proposition 3.8, we deduce that the sequence $(v_{b_n})_{n \in \mathbb{N}}$ decreases to a function v and by (H_1) the sequence $(v_{b_n} - b_n K_\Gamma(x, \infty))$ increases to v. Hence, v is a positive continuous function in $\overline{\Gamma}$. By using the monotone convergence theorem, we get

$$v(x) = \int_\Gamma G_\Gamma(x, y) f(y, v(y))\, dy, \quad \text{for all } x \in \Gamma.$$

Moreover, from Proposition 3.8 and (3), we have

$$v(x) \leq v_{\theta, b_n}(x) \leq \theta + b_n K_\Gamma(x, \infty) + V f(., \theta)(x), \quad \text{for all } x \in \Gamma. \tag{4.3}$$

Then it follows from Proposition 3.3, that

$$\lim_{x\longrightarrow\partial\Gamma} v(x) = \lim_{x\longrightarrow\infty} v(x) = 0.$$

Thus $v \in C_0(\overline{\Gamma} \cup \{\infty\})$ and v is a positive solution of the problem (1.2). By (H_1) and (4.3), we obtain for all $x \in \Gamma$

$$\int_\Gamma G_\Gamma(x, y)\, f(y, \sigma)\, dy \le v(x) \le \sigma.$$

Then it follows from Lemma 2.4 that for all $x \in \Gamma$

$$\frac{1}{A} \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n-2+2\alpha}} \int_\Gamma \frac{K_\Gamma(y,\infty)}{(|y|+1)^{n-2+2\alpha}}\, f(y,\sigma)\, dy \le v(x).$$

Since $K_\Gamma(.,\infty)$ is a positive harmonic function on the open connected set Γ, vanishing continuously on $\partial\Gamma$, it follows from the minimum principle that $K_\Gamma(.,\infty)$ is strictly positive on Γ and from (H_1), we conclude that $\frac{K_\Gamma(.,\infty)}{(|.|+1)^{n-2+2\alpha}} f(.,\sigma)$ is strictly positive on Γ. Hence we deduce from (H_2) and (3.5) that

$$0 < \int_\Gamma \frac{K_\Gamma(y,\infty)}{(|y|+1)^{n-2+2\alpha}}\, f(y,\sigma)\, dy < +\infty$$

and therefore we get for all $x \in \Gamma$,

$$A \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n+2\alpha-2}} \le v(x).$$

Since f is non-increasing with respect to the second variable, then we have

$$v(x) \le \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, f\left(y, A\frac{K_\Gamma(y,\infty)}{(|y|+1)^{n+2\alpha-2}}\right) dy\right). \qquad \square$$

EXAMPLE 4.2. Let $0 \le \alpha < 1$, $p < 2 < q$, $\Psi\colon]0,+\infty[\longrightarrow]0,+\infty[$ be a continuous and non-increasing function and suppose that $\omega(x) \simeq \delta_\Gamma(x)$, then the problem

$$\begin{cases} \Delta v(x) + \Psi(v(x)) \dfrac{(1+|x|)^{\alpha p - q}}{|x|^{p(\alpha-1)}\, \delta_\Gamma(x)^p} = 0 & x \in \Gamma, \\ v(x) > 0 & x \in \Gamma, \\ v = 0 & \text{on } \partial\Gamma, \end{cases} \tag{4.4}$$

has a unique positive solution $v \in C_0(\overline{\Gamma} \cup \{\infty\})$, satisfying for all $x \in \Gamma$,

$$\begin{aligned} &\frac{1}{A} \frac{|x|^{p^2(1-\alpha)} \delta_\Gamma(x)}{(1+|x|)^{n+2\alpha-2}} \\ &\le v(x) \qquad (4.5) \\ &\le \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, \varphi\Big(\frac{1}{A} \frac{|y|^{p^2(1-\alpha)} \delta_\Gamma(y)}{(1+|y|)^{n+2\alpha-2}}\Big) \frac{(1+|y|)^{\alpha p - q}}{|y|^{p(\alpha-1)} \delta_\Gamma(y)^p}\, dy\right). \end{aligned}$$

Proof. From [13] (Example 4.9),

$$J(x) = \frac{(1+|x|)^{\alpha p - q}}{|x|^{p(\alpha-1)} \delta_\Gamma(x)^p}$$

belongs to the Kato class $K(\Gamma)$. Moreover, we have

$$K_\Gamma(x,\infty) \simeq |x|^{p^2(1-\alpha)} \delta_\Gamma(x). \qquad (4.6)$$

It is clear that $f(x,t) = J(x)\Psi(t)$ satisfies the hypothesis (H_1)–(H_3), thus by theorem 1.4 the problem (4.4) has a unique positive solutions satisfying for all $x \in \Gamma$,

$$\begin{aligned} &A \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n+2\alpha-2}} \\ &\le v(x) \\ &\le \min\left(\sigma, \int_\Gamma G_\Gamma(x,y)\, J(y)\Psi\Big(A \frac{K_\Gamma(y,\infty)}{(|y|+1)^{n+2\alpha-2}}\Big)\, dy\right). \end{aligned}$$

By using (4.6) and the fact that Ψ is non-increasing the inequality (4.5) follows. □

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