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Some sufficient conditions for p-nilpotence of a finite group

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Abstract – In this paper, we give some new characterizations of finite p-nilpotent groups by using the notion of $\mathcal{H}C$-subgroups and extend several recent results.

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1. Introduction

In the present paper, we consider only finite groups. We use conventional notions and notation, as in Huppert (see [9]). G always denotes a finite group, $|G|$ is the order of G, p denotes a fixed prime, $\mathfrak{U}$ is the class of all supersoluble groups and $Z_{\mathfrak{U}}(G)$ is the product of all the normal subgroups of G whose G-chief factors have prime order. A normal subgroup E of G is said to be hypercyclically (resp. p-hypercyclically) embedded in G if every chief factor (resp. p-chief factor) of G below E is cyclic. If G/L is a supersoluble (resp. p-supersoluble), then G is supersoluble (resp. p-supersoluble) if and only if L is hypercyclically (resp. p-hypercyclically) embedded in G.

A subgroup H of G is said to be C-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of

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H in G (see [16]). A subgroup H of G is said to be an $\mathcal{H}$-subgroup of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$ (see [3]). Many people studied the structure of finite groups based on those two concepts and a lot of research has been given; see for example [1, 2, 3, 4, 6, 7, 11, 15, 16]. Recently, Wei and Guo (see [18]) introduced the following concept:

Definition 1.1. A subgroup H of G is said to be an $\mathcal{H}C$-subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$.

It is clear that each of C-normal subgroup and $\mathcal{H}$-subgroup implies that $\mathcal{H}C$-subgroup. The converse does not hold in general, see Examples 1 and 2 in [18]. In [17, 18], some conditions for a group to be supersolvable are given and many known results are generalized. In this paper, we give some new criteria for p-nilpotence of a finite group by assuming that some kind of subgroups having some fixed prime power order are $\mathcal{H}C$-subgroups.

2. Preliminaries

Lemma 2.1 ([18, Lemma 2.2]). *Suppose that H is an $\mathcal{H}C$-subgroup of G.*

(1) *If $H \leq K \leq G$, then H is an $\mathcal{H}C$-subgroup of K.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is an $\mathcal{H}C$-subgroup of G/N.*

(3) *If H is a p-subgroup and N is a normal p'-subgroup of G, then HN is an $\mathcal{H}C$-subgroup of G and HN/N is an $\mathcal{H}C$-subgroup of G/N.*

Proof. (1) and (2) is [18, Lemma 2.3]. (3) is [18, Lemma 2.4]. □

Lemma 2.2 ([17, Lemma 2.8]). *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If P is cyclic or P has a subgroup D with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is an $\mathcal{H}C$-subgroup of G, then G is p-nilpotent.*

Lemma 2.3. *Let P be a nontrivial normal p-subgroup of G. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is C-normal in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. It is a corollary of [13, Theorem]. □

Lemma 2.4 ([18, Lemma 2.5]). *Let K be a normal subgroup of G and H a normal subgroup of K. If H is an $\mathcal{H}C$-subgroup of G, then H is C-normal in G.*

Lemma 2.5 ([3, Theorem 6 (2)]). *Let H be an $\mathcal{H}$-subgroup of G. If H is subnormal in G, then H is normal in G.*

Lemma 2.6. *Let P be a nontrivial normal p-subgroup of G. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is an $\mathcal{H}$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Lemma 2.5, every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is normal in G. In view of Lemma 2.3, $P \leq Z_{\mathfrak{U}}(G)$. □

Lemma 2.7. *Let P be a nontrivial normal p-subgroup of G. If every maximal subgroup of P is an $\mathcal{H}C$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Lemma 2.4, every maximal subgroup of P is C-normal in G. In view of Lemma 2.3, $P \leq Z_{\mathfrak{U}}(G)$. □

Lemma 2.8. *Let P be a nontrivial normal p-subgroup of G. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is an $\mathcal{H}C$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. If every subgroup of P of order $|D|$ or 4 (if $|D| = 2$) is an $\mathcal{H}$-subgroup of G, then $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.6. Hence we may assume that there exists a subgroup H of P with $|H| = |D|$ such that H is not an $\mathcal{H}$-subgroup of G. By hypothesis, there exists a proper normal subgroup K of G such that $G = HK$ and $H^g \cap N_K(H) \leq H$ for all $g \in G$. Then we can pick a normal subgroup M of G such that $K \leq M$ and $|G : M| = p$. Obviously, $P \cap M$ is a maximal subgroup of P. If $|P : D| = p$, then $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.7. Hence we may assume that $|P : D| > p$. Then every subgroup of $P \cap M$ of order $|D|$ or 4 (if $|D| = 2$) is an $\mathcal{H}C$-subgroup of G. By induction, $P \cap M \leq Z_{\mathfrak{U}}(G)$. Since $|P/P \cap M| = p$, it follows that $P \leq Z_{\mathfrak{U}}(G)$. □

Lemma 2.9 ([18, Theorem 3.3]). *Let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is an $\mathcal{H}C$-subgroup of G.*

Lemma 2.10 ([5, Theorem 8.3.1]). *Let P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Then G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P.*

Lemma 2.11 ([8, Lemma 3.3]). *If G is p-supersoluble and $O_{p'}(G) = 1$, then G is supersoluble.*

For any group G, the generalized Fitting subgroup $F^*(G)$ is the set of all elements x of G which induce an inner automorphism on every chief factor of G. Clearly, $F^*(G)$ is a characteristic subgroup of G (see [10, X, 13]).

Lemma 2.12 ([14, Theorem C]). *Let E be a normal subgroup of G. If $F^*(E)$ is hypercyclically embedded in G, then E is also hypercyclically embedded in G.*

In the following, we shall denote by $F_p(G)$ the p-Fitting subgroup of G. In fact, $F_p(G) = O_{p'p}(G)$.

Lemma 2.13. *A p-soluble normal subgroup E of G is p-hypercyclically embedded in G if and only if $F_p(E)$ is p-hypercyclically embedded in G.*

Proof. We only need to prove the sufficiency. Suppose that the assertion is false and let (G, E) be a counterexample with $|G||E|$ minimal. We claim that $O_{p'}(E) = 1$. Indeed, since $F_p(E/O_{p'}(E)) = F_p(E)/O_{p'}(E)$, it is easy to verify that the hypothesis of the lemma holds for $(G/O_{p'}(E), E/O_{p'}(E))$. If $O_{p'}(E) \neq 1$, then the minimal choice of (G, E) implies that $E/O_{p'}(E)$ is p-hypercyclically embedded in $G/O_{p'}(E)$. Clearly $O_{p'}(E)$ is p-hypercyclically embedded in G. Therefore, E is p-hypercyclically embedded in G, a contradiction. Since E is p-soluble and $O_{p'}(E) = 1$, it follows that $F^*(E) = F(E) = F_p(E) = O_p(E)$, and so $F^*(E)$ is hypercyclically embedded in G. Applying Lemma 2.12, E is hypercyclically embedded in G, a contradiction again. □

3. Main Results

Theorem 3.1. *Let L be a p-soluble normal subgroup of G such that G/L is p-supersoluble, where p is a prime divisor of $|G|$. Suppose that for a Sylow p-subgroup P of $F_p(L)$, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is an $\mathcal{H}C$-subgroup of G. Then G is p-supersoluble. In particular, if p is the smallest prime divisor of $|G|$, then G is p-nilpotent.*

Proof. We distinguish two cases.

Case I: $O_{p'}(L) \neq 1$.

We consider the factor group $G/O_{p'}(L)$. Obviously, $(G/O_{p'}(L))/(L/O_{p'}(L)) \cong G/L$ is p-supersoluble. Since $O_{p'}(L/O_{p'}(L)) = 1$, we have

$$F_p(L/O_{p'}(L)) = O_p(L/O_{p'}(L)) = F_p(L)/O_{p'}(L) = PO_{p'}(L)/O_{p'}(L).$$

In view of Lemma 2.1(3), every subgroup of $F_p(L/O_{p'}(L))$ with order $|D|$ is an $\mathcal{H}C$-subgroup of $G/O_{p'}(L)$. Thus $G/O_{p'}(L)$ satisfies the hypothesis of the theorem. By induction, we have $G/O_{p'}(L)$ is p-supersoluble and so G is p-supersoluble.

Case II: $O_{p'}(L) = 1$.

Obviously, $F_p(L) = F(L) = O_p(L) = P$. Applying Lemma 2.8, $F_p(L)$ is hypercyclically embedded in G. In particular, $F_p(L)$ is p-hypercyclically embedded in G. In view of Lemma 2.13, L is p-hypercyclically embedded in G. Since G/L is p-supersoluble by hypothesis, it follows that G is p-supersoluble. □

Theorem 3.2. *Let L be a normal subgroup of G such that G/L is p-supersoluble, where p is the smallest prime divisor of $|L|$. Suppose that for a Sylow p-subgroup P of L, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is an $\mathcal{H}C$-subgroup of G. Then G is p-supersoluble. In particular, if p is also the smallest prime divisor of $|G|$, then G is p-nilpotent.*

Proof. By Lemma 2.1(1), it is easy to see that every subgroup H of P with $|H| = |D|$ (and order 4 if $|D| = 2$) is an $\mathcal{H}C$-subgroup of L. Applying Lemma 2.2, L is p-nilpotent. Then $O_{p'}(L)$ is the normal Hall p'-subgroup of L.

We distinguish two cases.

Case I: $O_{p'}(L) \neq 1$.

We consider the factor group $G/O_{p'}(L)$. Obviously,

$$(G/O_{p'}(L))/(L/O_{p'}(L)) \cong G/L$$

is p-supersoluble. In view of Lemma 2.1(3), every subgroup of $PO_{p'}(L)/O_{p'}(L))$ with order $|D|$ is an $\mathcal{H}C$-subgroup of $G/O_{p'}(L)$. Thus $G/O_{p'}(L)$ satisfies the hypothesis of the theorem. By induction, we have $G/O_{p'}(L)$ is p-supersoluble and so G is p-supersoluble.

Case II: $O_{p'}(L) = 1$.

Then L is a normal p-subgroup of G. Applying Lemma 2.8, L is hypercyclically embedded in G. Since G/L is p-supersoluble by hypothesis, it follows that G is p-supersoluble. □

Theorem 3.3. *Let p be an odd prime divisor of $|G|$. Suppose that G has a normal subgroup L such that G/L is p-nilpotent and P is a Sylow p-subgroup of L. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is an $\mathcal{H}C$-subgroup of G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

(1) $O_{p'}(G) = 1$.

Denote $T = O_{p'}(G)$. If $T > 1$, consider G/T. It is obvious that $(G/T)/(LT/T) \cong G/LT$ is p-nilpotent. Let HT/T be a subgroup of PT/T with order $|D|$, where H is a subgroup of P with order $|D|$. Since H is an $\mathcal{H}C$-subgroup of G, HT/T is an $\mathcal{H}C$-subgroup of G/T by Lemma 2.1(3). Again, $N_{G/T}(PT/T) = N_G(P)T/T$ is p-nilpotent since $N_G(P)$ is p-nilpotent. Hence G/T satisfies the hypothesis of the theorem. The choice of G implies that G/T is p-nilpotent, and hence G is p-nilpotent, a contradiction.

(2) Let K be a proper subgroup of G such that with $P \leq K$. Then K is p-nilpotent.

By Lemma 2.1(1), every subgroup H of P with order $|D|$ is an $\mathcal{H}C$-subgroup of K. Since $N_K(P) \leq N_G(P)$ and $N_G(P)$ is p-nilpotent, it follows that $N_K(P)$ is p-nilpotent. Hence K satisfies the hypothesis of the theorem. Then K is p-nilpotent by the minimal choice of G.

(3) $L = G$.

If $L < G$, then L is p-nilpotent by step (2). Let T be the normal p-complement of L. Then T char $L \trianglelefteq G$, so $T \trianglelefteq G$ and $T = 1$ by step (1). It follows that $L = P$ and $G = N_G(P)$ is p-nilpotent, a contradiction.

(4) $O_p(G) \neq 1$.

Consider the group $Z(J(P))$, where $J(P)$ is the Thompson subgroup of P. If $N_G(Z(J(P))) < G$, then $N_G(Z(J(P))$ is p-nilpotent by step (2). Then G is p-nilpotent by Lemma 2.10, a contradiction. Hence $N_G(Z(J(P))) = G$ and $1 < Z(J(P)) \leq O_p(G) < P$.

(5) $G/O_p(G)$ is p-nilpotent. In particular, $G/O_p(G)$ is p-supersoluble.

Let $\overline{G} = G/O_p(G)$, $\overline{P} = P/O_p(G)$, $\overline{K} = Z(J(\overline{P}))$ and $G_1/O_p(G) = N_{\overline{G}}(Z(J(\overline{P})))$. Since $O_p(\overline{G}) = 1$, we have $N_{\overline{G}}(Z(J(\overline{P})) < \overline{G}$. Thus $G_1 < G$.

By step (2), we have G_1 is p-nilpotent. Then $N_{\bar{G}}(Z(J(\bar{P}))$ is p-nilpotent. Thus $\bar{G}$ is p-nilpotent by Lemma 2.10.

(6) $G = PQ$, where Q is a Sylow q-subgroup of G with $p > q$.

Step (5) shows that G is p-soluble. Then there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G for any $q \in \pi(G)$ with $q \neq p$ by [5, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by step (1). Hence $Q \leq C_G(O_p(G)) \leq O_p(G)$ by [12, Theorem 9.3.1], a contradiction. Thus $PQ = G$. By virtue of Lemma 2.2, $p > q$.

(7) $|P| > p|D|$.

This follows from Lemma 2.9.

(8) $O_p(G)$ is a maximal subgroup of P.

By step (5), we may assume that $G/O_p(G)$ has a normal Hall p'-subgroup $T/O_p(G)$. Obviously, T is normal in G and G/T is p-group. Then there exists a normal subgroup M of G such that $T \leq M$ and $|G : M| = p$. It is easy to see that $P \cap M$ is a maximal subgroup of P and also a Sylow p-subgroup of M. If $N_G(P \cap M) < G$, then, by step (1), $N_G(P \cap M)$ is p-nilpotent and so is $N_M(P \cap M)$. From step (7) and Lemma 2.1(1), every subgroup H of $P \cap M$ with order $|D|$ is an $\mathcal{HC}$-subgroup of M. Consequently, M satisfies the hypothesis of our theorem. Hence M is p-nilpotent by the minimal choice of G. Then G is p-nilpotent. This contradiction shows that $P \cap M$ is a normal p-subgroup of G. Since $O_p(G) < P$, it follows that $P \cap M = O_p(G)$ and so $O_p(G)$ is a maximal of P.

(9) $O_p(G)$ is hypercyclically embedded in G.

By steps (7) and (8), $|D| < |O_p(G)|$. By the hypothesis of the theorem, every subgroup H of $O_p(G)$ with order $|D|$ is an $\mathcal{HC}$-subgroup of G. Applying Lemma 2.8, we have step (9).

(10) G is supersoluble.

Since $G/O_p(G)$ is p-supersoluble and $O_p(G)$ is hypercyclically embedded in G, it follows that G is p-supersoluble. By Lemma 2.11 and step (1), G is supersoluble.

(11) Final contradiction.

Since G possesses a Sylow tower of supersolvable type, it follows that P is normal in G by step (6). Therefore, $G = N_G(P)$ is p-nilpotent by hypothesis, a contradiction. □

THEOREM 3.4. *Let p be an odd prime divisor of $|G|$. Suppose that G has a normal subgroup L such that G/L is p-nilpotent and P is a Sylow p-subgroup of L. If there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with $|H| = |D|$ is an $\mathcal{H}C$-subgroup of G and $N_G(H)$ is p-nilpotent, then G is p-nilpotent.*

PROOF. We consider the following two case.

CASE I: $L = G$.

Assume the theorem is not true and let G be a counterexample of minimal order. With a similar argument as in steps (1) and (2) of the proof of Theorem 3.3, we have the following steps (1) and (2).

(1) $O_{p'}(G) = 1$.

(2) Let K be a proper subgroup of G such that with $P \le K$. Then K is p-nilpotent.

(3) P is a normal subgroup of G.

If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent by step (2). Applying Theorem 3.3, G is p-nilpotent. This contradiction implies P is normal in G.

(4) P is hypercyclically embedded in G.

By hypothesis every subgroup H of P with order $|D|$ is an $\mathcal{H}C$-subgroup of G, then, from Lemma 2.8, (4) holds.

(5) Let N be a minimal normal subgroup of G. Then $|N| < |D| < |P|$.

In view of step (3), G is p-soluble. Then N is a p-subgroup by step (1) and so $N \le P$. By virtue of step (4), $|N| = p$. If $|N| = |D|$, then $G = N_G(N)$ is p-nilpotent by the hypothesis of the theorem. This contradiction shows that $|N| < |D|$.

(6) Final contradiction.

By Lemma 2.1(2), it is easy to see that G/N satisfies the hypothesis of the theorem. Hence G/N is p-nilpotent by the minimal choice of G. Since the class of all p-nilpotent groups is a saturated formation, it follows that N is a unique minimal subgroup of G and $\Phi(G) = 1$. Consequently, $F(G) = N$. By steps (1) and (3), $F(G) = O_p(G) = P$. Hence $N = P$, contrary to step (5).

Case II: $L < G$.

By Lemma 2.1(1), every subgroup H of P with order $|D|$ is an $\mathcal{H}C$-subgroup of L. Obviously, $N_L(H)$ is p-nilpotent. By virtue of Case I, L is p-nilpotent. It follows that $L_{p'}$ is the normal Hall p'-subgroup of L. Clearly, $L_{p'} \trianglelefteq G$. If $L_{p'} \neq 1$, then it is easy to see that $G/L_{p'}$ satisfies the hypothesis of the theorem by virtue of Lemma 2.1(3). Hence $G/L_{p'}$ is p-nilpotent by induction. It follows that G is p-nilpotent. Hence we may assume that $L_{p'} = 1$. Then $L = P$. Since G/P is p-nilpotent, we may let V/P be the normal Hall p'-subgroup of G/P. By Schur-Zassenhaus Theorem, V has a Hall p'-subgroup $V_{p'}$. By Lemma 2.1(1), every subgroup H of P with order $|D|$ is an $\mathcal{H}C$-subgroup of V. Obviously, $N_V(H)$ is p-nilpotent. In view of Case I, $V = PV_{p'}$ is p-nilpotent and so $V_{p'}$ is normal in V. Obviously, $V_{p'}$ is also a normal p-complement of G and so G is p-nilpotent. □

Remark 3.5. Frobenius asserts that G is p-nilpotent if $N_G(H)$ is p-nilpotent for every p-subgroup H of G (see [9, Satz. IV.5.8]). In Theorem 3.4, we replace a condition of the Frobenius' theorem, namely, H is restricted to be a p-subgroup of a fixed order and we assume that H is an $\mathcal{H}C$-subgroup of G. Hence Theorem 3.4 can be considered as an extension of the Frobenius' theorem.

Corollary 3.6 ([1, Theorem 1.1]). *Let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is an $\mathcal{H}$-subgroup of G.*

Corollary 3.7 ([6, Theorem 3.1]). *Let p be an odd prime dividing $|G|$ and P a Sylow p-subgroup of G. If every maximal subgroup of P is C-normal in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

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