Rend. Sem. Mat. Univ. Padova, Vol. 139 (2018), 205–224
DOI 10.4171/RSMUP/139-8

# Definable categories and $\mathbb{T}$-motives

Luca Barbieri-Viale (*) – Mike Prest (**)

Abstract – Making use of Freyd's free abelian category on a preadditive category we show that if $T: D \to \mathcal{A}$ is a representation of a quiver $D$ in an abelian category $\mathcal{A}$ then there is an abelian category $\mathcal{A}(T)$, a faithful exact functor $F_T: \mathcal{A}(T) \to \mathcal{A}$ and an induced representation $\widetilde{T}: D \to \mathcal{A}(T)$ such that $F_T \widetilde{T} = T$ universally. We then can show that $\mathbb{T}$-motives as well as Nori's motives are given by a certain category of functors on definable categories.

Mathematics Subject Classification (2010). Primary: 14F99; Secondary: 19E15, 18E10, 03C60, 03G30.

Keywords. Motives, model theory, representations, abelian categories.

## Introduction

In [6], see also [8], Nori, starting with the category $\mathcal{V}_k$ of algebraic varieties, i.e. separated schemes of finite type, over $k$ a subfield of the complex numbers $\mathbb{C}$, constructed an abelian category which is an avatar of the hypothesized category of effective homological mixed motives. See [1] for an introduction to the motivic world.

First he builds the diagram, or *quiver*, which has, for vertices, triples $(X, Y, i)$ with $X \in \mathcal{V}_k$, $Y$ a closed subvariety of $X$ and $i \geq 0$ an integer. If $f: X \to X'$ is a morphism in $\mathcal{V}_k$ with $f(Y) \subseteq Y'$ then there is a corresponding arrow from $(X, Y, i)$ to $(X', Y', i)$ for each $i$. There is also an arrow, for each pair of closed

(*) *Indirizzo dell'A.*: Dipartimento di Matematica "F. Enriques," Università degli Studi di Milano, Via C. Saldini 50, I-20133 Milano, Italy
E-mail: luca.barbieri-viale@unimi.it

(**) *Indirizzo dell'A.*: School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK
E-mail: mprest@manchester.ac.uk

subvarieties $Z \subseteq Y \subseteq X$, from $(X, Y, i)$ to $(Y, Z, i-1)$ corresponding to the boundary map in the long exact sequence of singular homology groups. He then considers the representation of this diagram, given by singular homology of a pair, mapping $(X, Y, i)$ to the finitely generated abelian group $H_i^{\mathrm{sing}}(X(\mathbb{C}), Y(\mathbb{C}))$. From this, he constructs an abelian category EHM through which this, and any other reasonable (co)homology theory should factor.

More generally, in fact, such a universal abelian category exists for any representation in the category $R\text{-}\mathrm{mod}$, of finitely generated modules over a commutative Noetherian ring $R$.

Theorem (Nori). *Let $D$ be a quiver and $T\colon D \to R\text{-}\mathrm{mod}$ a representation of $D$. There is an abelian $R$-linear category $\mathcal{C}(T)$, a $R$-linear faithful exact functor $F_T\colon \mathcal{C}(T) \to R\text{-}\mathrm{mod}$ and a representation $\widetilde{T}\colon D \to \mathcal{C}(T)$ such that $F_T\widetilde{T} = T$ universally.*

See [8, Chapter 7] for a detailed proof. The universal property goes as follows. Let $\mathcal{B}$ be an abelian $R$-linear category, $G\colon \mathcal{B} \to R\text{-}\mathrm{mod}$ a $R$-linear faithful exact functor, and $S\colon D \to \mathcal{B}$ a representation of $S$ in $\mathcal{B}$ such that $GS = T$. Then there is a $R$-linear faithful exact functor $F_S\colon \mathcal{C}(T) \to \mathcal{B}$ unique (up to unique isomorphism) such that the following diagram commutes (up to isomorphism):

$$\begin{tikzcd} & \mathcal{C}(T) \arrow[d, dotted, "F_S"] \arrow[dr, bend left, "F_T"] & \\ D \arrow[ur, bend left, "\widetilde{T}"] \arrow[r, "S"] \arrow[rr, bend right, "T"'] & \mathcal{B} \arrow[r, "G"] & R\text{-}\mathrm{mod} \end{tikzcd}$$

The original proof of Nori's theorem is not straightforward and goes *via* a construction of $\mathcal{C}(T)$ involving the algebra of endomorphisms of the representation, more precisely: when $D$ is finite, one takes $\mathcal{C}(T) := \mathrm{End}(T)\text{-}\mathrm{mod}$ and then, in general, $\mathcal{C}(T)$ is obtained by taking the 2-colimit of the abelian categories $\mathrm{End}(T|_E)\text{-}\mathrm{mod}$ as $E$ varies among the finite subquivers of $D$. Under additional conditions on $R$, e.g. if it is a field $K$, we have that $\mathcal{C}(T)$ itself is a category of comodules over a coalgebra. Note that in [2] it is proven that Deligne's 1-motives can be obtained via Nori's construction.

In [9], Ivorra generalizes Nori's theorem by allowing representations of the form $T\colon D \to \mathcal{A}$ where $\mathcal{A}$ is a $K$-linear abelian category which is finite and Hom finite: these categories $\mathcal{A}$ are precisely the categories of finite dimensional comodules over some $K$-coalgebra. The characterization of these abelian categories $\mathcal{A}$

relies on Nori's key result that a faithful exact functor $T\colon \mathcal{A} \to K\text{-}\mathrm{mod}$ yields an equivalence $\widetilde{T}\colon \mathcal{A} \xrightarrow{\simeq} \mathcal{C}(T)$, see [9, §4] for details. The proof of the above mentioned generalization is by reducing the general case to the case considered by Nori. See also [10] for a reconstruction of Laumon's 1-motives over $\mathbb{Q}$ by using Nori's formalism.

In [3], Caramello gives a proof of Nori's theorem by a very different, more general and rather direct construction, obtaining the category $\mathcal{C}(T)$ as the Barr exact completion or effectivization of the syntactic category of the regular theory of the representation $T$. This category is indeed abelian and it has the required universal property even for representations in all $R$-modules, the latter following from general properties of models of a regular theory.

Furthermore, in [4], for any fixed base category $\mathcal{C}$ along with a distinguished subcategory $\mathcal{M}$, a regular homological theory $\mathbb{T}$ is introduced on a signature strongly related to Nori's diagram when $\mathcal{C}$ is $\mathcal{V}_k$ and $\mathcal{M}$ is given by closed immersions. The exact completion of the regular syntactic category of $\mathbb{T}$ is an abelian category $\mathcal{A}[\mathbb{T}]$ which is universal with respect to $\mathbb{T}$-models in abelian categories. Nori's category EHM can be obtained as $\mathcal{A}[\mathbb{T}_{H^{\mathrm{sing}}}]$ for the regular theory $\mathbb{T}_{H^{\mathrm{sing}}}$ of the model $H^{\mathrm{sing}}$ (singular homology) by adding to $\mathbb{T}$ all regular axioms which are valid in $H^{\mathrm{sing}}$. Notably, the category $\mathcal{A}[\mathbb{T}]$ (resp. its variant $\mathcal{A}[\mathbb{T}_{H^{\mathrm{sing}}}]$) is determined by the motivic topos associated to the homological theory $\mathbb{T}$ (resp. $\mathbb{T}_{H^{\mathrm{sing}}}$) as the category of (effective) *constructible* $\mathbb{T}$-motives (resp. $\mathbb{T}_{H^{\mathrm{sing}}}$-motives, i.e. Nori motives): see [4], precisely Corollary 4.2.2, Lemma 4.3.1, and Definition 4.3.2.

In fact, in the additive context, a direct algebraic construction may be given of both $\mathcal{C}(T)$ and $\mathcal{A}[\mathbb{T}]$, which makes use of Freyd's free abelian category on a preadditive category [7]. We use [13, Chapter 4] as a reference for this, see also [14]. We also make use of definable additive categories which are exactly the categories of models of regular additive theories in abelian groups. In fact, all the previously mentioned constructions can be deduced from the following result.

THEOREM. *Let $D$ be a quiver. There is an abelian category* $\mathrm{Ab}(D)$ *and a universal representation* $\Delta\colon D \to \mathrm{Ab}(D)$, *i.e. if* $T\colon D \to \mathcal{A}$ *is a representation of* $D$ *in an abelian category* $\mathcal{A}$ *then there is a unique* (*up to equivalence*) *exact functor* $F\colon \mathrm{Ab}(D) \to \mathcal{A}$ *such that* $F\Delta = T$. *Furthermore, there is a Serre quotient* $\pi\colon \mathrm{Ab}(D) \twoheadrightarrow \mathcal{A}(T)$ *along with a faithful exact functor* $F_T\colon \mathcal{A}(T) \to \mathcal{A}$ *and an induced representation* $\widetilde{T}\colon D \to \mathcal{A}(T)$ *such that* $F_T\widetilde{T} = T$ *universally.*

We shall refer to the above as the *universal representation Theorem*. The universal property here can be visualized by the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Ab}(D) \arrow[d, "\pi"] \arrow[dr, bend left, "F"] & \\
D \arrow[ur, bend left, "\Delta"] \arrow[r, "\widetilde{T}"] \arrow[rr, bend right, "T"'] \arrow[dr, bend right, "S"'] & \mathcal{A}(T) \arrow[r, "F_T"] \arrow[d, dotted, "F_S"] & \mathcal{A} \\
& \mathcal{B} \arrow[ur, bend right, "G"'] &
\end{tikzcd}$$

where $GS = T$, $\mathcal{B}$ is abelian, $G$ and $F_S$ are faithful exact. The functors $F$, $F_T$ and $F_S$ are unique up to natural equivalence. Note that here we have that $F_S$ factors through an equivalence $\mathcal{A}(T) \xrightarrow{\simeq} \mathcal{A}(S)$ where $\mathcal{A}(S)$ is the corresponding universal abelian category associated to the representation $S\colon D \to \mathcal{B}$.

There is an easy $R$-linear variant of this Theorem, where $\mathrm{Ab}(D)$ has to be replaced by a universal $R$-linear abelian category $\mathrm{Ab}_R(D)$, showing Nori's Theorem in the particular case when $\mathcal{A} = R$- mod.

For example, for $D = \mathcal{D}$ an abelian $R$-linear category we easily see that $\widetilde{T}\colon \mathcal{D} \xrightarrow{\simeq} \mathcal{A}(T)$ is an equivalence for any $T\colon \mathcal{D} \to \mathcal{A}$ which is a faithful exact $R$-linear functor, e.g. by taking $\mathcal{B} = \mathcal{D}$, $S = id_{\mathcal{D}}$ and $G = T$ in the diagram above. We remark that this fact is [8, Theorem 7.1.20] (with a considerably longer proof there).

The construction of this abelian category $\mathcal{A}(T)$ attached to a representation $T$ gives us, in addition, an interpretation of $\mathcal{A}(T)$ as a certain category of functors on the definable category generated by $T$ (that is, the category of models of its regular theory).

We also describe another interpretation of $\mathcal{A}(T)$, as the category of pp-pairs and pp-defined maps for the theory of $T$, already present through the previous construction (we remark that "regular formula" and "pp formula" are alternative and equivalent terminologies). This interpretation sheds some light on the category $\mathcal{A}[\mathbb{T}]$, presented here as a Serre quotient of $\mathrm{Ab}(D)$ where $D$ is the canonical diagram associated to a pair $(\mathcal{C}, \mathcal{M})$ as mentioned above, and also on its Serre quotients $\mathcal{A}[\mathbb{T}']$ obtained by adding regular axioms to the homological theory $\mathbb{T}$ as explained in [4].

The plan of this paper is the following. In Section 1 we provide a proof of the universal representation Theorem and its $R$-linear variant. In Section 2 we give a description of $\mathrm{Ab}(D)$ and $\mathcal{A}(T)$ in terms of definable categories and we show the link with $\mathbb{T}$-motives.

*Notation*

We shall denote by Ab the category of abelian groups. For $\mathcal{A}$ and $\mathcal{B}$ preadditive categories $(\mathcal{A}, \mathcal{B})$ shall denote the category of additive functors from $\mathcal{A}$ to $\mathcal{B}$ where we tacitly assume that $\mathcal{A}$ is skeletally small, i.e. it has a set of objects up to isomorphism.

## 1. The universal representation of a quiver

Here we construct the universal representation $\Delta: D \to \mathrm{Ab}(D)$ of a quiver. We show how any representation $T$ of $D$ in an abelian category $\mathcal{A}$ lifts to an exact functor $F: \mathrm{Ab}(D) \to \mathcal{A}$ between abelian categories. We also describe an $R$-linear variant of this construction.

### 1.1 – *Freyd's free abelian category*

A category is *preadditive* if the set of morphisms between any two objects has an abelian group structure and composition of maps is bilinear. Note that a preadditive category with just one object is essentially a ring, namely the endomorphism ring of that object. Moreover, additive functors from that category to Ab are essentially modules over that ring. For this reason a preadditive category $\mathcal{R}$ may be referred to as a 'ring with many objects' (see, for instance, [11]) and the additive functors from $\mathcal{R}$ to the category Ab of abelian groups may be regarded as (left) $\mathcal{R}$-modules.

Let us suppose that we have an additive functor $M: \mathcal{R} \to \mathcal{A}$ with $\mathcal{A}$ abelian and $\mathcal{R}$ preadditive. Then there is an embedding of $\mathcal{R}$ into an abelian category (which depends just on $\mathcal{R}$) and an essentially unique lift to an exact functor, as stated next.

Theorem 1.1 (Freyd, [7, 4.1]). *Given a skeletally small preadditive category $\mathcal{R}$, there is a full and faithful embedding $\mathcal{R} \to \mathrm{Ab}(\mathcal{R})$ into an abelian category such that, for any additive functor $M: \mathcal{R} \to \mathcal{A}$, where $\mathcal{A}$ is abelian, there is a unique-to-natural-equivalence exact extension $F: \mathrm{Ab}(\mathcal{R}) \to \mathcal{A}$ as follows*

$$\begin{tikzcd} \mathcal{R} \arrow[r] \arrow[d, "M"'] & \mathrm{Ab}(\mathcal{R}) \arrow[dl, dotted, "F"] \\ \mathcal{A} & \end{tikzcd}$$

We will outline a construction and proof of this theorem in Section 1.3.

It might be that the functor $F$ above is not faithful but we may factor through a quotient abelian category which does depend on $M$.

Corollary 1.2. *Given a skeletally small preadditive category* $\mathcal{R}$ *and an additive functor* $M\colon \mathcal{R} \to \mathcal{A}$*, there is a small abelian category* $\mathcal{A}(M) := \mathrm{Ab}(\mathcal{R})/\ker(F)$*, i.e. the Serre quotient of* $\mathrm{Ab}(\mathcal{R})$ *by* $\ker(F)$*, and a commutative diagram as shown, where* $F_M$ *is exact and is faithful.*

$$\begin{tikzcd}
\mathcal{R} \arrow[r] \arrow[dr, "M"'] & \mathrm{Ab}(\mathcal{R}) \arrow[r, two heads, "\pi"] \arrow[d, "F"'] & \mathcal{A}(M) \arrow[dl, dotted, "F_M"] \\
& \mathcal{A} &
\end{tikzcd}$$

Proof. Recall that $\ker(F) := \{A \in \mathrm{Ab}(\mathcal{R})\colon F(A) = 0\}$. Since $F$ is exact, $\ker(F)$ is a *Serre subcategory* of $\mathrm{Ab}(\mathcal{R})$, meaning that if $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathrm{Ab}(\mathcal{R})$, then $B$ is in $\ker(F)$ iff $A, C \in \ker(F)$. Given any abelian category and a Serre subcategory $\mathcal{S}$, there is (e.g. [12, 4.3.3]) an exact functor to a quotient abelian category which is universal for exact functors with kernel containing $\mathcal{S}$. For $\mathcal{S} = \ker(F)$ we get the factorisation as claimed, where the induced functor $F_M\colon \mathcal{A}(M) \to \mathcal{A}$ is faithful and exact. □

We may call $\mathcal{A}(M)$ the *universal abelian category defined by* $M$. A universal property of $\mathcal{A}(M)$, generalising that of $\mathrm{Ab}(\mathcal{R})$ in Theorem 1.1, will be also given as Theorem 2.6. It replaces the category of all $\mathcal{R}$-modules by the definable category generated by $M$.

## 1.2 – *Representations of quivers and modules over path algebras*

Let's start with a diagram, or quiver, $D$, that is, a directed graph, given by a set $D_0$ of vertices and a set $D_1$ of arrows, plus source and target maps from $D_1$ to $D_0$. From $D$ we can build its *path category* $\bar{D}$. This has the same objects as $D$ and the morphisms are the identity maps and all the paths in $D$ where, by a *path* in $D$ we mean a sequence $\alpha_n \dots \alpha_1$ where the source of $\alpha_{i+1}$ is the target of $\alpha_i$ for each $i$ (so if $\alpha$ is a loop then every power of $\alpha$ is a path).

Given any category $\mathcal{C}$ we may define its enrichment $\mathbb{Z}\mathcal{C}$ in the category of abelian groups. This has the same objects as $\mathcal{C}$ and has, for the group of morphisms from $c$ to $d$, the free $\mathbb{Z}$-module on $\mathcal{C}(c, d)$, with composition being defined in the obvious way. Thus, in particular $\mathbb{Z}\bar{D}$, is a preadditive category.

A *representation* $T$ of a quiver $D$ in an abelian (or more generally, preadditive) category $\mathcal{A}$ is an assignment of, for each $s \in D_0$, an object $T_s \in \mathcal{A}$, and for each arrow $\alpha\colon s \to t$ in $D_1$, a morphism $T_\alpha\colon T_s \to T_t$ of $\mathcal{A}$. These are the objects of the category $\mathrm{Rep}_{\mathcal{A}}(D)$ of $\mathcal{A}$-representations of $D$. A morphism $f\colon T \to S$ between representations is a collection $(f_s)_{s \in D_0}$ of morphisms in $\mathcal{A}$ with $f_s\colon T_s \to S_s$ such that, for every $\alpha \in D_1$, $\alpha\colon s \to t$, we have

$$\begin{tikzcd} T_s \arrow[r, "T_\alpha"] \arrow[d, "f_s"'] & T_t \arrow[d, "f_t"] \\ S_s \arrow[r, "S_\alpha"'] & S_t \end{tikzcd}$$

a commutative square in the category $\mathcal{A}$.

Note that any representation $T\colon D \to \mathcal{A}$ extends uniquely to a functor $\overline{T}\colon \overline{D} \to \mathcal{A}$ from the path category $\overline{D}$ of $D$. In turn, the functor $\overline{T}$ extends uniquely to an additive functor, $M$ say, from $\mathbb{Z}\overline{D}$ to $\mathcal{A}$. Explicitly, on objects, $M$ agrees with $T$ and, if $\alpha_i \in D(d, d')$ and $n_i \in \mathbb{Z}$, then $M(\sum_i n_i\alpha_i) = \sum_i n_i T_{\alpha_i}$. In general, if we have a skeletally small preadditive category $\mathcal{R}$, then a *representation* of $\mathcal{R}$ in $\mathcal{A}$ will mean an *additive* functor from $\mathcal{R}$ to $\mathcal{A}$ and, again extending from the case where $\mathcal{R}$ has just one object, this is usually referred to as a *left $\mathcal{R}$-module* $M\colon \mathcal{R} \to \mathcal{A}$.

If $D$ is our diagram then the enrichment $\mathbb{Z}\overline{D}$ of the path category in Ab may be identified with the *$\mathbb{Z}$-path algebra*, $\mathbb{Z}D$, of $D$. This is formed by taking the free $\mathbb{Z}$-module on basis $D_0 \cup D_1^*$ where the latter denotes the set of all paths in $D$. To define the multiplication on $\mathbb{Z}D$ it is enough to define it on basis elements, where it is composition when defined, and 0 when not (i.e. if the target of the path $p$ is a different vertex from the source of the path $q$ then $qp = 0$). The arrows corresponding to the vertices of $D$ act as local identities. If there are just finitely vertices then $\mathbb{Z}D$ is a ring with 1 (the sum of the local identities).

Lemma 1.3. *The category* $\mathrm{Rep}_{\mathcal{A}}(D)$ *is naturally equivalent to the category* $(\mathbb{Z}D, \mathcal{A})$ *of additive functors from $\mathbb{Z}D$ to $\mathcal{A}$. In particular, the category of representations of $D$ in* Ab *is naturally equivalent to the category* $\mathbb{Z}D$-Mod *of left $\mathbb{Z}D$-modules.*

Proof. If we think of a $\mathbb{Z}D$-module as an additive functor from the preadditive category $\mathbb{Z}\overline{D}$ then, from representations of $D$ to $\mathbb{Z}D$-modules is the construction above, with the inverse being restriction of a $\mathbb{Z}D$-module to its values on the vertices and arrows of $D$. □

1.3 – *Proof of the universal representation Theorem*

We prove the theorem stated in the introduction, at the same time explaining (following [13, Chapter 4]) the construction and proof of Freyd's Theorem 1.1 in the context of the preadditive category $\mathcal{R} = \mathbb{Z}D$. We denote the category $\mathrm{Ab}(\mathbb{Z}D)$ by $\mathrm{Ab}(D)$ for short.

Note that the functor $M$ from $\mathbb{Z}D$ to $\mathcal{A}$ induced by $T \in \mathrm{Rep}_{\mathcal{A}}(D)$ has a unique extension to an additive functor $M^+$ from the additive completion $\mathbb{Z}D^+$ of $\mathbb{Z}D$. The objects of $\mathbb{Z}D^+$ are finite sequences of objects of $D$ and the arrows are rectangular matrices of arrows (with appropriate domains and codomains) from $\mathbb{Z}D$. The functor $M^+$ takes $(d_1, \ldots, d_n)$ to $T_{d_1} \oplus \cdots \oplus T_{d_n}$ and takes a morphism $(\alpha_{ij})_{ij}$ to $(M\alpha_{ij})_{ij}$. This may be further extended uniquely to an additive functor $M^{++}$ from the idempotent-splitting (= Karoubian = pseudoabelian) completion $\mathbb{Z}D^{++}$ to $\mathcal{A}$ (see, e.g. [11, p. 12] for a construction).

Lemma 1.4. *There is a natural equivalence between Grothendieck abelian categories*

$$\mathrm{Rep}_{\mathrm{Ab}}(D) \simeq \mathbb{Z}D\text{-}\mathrm{Mod} = (\mathbb{Z}D, \mathrm{Ab}) \simeq (\mathbb{Z}D^+, \mathrm{Ab}) \simeq (\mathbb{Z}D^{++}, \mathrm{Ab})$$

Proof. This is easily checked using Lemma 1.3; see, for example, [13, Chapter 2]. □

We tacitly keep using the equivalence

$$\mathbb{Z}D\text{-}\mathrm{Mod} \simeq \mathbb{Z}D^{++}\text{-}\mathrm{Mod} := (\mathbb{Z}D^{++}, \mathrm{Ab})$$

in what follows. Let $\mathbb{Z}D$-mod denote the category $(\mathbb{Z}D, \mathrm{Ab})^{\mathrm{fp}}$ of finitely presented left $\mathbb{Z}D$-modules. The general definition of an object $X$ being finitely presented, respectively finitely generated, is that the representable functor $(X, -)$ should commute with direct limits, resp. direct limits of monomorphisms; these coincide with the more familiar notions when those make sense. The Yoneda embedding $X \mapsto (X, -)$ from $\mathbb{Z}D^{++}$ to $(\mathbb{Z}D^{++}\text{-}\mathrm{Mod})^{\mathrm{op}}$ is an anti-equivalence of $\mathbb{Z}D^{++}$ with the category of finitely generated projectives in $\mathbb{Z}D$-Mod, and these are generating (e.g. [15, 10.1.12, 10.1.13]). So, in this context, the finitely presented modules are those with a projective presentation as the cokernel of

$$(Y, -) \xrightarrow{(\gamma, -)} (X, -)$$

for some $\gamma\colon X \to Y$ in $\mathbb{Z}D^{++}$.

We extend the $\mathbb{Z}D$-module $M$, equivalently $M^{++}$, to a left exact functor

$$F'\colon (\mathbb{Z}D\text{-}\mathrm{mod})^{\mathrm{op}} \longrightarrow \mathcal{A}$$

that is, to a right exact functor from $\mathbb{Z}D$-mod to $\mathcal{A}$, as follows (cf. the proof of Theorem 4.3 in [13] and the commentary before that). Let $N \in \mathbb{Z}D$-mod. Take $\gamma\colon X \to Y$ in $\mathbb{Z}D^{++}$ inducing, under the Yoneda embedding, a projective presentation

$$(Y, -) \longrightarrow (\gamma, -)(X, -) \longrightarrow N \longrightarrow 0$$

of $N$. Then define

$$F'(N) := \ker\big(M^{++}X \longrightarrow M^{++}\gamma M^{++}Y\big)$$

Using projectivity of representable functors, there is an induced action on morphisms and one checks that the action of $F'$ on objects is independent of the chosen projective presentation and that the action on morphisms is well defined.

Again we have a Yoneda functor $N^{\mathrm{o}} \to (N, -)$, fully embedding $(\mathbb{Z}D\text{-}\operatorname{mod})^{\mathrm{op}}$ in $(\mathbb{Z}D\text{-}\operatorname{mod}, \mathrm{Ab})^{\mathrm{fp}}$, where we use superscript $^{\mathrm{o}}$ to indicate objects and morphisms in the opposite category. The final step is to extend $F'$ to a well-defined, right exact, indeed exact, functor

$$F\colon (\mathbb{Z}D\text{-}\operatorname{mod}, \mathrm{Ab})^{\mathrm{fp}} \longrightarrow \mathcal{A}$$

which is defined by sending $\Theta \in (\mathbb{Z}D\text{-}\operatorname{mod}, \mathrm{Ab})^{\mathrm{fp}}$ to

$$F(\Theta) := \operatorname{coker}\big(F'N^{\mathrm{o}} \xrightarrow{F'g^{\mathrm{o}}} F'P^{\mathrm{o}}\big)$$

where $P \xrightarrow{g} N$ in $\mathbb{Z}D^{++}\text{-}\operatorname{mod}$ is such that $(N, -) \xrightarrow{(g,-)} (P, -) \to \Theta \to 0$ is a projective presentation in $(\mathbb{Z}D\text{-}\operatorname{mod}, \mathrm{Ab})^{\mathrm{fp}}$. Exactness of $F$ follows since it can be checked that application of $F$ to an exact sequence of functors is evaluation of those functors at $M$.

In summary, we obtain the following commutative diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[rrr, bend left, "\Delta"] \arrow[drr, bend right, "T"'] & \mathbb{Z}D \arrow[r] \arrow[dr, "M"'] & (\mathbb{Z}D\text{-}\operatorname{mod})^{\mathrm{op}} \arrow[r] \arrow[d, "F'"] & (\mathbb{Z}D\text{-}\operatorname{mod}, \mathrm{Ab})^{\mathrm{fp}} := \mathrm{Ab}(D) \arrow[dl, bend left, "F"] \\
& & \mathcal{A} &
\end{tikzcd}$$

where the functor from $\mathbb{Z}D$ to $\mathrm{Ab}(D)$ factors through $(\mathbb{Z}D^{++}\text{-}\operatorname{mod})^{\mathrm{op}}$ and/or equivalently $(\mathbb{Z}D\text{-}\operatorname{mod})^{\mathrm{op}}$ and it is the composition of two Yoneda embeddings. Thus, given $M\colon \mathbb{Z}D \to \mathcal{A}$ induced by $T \in \mathrm{Rep}_{\mathcal{A}}(D)$ with $\mathcal{A}$ abelian, one obtains the exact functor $F$.

Now let $\ker(F) := \{\Theta \in \mathrm{Ab}(D)\colon F(\Theta) = 0\}$ as in Corollary 1.2. This is a Serre subcategory of $\mathrm{Ab}(D)$ and so, by the universal property of the quotient

$$\pi\colon \mathrm{Ab}(D) \longrightarrow\!\!\!\!\rightarrow \mathcal{A}(T) := \mathcal{A}(M) = \mathrm{Ab}(D)/\ker(F)$$

there is an essentially unique, exact, functor $F_T\colon \mathcal{A}(T) \to \mathcal{A}$ with $F_T\pi = F$ as claimed. Clearly, the induced representation $\widetilde{T} := \pi\Delta$ from $D$ to $\mathcal{A}(T)$ is such that $F_T\widetilde{T} = F_T\pi\Delta = F\Delta = T$.

Furthermore, if we have $S \in \mathrm{Rep}_{\mathcal{B}}(D)$ with $\mathcal{B}$ abelian and $G: \mathcal{B} \to \mathcal{A}$ faithful exact such that $GS = T$ then let $H: \mathrm{Ab}(D) \to \mathcal{B}$ be the exact functor induced by Freyd's Theorem 1.1 such that $H\Delta = S$. So $GH\Delta = T$ and by uniqueness of $F$ we have a natural equivalence $GH \cong F$. This implies that $\mathcal{A}(T) = \mathcal{A}(S)$ since $\ker(F) = \ker(H)$, $G$ being faithful exact. We then have $\widetilde{T} = \widetilde{S}$ and get $F_S: \mathcal{A}(T) \to \mathcal{B}$ faithful exact such that $GF_S$ is naturally equivalent to $F_T$. The universal representation Theorem is then clear.

### 1.4 – *Nori's category and the $R$-linear case*

Note that for $R$ a commutative unitary ring we now can get an $R$-linear structure $RD$ in the same way as we did for $\mathbb{Z}D$ by considering the *$R$-path algebra*. This is the small preadditive $R$-linear category $RD$ given by the path category $\overline{D}$. We here define $RD$ to have the same objects as $\overline{D}$ and to have, for $R$-module of morphisms from $c$ to $d$, the free $R$-module on $\overline{D}(c, d)$, with composition being defined as usual.

Given this, and an additive $R$-linear category $\mathcal{A}$, the category $\mathrm{Rep}_{\mathcal{A}}(D)$ is then naturally equivalent to the category $(RD, \mathcal{A})$ of additive $R$-linear functors from $RD$ to $\mathcal{A}$. Now we have natural equivalences

$$(RD, R\text{-}\mathrm{Mod}) \simeq (RD^{+}, R\text{-}\mathrm{Mod}) \simeq (RD^{++}, R\text{-}\mathrm{Mod})$$

as above. Consider the category $(RD, R\text{-}\mathrm{Mod})^{\mathrm{fp}}$ of finitely presented $RD$-modules and set

$$\mathrm{Ab}_R(D) := ((RD, R\text{-}\mathrm{Mod})^{\mathrm{fp}}, R\text{-}\mathrm{Mod})^{\mathrm{fp}}$$

Thus given $M: RD \longrightarrow \mathcal{A}$ induced by $T \in \mathrm{Rep}_{\mathcal{A}}(D)$ with $\mathcal{A}$ abelian and $R$-linear, one obtains, as above, the functor $F: \mathrm{Ab}_R(D) \to \mathcal{A}$ which is exact and $R$-linear. We then set

$$\mathcal{A}(T) := \mathrm{Ab}_R(D)/\ker F$$

so that $F_T: \mathcal{A}(T) \to \mathcal{A}$ is also faithful and $R$-linear.

In particular, for $D = \mathcal{D}$ an abelian $R$-linear category we get an exact functor $\pi_R: \mathrm{Ab}_R(\mathcal{D}) \to \mathcal{D}$ which is a section of $\Delta_R: \mathcal{D} \to \mathrm{Ab}_R(\mathcal{D})$. For $T \in \mathrm{Rep}_{\mathcal{A}}(\mathcal{D})$ which is a faithful exact $R$-linear functor we have that the corresponding $F$ is given by the composition $T\pi_R$ so that $F_T = T$ and $\mathcal{A}(T) = \mathcal{D}$ in this case. Note that for any $S \in \mathrm{Rep}_{\mathcal{B}}(\mathcal{D})$ and $G: \mathcal{B} \to \mathcal{A}$ a faithful exact $R$-linear functor such that $GS = T$ we get that $S$ itself is a faithful exact $R$-linear functor and $F_S := S$ is justifying the universal property.

As a particular case, we get Nori's Theorem as a corollary of our universal representation Theorem for $\mathcal{A} = R$-mod and a Noetherian ring $R$. In fact, for $D$ a finite quiver and $T: D \to R$-mod, by universality of $\mathcal{A}(T)$ we get a faithful exact comparison functor $\iota: \mathcal{A}(T) \to \mathcal{C}(T) := \mathrm{End}(T)$-mod. It's not difficult but rather tedious to verify directly that $\iota$ is also full and essentially surjective. Alternatively, one can use the universal property of $\mathcal{C}(T)$ to get a quasi-inverse. We give another alternative proof in Proposition 2.3.

Corollary 1.5. *For a Noetherian ring $R$ and $\mathcal{A} = R$-mod we get $\mathcal{C}(T) \simeq \mathcal{A}(T)$.*

Let us also note that when the preadditive category $\mathcal{R}$ has an $R$-linear structure then so does $\mathrm{Ab}(\mathcal{R})$ and hence $\mathrm{Ab}_R(\mathcal{R})$ is naturally equivalent (as an $R$-linear category) to $\mathrm{Ab}(\mathcal{R})$ equipped with that $R$-linear structure. To see the $R$-linear structure on $\mathrm{Ab}(\mathcal{R})$ directly we can follow its construction. A finitely presented $\mathcal{R}$-module is the cokernel of a morphism $(Y, -) \xrightarrow{(\gamma,-)} (X, -)$ with $\gamma: X \to Y$ in $\mathcal{R}$. The $R$-linear structure of $\mathcal{R}$ means that $R$ acts as endomorphisms of the identity functor of $\mathcal{R}$, hence any additive functor from $\mathcal{R}$ to Ab factors through $R$-Mod. So there are induced $R$-module structures on $(Y, -)$ and $(X, -)$, and $(\gamma, -)$ will be a morphism of $R$-modules, so the cokernel also carries an $R$-module structure. Similarly for the second stage of the construction. We deduce that every object of $\mathrm{Ab}(\mathcal{R})$ is an $R$-module and all morphisms of $\mathrm{Ab}(\mathcal{R})$ are $R$-linear.

## 2. Theoretical motives via definable categories

Every skeletally small abelian category $\mathcal{A}$ has the form $\mathcal{A}(M)$ for some module $M$ over some small preadditive category $\mathcal{R}$ (see [16, 2.18]). Moreover, every exact functor from $\mathcal{A}$ to Ab has the form $F_M$ for some module $M$ (see [13, 10.8]). We may fix $\mathcal{A}$ and consider the category, $\mathrm{Ex}(\mathcal{A}, \mathrm{Ab})$, of exact functors from $\mathcal{A}$ to Ab. This is a typical *definable category*. These categories can be characterised in very different ways and the relation between $\mathrm{Ex}(\mathcal{A}, \mathrm{Ab})$ and $\mathcal{A}$ leads to alternative constructions of $\mathcal{A}(M)$ from $M$. This uses the equivalent views of these abelian categories as, on the one hand categories of pp-sorts and, on the other, as localisations of the category of functors on finitely presented modules.

### 2.1 – *Definable additive categories*

Definable categories first arose in the model theory of modules. We fix a ring, or skeletally small preadditive category, $\mathcal{R}$ and set up a formal language for

$\mathcal{R}$-modules (e.g. [13, §18]). Those formulas which are obtained by existentially quantifying out some variables from a homogeneous system of $\mathcal{R}$-linear equations are termed pp (for "positive primitive").

Each such formula $\phi$ defines, by taking an $\mathcal{R}$-module $M$ to its solution set $\phi(M)$, an additive functor $F_\phi$ from the category of $\mathcal{R}$-modules to Ab (solution sets to pp formulas are preserved by morphisms, so there is an induced action giving the value of $F_\phi$ on morphisms). When $\mathcal{R}$ is a ring with one object, $\phi(M)$ is a subgroup of $M^n$, $n$ being the number of free variables in $\phi$; in general it is a subgroup of the product of the sorts of $M$ (the groups which are the values of $M$ on the various objects of $\mathcal{R}$) corresponding to the sequence $\bar{x}$ of free variables appearing in the formula.

If a pp formula $\psi$ defines a subfunctor of $\phi$, that is if the implication $\psi \to \phi$ is true in every module (it is enough that it be true in every finitely presented module), then we may form the quotient functor $F_\phi/F_\psi$. In this case we refer to the "pp-pair" $\phi/\psi$ (or $\phi(\bar{x})/\psi(\bar{x})$ in order to show the free variables) and take the pp-pairs to be the objects of the *category*, denoted ${}_{\mathcal{R}}\mathbb{L}^{\mathrm{eq}+}$, *of pp-pairs*; the arrows are the pp-defined maps between pp-pairs - that is, the pp-defined relations between such pairs which are functional. This has a model-theoretic meaning as the category of pp-sorts and pp-definable maps for $\mathcal{R}$-modules (see [13, §22]) but it is also equivalent to Ab($\mathcal{R}$) as constructed above.

Theorem 2.1 (e.g. [15, 10.2.30, 10.2.37]). *The category* ${}_{\mathcal{R}}\mathbb{L}^{\mathrm{eq}+}$ *of pp-pairs is equivalent to* Ab($\mathcal{R}$).

The construction above localises to any given $\mathcal{R}$-module $M$ by factoring out the Serre subcategory consisting of those pairs $\phi/\psi$ which are *closed* on $M$, that is, with $\phi(M)/\psi(M) = 0$.

There is an equivalent categorical logic terminology, which is used in [3] and [4], with "regular formulas" and "regular sequents" replacing "pp formulas" and "pp-pairs." A pp-pair as above is replaced by the sequent (implication) $\vdash_{\bar{x}} \phi \to \psi$ where, without of loss of generality, it can be assumed that $\vdash_{\bar{x}} \psi \to \phi$ already is valid in every $\mathcal{R}$-module; then $\phi/\psi$ being closed on $M$ means that the sequent $\vdash_{\bar{x}} \phi \to \psi$ is valid in $M$. This is the context of Caramello's construction in [3] but that is done without assuming additivity from the outset. There, the abelian category $\mathcal{A}(M)$ is obtained as the effectivisation of the regular syntactic category for the theory of the module $M$. Essentially by the definitions, that category is naturally equivalent to the category ${}_{\mathcal{R}}\mathbb{L}^{\mathrm{eq}+}$ defined above. We will see next how, in the additive context, the link between regular theories and their categories of models can be expressed in terms of Serre subcategories and corresponding localisations of the free abelian category.

Given an $\mathcal{R}$-module $M$, we may, similarly to the construction of ${}_{\mathcal{R}}\mathbb{L}^{\mathrm{eq}+}$, define the category $\mathbb{L}^{\mathrm{eq}+}(M)$ of pp sorts and function symbols for the canonical language for $M$ (and for the modules in the definable category that $M$ generates, see below). The objects are pp-pairs as before, the morphisms are the pp-defined relations between pp-sorts which are functional when evaluated on $M$, equivalently which are provably functional in the regular theory of $M$, equivalently which are functional on every object in the definable category generated by $M$. The local version of 2.1 holds.

Theorem 2.2 (e.g. [15, 12.3.20]). *For any module $M$, the category $\mathbb{L}^{\mathrm{eq}+}(M)$ of pp-pairs for $M$ is equivalent to $\mathcal{A}(M)$.*

This gives us an interpretation of the exact functor which we have denoted $F_M$ in 1.2 as the unique extension, usually denoted $M^{\mathrm{eq}+}$, of a module $M$ to a structure for its canonical language (the language based on the the category of pp-sorts). This assigns to each object $\phi/\psi$ of $\mathbb{L}^{\mathrm{eq}+}(M)$ its value, $\phi(M)/\psi(M)$, on $M$ and, to each arrow from $\phi(\bar{x})/\psi(\bar{x})$ to $\phi'(\bar{y})/\psi'(\bar{y})$, it assigns the corresponding additive function from $\phi(M)/\psi(M)$ to $\phi'(M)/\psi'(M)$ (where this function is definable by some pp formula $\theta(\bar{x}, \bar{y})$). For more discussion of these languages see [14].

The kernel, $\mathcal{S}_M$ of the above localisation $\pi\colon \mathrm{Ab}(\mathcal{R}) \to \mathcal{A}(M)$ consists of the pp-pairs which are closed on $M$. A set of pairs which generates $\mathcal{S}_M$ as a Serre subcategory is exactly a set of regular sequents which generates the regular theory, $\mathbb{T}_M$, of $M$. Recall that the standard construction of a localised category such as $\mathcal{A}/\mathcal{S}_M$ changes the morphisms but not the objects. Therefore, given an $\mathcal{R}$-module $M$, we may take the objects of the localised category $\mathcal{A}(M)$ to be the pp-pairs in ${}_{\mathcal{R}}\mathbb{L}^{\mathrm{eq}+}(\simeq \mathrm{Ab}(\mathcal{R}))$, but there will be more maps (in particular every object of $\mathcal{S}_M$ will be isomorphic to 0).

Here is a proof of the assertion before Corollary 1.5.

Proposition 2.3. *Suppose that $M$ is a module over $RD$ where $R$ is a commutative noetherian ring and $D$ is a finite quiver, such that $M$ is finitely generated as an $R$-module. Set $S = \mathrm{End}(M)$ to be the endomorphism ring of the $RD$-module $M$ (with $S$ acting on the left). Then the abelian category $\mathcal{A}(M)$ associated to $M$ is equivalent to the category $S\text{-}\mathrm{mod}$ of finitely generated $S$-modules.*

Proof. First note that $S$ is a finitely generated $R$-module, so is Noetherian.

Consider the objects of $\mathcal{A}(M)$ as pp-pairs. We use the fact that every pp-definable subgroup of $M$ is a (finitely generated) $S$-submodule (see [15, 1.1.8]) so evaluation at $M$, $\mathrm{ev}_M$, is a functor from $\mathcal{A}(M)$ to $S$-mod. It is faithful, by the construction of $\mathcal{A}(M)$ (we already factored out the pp-pairs which are 0 when evaluated at $M$). Moreover, since $M$ is finitely presented over its endomorphism ring, $\mathrm{ev}_M$ is also full by (the argument for) [15, 6.1.21]. Since $\mathrm{ev}_M$ is exact, we therefore have an exact and full embedding of $\mathcal{A}(M)$ into the category $S$-mod. If we can show that the image contains the free $S$-module ${}_SS$ then we will be done, since the smallest abelian category containing ${}_SS$ is $S$-mod.

Suppose that $M_R = \sum_{i=1}^n a_i R$ and consider $\bar{a} = (a_1, \ldots, a_n) \in M^n$. If $f \in S$ and $f(\bar{a}) = 0$ then $f = 0$, so $S\bar{a} \simeq S$. Since $M^n$ is a finitely presented $RD$-module, there is a pp formula $\phi$ such that $\phi(M) = S\bar{a}$ [15, 1.2.12]. Thus we have the free $S$-module occurring as the value of the pp-pair $\phi(\bar{x})/\bar{x} = \bar{0}$, that is, as the value of $\mathrm{ev}_M$ at an object of $\mathcal{A}(M)$, as required. □

Note that the initial formal language, and hence the meaning of pp=regular formula and of the syntactic category, does depend on the choice of language. For instance the language based on $\mathbb{Z}D$ will have fewer sorts (there is some discussion of sorts in [13, §18, §22]) than that based on $\mathbb{Z}D^{++}$ but they are equivalent in that the solution set of any (pp) formula of the larger language can be defined, uniformly for all $\mathbb{Z}D$-modules, using (pp) formulas in the smaller language. In model theory this addition of definable new sorts is known as the imaginaries, or ${}^{\mathrm{eq}}$ construction (the notation ${}^{\mathrm{eq}+}$ indicates the construction restricted to pp formulas). New sorts are formed by introducing finite products of existing sorts, definable subsets of these and factoring by definable equivalence relations – the process referred to as effectivisation in the more category-theoretic model theory literature. The various initial choices for language/syntactic category do all lead to the same abelian category of pp-sorts which may, therefore, be regarded as the category underlying the richest, and canonical, language for $\mathcal{R}$-modules.

Now, given any set $\Phi$ of pp-pairs, we may consider the full subcategory of $\mathcal{R}$-modules consisting of the modules on which each pp-pair in $\Phi$ is closed. This is a typical *definable subcategory* of the category of $\mathcal{R}$-modules. In particular, given an $\mathcal{R}$-module $M$, we may take $\Phi$ to be the set of all pp-pairs closed on $M$ and then refer to the corresponding definable subcategory as the *definable additive category* $\langle M \rangle$ *generated by* $M$. It is the full subcategory of $\mathcal{R}$-Mod on those objects which are models of the regular theory of the $\mathcal{R}$-module $M$. These are the modules which satisfy all the sequents $\vdash \phi \to \psi$ in $\mathbb{T}_M$, equivalently those modules on which each pp-pair $\phi/\psi$ in $\mathbb{S}_M$ is closed. This category has the following algebraic characterisations, where an embedding of left $\mathcal{R}$-modules $N \xrightarrow{f} M$ is *pure* if, for every right $\mathcal{R}$-module $L$ the morphism $N \otimes_{\mathcal{R}} L \xrightarrow{f \otimes 1_L} M \otimes_{\mathcal{R}} L$ is monic; there is an equivalent definition in terms of pp formulas (e.g. [13, 5.2]).

THEOREM 2.4 (see [15, 3.4.7]). *The definable additive subcategory $\langle M\rangle$ of $\mathcal{R}$-* Mod *generated by an $\mathcal{R}$-module $M$ is the smallest full subcategory of $\mathcal{R}$-* Mod *containing $M$ which is closed under direct products, direct limits and pure submodules. It is also equivalent to the category of exact functors from $\mathcal{A}(M)$ to* Ab, *i.e.*

$$\langle M\rangle \simeq \mathrm{Ex}(\mathcal{A}(M), \mathrm{Ab})$$

(*this follows from the universal property of the abelian categories involved*).

As described above, the equivalence of the category $\mathcal{A}(M)$ with the category of pp-pairs and pp-definable maps for the regular theory of $M$ allows us to view $\mathcal{A}(M)$ as a category of functors from $\langle M\rangle$ to Ab. Indeed, as stated next, it is the category of those additive functors from $\langle M\rangle$ to Ab which commute with direct products and direct limits. Since every definable additive category has the form $\langle M\rangle$ for some $M$ ([15, 3.4.12]), we have the following where, for any definable category $\mathcal{D}$, we write $\mathrm{fun}(\mathcal{D}) = (\mathcal{D}, \mathrm{Ab})^{\prod\rightarrow}$ - the category of functors from $\mathcal{D}$ to Ab which commute with direct products and direct limits.

THEOREM 2.5 ([13, 12.10]). *For any definable additive category $\mathcal{D}$, there is an equivalence* $\mathrm{fun}(\mathcal{D}) \simeq \mathcal{A}(M)$ *where $M$ is any module such that $\langle M\rangle = \mathcal{D}$.*

*As seen already, in the other direction, $\mathcal{D} = \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathrm{Ab})$.*

Each module $M'$ in $\langle M\rangle$ has regular theory containing that of $M$; these will be equal iff $\langle M'\rangle = \langle M\rangle$, that is, in the regular theories terminology, if and only if $M'$ is a conservative model of the regular theory of $M$. Otherwise $M'$ satisfies more regular sequents than $M$, so $\mathcal{S}_{M'} \supsetneq \mathcal{S}_M$. In that case, $\langle M'\rangle$ will be a proper definable subcategory of $\langle M\rangle$ and, on the functor category side, $\mathcal{A}(M')$ will be a proper Serre quotient of $\mathcal{A}(M)$. A Serre-generating subset of the kernel, $\mathcal{S}_{M'}/\mathcal{S}_M$, of that quotient map is a set of regular axioms which must be added to the regular theory of $M$ in order to logically generate the regular theory of $M'$. We remark that every proper definable subcategory of $\langle M\rangle$ has the form $\langle M'\rangle$ for some $M' \in \langle M\rangle$. We may express this relation between $\langle M\rangle$ and $\mathcal{A}(M)$ by the following universal property.

THEOREM 2.6. *Suppose that $\mathcal{R}$ is an $R$-linear preadditive category, where $R$ is a commutative unital ring. Let $M\colon \mathcal{R} \to R$-* Mod *be an $R$-linear representation. Let $M' \in \langle M\rangle$, that is, suppose that $M'$ is a model of the regular theory of $M$. Then there is a unique, to natural equivalence, exact $R$-linear functor $G_{M'}\colon \mathcal{A}(M) \to$ $R$-* Mod *such that the following diagram*

$$
\begin{tikzcd}
\mathcal{R} \arrow[r] \arrow[d, "M'"'] & \mathcal{A}(M) \arrow[dl, dotted, "G_{M'}"] \\
R\text{-}\mathrm{Mod} &
\end{tikzcd}
$$

*commutes. The functor $G_{M'}$ will be faithful precisely if $\langle M' \rangle = \langle M \rangle$, that is, if and only if $M$ and $M'$ have the same regular theory.*

Proof. Recall from Section 1.4 that $\mathrm{Ab}(\mathcal{R})$ and hence $\mathcal{A}(M)$ carries an $R$-linear structure.

For existence, we take the composition of the canonical localisation

$$\pi_0 \colon \mathcal{A}(M) \longrightarrow \mathcal{A}(M')$$

(which exists since, by hypothesis, $\mathcal{S}_M \subseteq \mathcal{S}_{M'}$) with the functor $F_{M'}$ from the diagram after Theorem 1.1.

Suppose that the exact functor $G \colon \mathcal{A}(M) \to R\text{-}\mathrm{Mod}$ also makes the diagram commute. Precomposing with the localisation $\pi \colon \mathrm{Ab}(\mathcal{R}) \to \mathcal{A}(M)$, we deduce from Theorem 1.1 that $G\pi$ and $G_{M'}\pi$ are naturally equivalent and hence, by the universal property of the localisation $\pi$, that $G$ is naturally equivalent to $G_{M'}$.

Since $G_{M'}$ factors through the localisation $\mathcal{A}(M) \to \mathcal{A}(M')$ the last statement is direct from the discussion above, describing the kernel of this localisation in terms of regular theories. □

It can be useful to have a statement like Theorem 2.6 but starting with an $R$-linear representation $M \colon \mathcal{R} \to \mathcal{G}$ where $\mathcal{G}$ is any Grothendieck abelian $R$-linear category. In fact, the proof above works just as well in that case. Here is the diagram (which also illustrates the proof of the Theorem 2.6).

$$
\begin{tikzcd}
 & \mathcal{R} \arrow[r, "\Delta"] \arrow[dl, "M"'] \arrow[dd, "M'"] & \mathrm{Ab}(\mathcal{R}) \arrow[dr, "\pi"] \arrow[dd, "\pi'"] & \\
\mathcal{G} \arrow[dr, equal] & & & \mathcal{A}(M) \arrow[lll, "F_M"'] \arrow[dll, dotted, "G"'] \arrow[dl, dotted, "\pi_0"] \\
 & \mathcal{G} & \mathcal{A}(M') \arrow[l, "F_{M'}"] &
\end{tikzcd}
$$

## 2.2 – $\mathbb{T}$-*motives*

Consider now a category $\mathcal{C}$ and a distinguished subcategory $\mathcal{M}$ of $\mathcal{C}$. Let $\mathcal{C}^{\square}$ be the *category of pairs* with objects the arrows in $\mathcal{M}$ and morphisms the commutative squares of $\mathcal{C}$. We shall denote $(X, Y)$ an object of $\mathcal{C}^{\square}$, i.e. a morphism $f: Y \to X$ of $\mathcal{M}$, and $\square: (X, Y) \to (X', Y')$ a commutative square.

Following Nori, we can define a quiver $D$, which we can call the *Nori diagram of* $\mathcal{C}^{\square}$, with vertices the triples $(X, Y, i)$ for each $i \in \mathbb{Z}$ and arrows $(X, Y, i) \to (X', Y', i)$ associated to arrows $\square$ in $\mathcal{C}^{\square}$ with additional arrows $(X, Y, i) \to (Y, Z, i-1)$ corresponding to $\partial: (Y, Z) \to (X, Y)$ the morphism of $\mathcal{C}^{\square}$ given by $f: Z \to Y$ and $g: Y \to X$ objects of $\mathcal{C}^{\square}$.

We let $\mathbb{T}$ be the regular homological theory defined in [4]. The reader who is not familiar with the categorical logic terminology shall find the essential facts on regular theories in [5]. See also [4, §1] where one can find all needed preliminaries on theories and models along with other key references. In fact, for such a category $\mathcal{C}^{\square}$ and/or its Nori diagram $D$ there is a corresponding, slightly richer, signature $\Sigma$ where vertices are sorts and arrows are function symbols, see [4, §2]. We let $\mathcal{A}[\mathbb{T}]$ be the abelian category of constructible $\mathbb{T}$-motives as defined in [4, §4.1]. This is the Barr exact completion (or effectivization) of the regular syntactic category.

A model $H$ of the regular theory $\mathbb{T}$ in an abelian category $\mathcal{A}$ yields a representation $H \in \mathrm{Rep}_{\mathcal{A}}(D)$. In particular, for $\mathcal{A}[\mathbb{T}]$ and the universal model $H^{\mathbb{T}}$ (see [4, 4.1.5]) we get a representation $H^{\mathbb{T}} \in \mathrm{Rep}_{\mathcal{A}[\mathbb{T}]}(D)$. We can apply our universal representation Theorem to $D$, the Nori diagram of $\mathcal{C}^{\square}$, and we thus obtain: i) an exact functor $F: \mathrm{Ab}(D) \to \mathcal{A}[\mathbb{T}]$ lifting $H^{\mathbb{T}}$, ii) the abelian category $\mathcal{A}(\mathbb{T})$ defined to be the Serre quotient of $\mathrm{Ab}(D)$ by $\ker(F)$ and iii) the induced faithful exact functor $F_{\mathbb{T}}: \mathcal{A}(\mathbb{T}) \to \mathcal{A}[\mathbb{T}]$ filling in the following commutative diagram

$$\begin{tikzcd} & \mathrm{Ab}(D) \arrow[d, "\pi"] \arrow[dr, bend left, "F"] & \\ D \arrow[ur, bend left, "\Delta"] \arrow[r, "\widetilde{H}^{\mathbb{T}}"] \arrow[rr, bend right, "H^{\mathbb{T}}"'] & \mathcal{A}(\mathbb{T}) \arrow[r, dotted, "F_{\mathbb{T}}"] & \mathcal{A}[\mathbb{T}] \end{tikzcd}$$

THEOREM 2.7. *The functor* $F_{\mathbb{T}}: \mathcal{A}(\mathbb{T}) \xrightarrow{\simeq} \mathcal{A}[\mathbb{T}]$ *is an equivalence.*

PROOF. Since $F_{\mathbb{T}}$ is a faithful exact functor it preserves and reflects validity of regular sequents (or axioms, see [4, §1] for this terminology). Since $F_{\mathbb{T}}\widetilde{H}^{\mathbb{T}} = H^{\mathbb{T}}$ we then obtain that $\widetilde{H}^{\mathbb{T}} := \pi\Delta \in \mathrm{Rep}_{\mathcal{A}(\mathbb{T})}(D)$ is actually a model of $\mathbb{T}$ in the abelian category $\mathcal{A}(\mathbb{T})$. Now recall (see [4, 4.1.3]) that there is a natural equivalence, where $\mathbb{T}$-Mod$(\mathcal{A}(\mathbb{T}))$ denotes the category of $\mathbb{T}$-models in $\mathcal{A}(\mathbb{T})$,

$$\mathbb{T}\text{-Mod}(\mathcal{A}(\mathbb{T})) \cong \mathrm{Ex}(\mathcal{A}[\mathbb{T}], \mathcal{A}(\mathbb{T}))$$

so that we also obtain an exact functor $G: \mathcal{A}[\mathbb{T}] \to \mathcal{A}(\mathbb{T})$ corresponding to the $\mathbb{T}$-model $\widetilde{H}^{\mathbb{T}}$ above. By naturality, the composition $F_{\mathbb{T}}G$ is the identity since

$F_{\mathbb{T}}\widetilde{H}^{\mathbb{T}} = H^{\mathbb{T}}$ and the universal model corresponds to the identity. On the other hand, by construction of the equivalence between $\mathbb{T}$-models and exact functors, we have that $GH^{\mathbb{T}} \cong \widetilde{H}^{\mathbb{T}}$. Therefore, by uniqueness, using the universal representation Theorem, applied to $\widetilde{H}^{\mathbb{T}} \in \mathrm{Rep}_{\mathcal{A}(\mathbb{T})}(D)$, we then get that $GF_{\mathbb{T}}$ is naturally isomorphic to the identity. $\square$

A similar argument works for the theories $\mathbb{T}'$ obtained from $\mathbb{T}$ by adding regular axioms on the same signature. Since the universal model $H^{\mathbb{T}'}$ of $\mathbb{T}'$ in $\mathcal{A}[\mathbb{T}']$ is also a $\mathbb{T}$-model we get an exact functor $\mathcal{A}[\mathbb{T}] \to \mathcal{A}[\mathbb{T}']$ sending $H^{\mathbb{T}}$ to $H^{\mathbb{T}'}$. Recall that the universal model is a conservative model of the regular theory, i.e. the regular theory $\mathbb{T}'$ obtained by adding all regular axioms which are satisfied by the universal model is a conservative extension of $\mathbb{T}$ (see [5, Proposition 6.4]). We also easily obtain from Theorem 2.7 the following result.

Corollary 2.8. *If $\mathbb{T}'$ is obtained from $\mathbb{T}$ by adding regular axioms on the same signature then $\mathcal{A}[\mathbb{T}']$ is a Serre quotient of $\mathcal{A}[\mathbb{T}]$. Furthermore, $\mathbb{T}'$ is a conservative extension of $\mathbb{T}$ if and only if $\mathcal{A}[\mathbb{T}] \cong \mathcal{A}[\mathbb{T}']$ is an equivalence.*

Note that this is also the case for the regular theory $\mathbb{T}_H$ of any model $H \in \mathbb{T}\text{-Mod}(\mathcal{A})$ obtained by adding all regular axioms which are valid in the model $H$. As we regard a model as a representation we also get a corresponding commutative diagram

$$\begin{tikzcd}
& \mathrm{Ab}(D) \arrow[d, "\pi"] \arrow[dr] \arrow[drr] \arrow[drrr, bend left, "F"] & & & \\
D \arrow[ur, bend left, "\Delta"] \arrow[r, "\widetilde{H}"] \arrow[rrrr, bend right, "H"'] & \mathcal{A}(H) \arrow[r, dotted] & \mathcal{A}(\mathbb{T}_H) \arrow[r, dotted] & \mathcal{A}[\mathbb{T}_H] \arrow[r, dotted, "r_H"] & \mathcal{A}
\end{tikzcd}$$

where the functor $F$ is induced by $H \in \mathrm{Rep}_{\mathcal{A}}(D)$ and the exact functor

$$r_H : \mathcal{A}[\mathbb{T}_H] \longrightarrow \mathcal{A}$$

corresponding to $H \in \mathbb{T}\text{-Mod}(\mathcal{A})$ is faithful as, clearly, $H$ is a conservative $\mathbb{T}_H$-model. Arguing as above, $\mathcal{A}(\mathbb{T}_H) \to \mathcal{A}[\mathbb{T}_H]$ also is an equivalence. Since $r_H$ is faithful, $\mathcal{A}(H)$ and $\mathcal{A}[\mathbb{T}_H]$ are quotients of $\mathrm{Ab}(D)$ by the same Serre subcategory, hence are equivalent, and, so finally:

Corollary 2.9. *The functors $\mathcal{A}(H) \xrightarrow{\simeq} \mathcal{A}(\mathbb{T}_H) \xrightarrow{\simeq} \mathcal{A}[\mathbb{T}_H]$ are equivalences and the composition with $r_H$ coincides with the canonical faithful exact functor $F_H$ attached to $H \in \mathrm{Rep}_{\mathcal{A}}(D)$.*

Moreover, for any $H' \in \mathrm{Rep}_{\mathcal{A}'}(D)$, i.e. $H' \colon D \to \mathcal{A}'$, and $G \colon \mathcal{A}' \to \mathcal{A}$ faithful exact such that $GH' = H$ we have that $\mathcal{A}'(H') \xrightarrow{\simeq} \mathcal{A}(H)$. In particular, applying this to the case of Nori's singular homology representation $H = H^{\mathrm{sing}}$ in $\mathcal{A} = \mathrm{Ab}$ or $\mathbb{Z}$-mod we get (see [4, §4.2] and cf. [3])

Corollary 2.10. $\mathrm{EHM} \cong \mathcal{A}(H^{\mathrm{sing}}) \cong \mathcal{A}(\mathbb{T}_{H^{\mathrm{sing}}}) \cong \mathcal{A}[\mathbb{T}_{H^{\mathrm{sing}}}]$

Finally, as an immediate application of the universal representation Theorem, we see that the abelian category EHM can be obtained from several representations, e.g. for any $H' \in \mathrm{Rep}_{\mathcal{A}'}(D)$ such that $GH' = H^{\mathrm{sing}}$ under a faithful exact functor $G$. For example, by taking $\mathcal{A}' = \mathrm{MHS}$ the abelian category of (graded-polarizable) mixed Hodge structures, $H' = H^{\mathrm{MHS}} \in \mathrm{Rep}_{\mathcal{A}'}(D)$ Deligne's mixed Hodge structure representation of Nori's diagram and $G$ the forgetful functor from MHS to (finitely generated) abelian groups.

*Acknowledgments.* L. Barbieri-Viale acknowledges the support of the *Ministero dell'Istruzione, dell'Università e della Ricerca* (MIUR) through the Research Project (PRIN 2010-11) "Arithmetic Algebraic Geometry and Number Theory." M. Prest acknowledges the support of the *Engineering and Physical Sciences Research Council* (EPSRC) through the research grant EP/K022490/1, "Interpretation functors and infinite-dimensional representations of finite-dimensional algebras."

## References

[1] Y. André, *Une introduction aux motifs* (*motifs purs, motives mixtes, périodes*), Panoramas et Synthèses, 17. Société Mathématique de France, Paris, 2004.

[2] J. Ayoub – L. Barbieri-Viale, *Nori* 1*-motives*, Math. Ann. **361** (2015), pp. 367–402.

[3] L. Barbieri-Viale – O. Caramello – L. Lafforgue, *Syntactic categories for Nori motives*, preprint 2015, arXiv:1506.06113 [math.AG]

[4] L. Barbieri-Viale, $\mathbb{T}$*-motives*, J. Pure Appl. Algebra **221** (2017) pp. 1495–1898.

[5] C. Butz, *Regular categories & regular logic*, BRICS Lecture Series LS-98-2 (1998). http://www.brics.dk/LS/98/2/BRICS-LS-98-2.pdf

[6] N. Fakhruddin, *Notes of Nori's lectures on mixed motives*, TIFR, Mumbai, 2000.

[7] P. Freyd, *Representations in abelian categories*, in S. Eilenberg, D. K. Harrison, S. MacLane, H. Röhrl (eds.), *Proceedings of the Conference on Categorical Algebra, La Jolla, 1965*, Springer-Verlag New York, New York 1966, pp. 95–120.

[8] A. Huber – S. Müller-Stach, *Periods and Nori motives*, with contributions by B. Friedrich and J. von Wangenheim, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 65. Springer, Cham, 2017.

[9] F. Ivorra, *Perverse Nori motives*, to appear in Math. Research Letters.

[10] F. Ivorra – T. Yamazaki, *Nori motives of curves with modulus and Laumon* 1*-motives*, preprint 2016. arXiv:1609.08279 [math.AG]

[11] B. Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), pp. 1–161.

[12] N. Popescu, *Abelian categories with applications to rings and modules*, London Mathematical Society Monographs, 3, Academic Press, 1973.

[13] M. Prest, *Definable additive categories*: *purity and model theory*, Mem. Amer. Math. Soc. **210** (2011), no. 987.

[14] M. Prest, *Model theory in additive categories*, in B. Hart, Th. G. Kucera, A. Pillay, Ph. J. Scott, and R. A. G. Seely (eds.), *Models, logics, and higher-dimensional categories,* A tribute to the work of M. Makkai, Proceedings of the meeting held at the Université de Montréal, Montréal, QC, June 18–20, 2009, CRM Proceedings & Lecture Notes, 53, American Mathematical Society, Providence, R.I., 2011, pp. 231–244.

[15] M. Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, 121, Cambridge University Press, Cambridge, 2009.

[16] M. Prest, *Abelian categories and definable additive categories*, preprint 2012. arXiv:1202.0426 [math.CT]

Manoscritto pervenuto in redazione il 3 ottobre 2016.