

Cyclic subgroup commutativity degrees of finite groups

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ABSTRACT – In this paper we introduce and study the concept of cyclic subgroup commutativity degree of a finite group G . This quantity measures the probability of two random cyclic subgroups of G commuting. Explicit formulas are obtained for some particular classes of groups. A criterion for a finite group to be an Iwasawa group is also presented.

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1. Introduction

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects which have been studied is the probability that two elements of a finite group G commute. It is called the *commutativity degree* of G and has been investigated in many papers, such as [2, 3] and [5]–[9]. Inspired by this concept, in [16] we introduced a similar notion for the subgroups of G , called the *subgroup commutativity degree* of G . This quantity is defined by

$$(1) \quad \begin{aligned} \text{sd}(G) &= \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}| \\ &= \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK \in L(G)\}| \end{aligned}$$

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(where $L(G)$ denotes the subgroup lattice of G) and it measures the probability that two subgroups of G commute, or equivalently the probability that the product of two subgroups of G be a subgroup of G (recall also the natural generalization of $\text{sd}(G)$, namely the *relative subgroup commutativity degree* of a subgroup of G , introduced and studied in [17]).

Another two probabilistic notions on $L(G)$ have been investigated in [20] and [21]: the *normality degree* and the *cyclicity degree* of G . They are defined by

$$\text{ndeg}(G) = \frac{|N(G)|}{|L(G)|} \quad \text{and} \quad \text{cdeg}(G) = \frac{|L_1(G)|}{|L(G)|},$$

where $N(G)$ and $L_1(G)$ denote the normal subgroup lattice and the poset of cyclic subgroups of G , and measure the probability of a random subgroup of G to be normal or cyclic, respectively.

Clearly, in the definition of $\text{sd}(G)$ we may restrict to one of the above remarkable subsets of $L(G)$. In the case of $N(G)$ nothing can be said, since normal subgroups commute with all subgroups of G . By taking $L_1(G)$ instead of $L(G)$ in (1) a new significant quantity is obtained, namely

$$\begin{aligned} \text{csd}(G) &= \frac{1}{|L_1(G)|^2} |\{(H, K) \in L_1(G)^2 \mid HK = KH\}| \\ &= \frac{1}{|L_1(G)|^2} |\{(H, K) \in L_1(G)^2 \mid HK \in L(G)\}|. \end{aligned}$$

This measures the probability that two cyclic subgroups of G commute and will be called the *cyclic subgroup commutativity degree* of G . Its study is the purpose of the current paper.

The paper is organized as follows. Some basic properties and results on cyclic subgroup commutativity degree are presented in Section 2. Section 3 deals with cyclic subgroup commutativity degrees for some special classes of finite groups: P -groups, dihedral groups and p -groups possessing a cyclic maximal subgroup. As an application, in Section 4 we give a criterion for a finite group to be an Iwasawa group. In the final section some further research directions and a list of open problems are indicated.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [4, 14]. For subgroup lattice concepts we refer the reader to [13, 15, 19].

2. Basic properties of cyclic subgroup commutativity degree

Let G be a finite group. First of all, we remark that the cyclic subgroup commutativity degree $\text{csd}(G)$ satisfies the following relation

$$0 < \text{csd}(G) \leq 1.$$

Moreover, by consequence (9) on page 202 of [13], the permutability of a subgroup $H \in L_1(G)$ with all cyclic subgroups of G is equivalent with the permutability of H with all subgroups of G . This shows that

$$\text{csd}(G) = 1 \iff \text{sd}(G) = 1$$

and therefore the finite groups G satisfying $\text{csd}(G) = 1$ are in fact the Iwasawa groups, i.e. the nilpotent modular groups (see [13, Exercise 3, p. 87]). Notice that a complete description of these groups is given by a well-known Iwasawa's result (see Theorem 2.4.13 of [13]). In particular, we infer that $\text{csd}(G) = 1$ for all Dedekind groups G .

Given $H \in L_1(G)$, we will denote by $C_1(H)$ the set consisting of all cyclic subgroups of G commuting with H , that is

$$C_1(H) = \{K \in L_1(G) \mid HK = KH\}.$$

Then

$$(2) \quad \text{csd}(G) = \frac{1}{|L_1(G)|^2} \sum_{H \in L_1(G)} |C_1(H)|,$$

which leads to a precise expression of $\text{csd}(G)$ for finite groups G whose cyclic subgroup structure is known.

EXAMPLE 2.1. The alternating group A_4 has eight cyclic subgroups, namely: the trivial subgroup H_1 , three subgroups $H_i \cong \mathbb{Z}_2, i = 2, 3, 4$, and four subgroups $H_i \cong \mathbb{Z}_3, i = 5, 6, 7, 8$. We can easily see that $|C_1(H_1)| = 8, |C_1(H_i)| = 4$ for $i = 2, 3, 4$, and $|C_1(H_i)| = 5$ for $i = 5, 6, 7, 8$. Hence

$$\text{csd}(A_4) = \frac{1}{64} (8 + 3 \cdot 4 + 4 \cdot 5) = \frac{5}{8}.$$

Clearly, we have $L(H) \cup (N(G) \cap L_1(G)) \subseteq C_1(H)$, for all $H \in L_1(G)$, implying that

$$\text{csd}(G) \geq \frac{1}{|L_1(G)|^2} \sum_{H \in L_1(G)} |L(H) \cup (N(G) \cap L_1(G))|.$$

By this inequality some lower bounds for $\text{csd}(G)$ can be inferred, namely

$$\text{csd}(G) \geq \frac{1}{|L_1(G)|^2} \sum_{H \in L_1(G)} |N(G) \cap L_1(G)| = \frac{|N(G) \cap L_1(G)|}{|L_1(G)|}$$

and

$$\text{csd}(G) \geq \frac{1}{|L_1(G)|^2} \sum_{H \in L_1(G)} |L(H)| \geq \frac{2|L_1(G)| - 1}{|L_1(G)|^2},$$

since $|L(H)| \geq 2$ for every non-trivial cyclic subgroup H of G . Another lower bound for $\text{csd}(G)$ follows by the simple remark that for every subgroup M of G we have

$$\{(H, K) \in L_1(G)^2 \mid HK = KH\} \supseteq \{(H, K) \in L_1(M)^2 \mid HK = KH\}.$$

Thus

$$\text{csd}(G) \geq \left(\frac{|L_1(M)|}{|L_1(G)|} \right)^2 \text{csd}(M).$$

In particular, if M is abelian, then $\text{csd}(M) = 1$ and so

$$\text{csd}(G) \geq \left(\frac{|L_1(M)|}{|L_1(G)|} \right)^2.$$

Assume next that G_1 and G_2 are two finite groups. If $G_1 \cong G_2$, then $\text{csd}(G_1) = \text{csd}(G_2)$. The same thing cannot be said in the case when G_1 and G_2 are only lattice-isomorphic, as shows the following elementary example.

EXAMPLE 2.2. It is well known that the subgroup lattices of $G_1 = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $G_2 = S_3$ are isomorphic. On the other hand, we have $\text{csd}(G_1) = 1$ because G_1 is abelian, but $\text{csd}(G_2) \neq 1$ because G_2 is not nilpotent (more precisely, we can easily check that $\text{csd}(G_2) = 19/25$).

By a direct calculation, one obtains

$$\text{csd}(S_3 \times \mathbb{Z}_3) = \frac{85}{121} \neq \frac{19}{25} = \text{csd}(S_3) \text{csd}(\mathbb{Z}_3)$$

and consequently in general we don't have $\text{csd}(G_1 \times G_2) = \text{csd}(G_1) \text{csd}(G_2)$. A sufficient condition in order to this equality holds is that G_1 and G_2 be of coprime orders, because in this case any two cyclic subgroups H, K of $G_1 \times G_2$ are of type

$$H = H_1 \times H_2 \quad \text{and} \quad K = K_1 \times K_2 \quad \text{with} \quad H_i, K_i \leq G_i, \quad i = 1, 2$$

and

$$HK = KH \iff H_i K_i = K_i H_i, \quad i = 1, 2.$$

This remark can naturally be extended to arbitrary finite direct products.

PROPOSITION 2.3. *Let $(G_i)_{i=\overline{1,k}}$ be a family of finite groups having coprime orders. Then*

$$(3) \quad \text{csd} \left(\bigtimes_{i=1}^k G_i \right) = \prod_{i=1}^k \text{csd}(G_i).$$

The following immediate consequence of Proposition 2.3 shows that computing the cyclic subgroup commutativity degree of a finite nilpotent group is reduced to finite p -groups.

COROLLARY 2.4. *If G is a finite nilpotent group and $(G_i)_{i=\overline{1,k}}$ are the Sylow subgroups of G , then*

$$\text{csd}(G) = \prod_{i=1}^k \text{csd}(G_i).$$

REMARK 2.5. The condition in the hypothesis of Proposition 2.3 is not necessary to obtain the equality (3). For example, we have

$$\text{csd}(S_3 \times \mathbb{Z}_2) = \frac{19}{25} = \text{csd}(S_3) \text{csd}(\mathbb{Z}_2),$$

even though the groups S_3 and \mathbb{Z}_2 are not of coprime orders.

3. Cyclic subgroup commutativity degrees for some classes of finite groups

In this section we will compute explicitly the cyclic subgroup commutativity degree of several semidirect products for which we are able to describe the cyclic subgroup structure.

3.1 – The cyclic subgroup commutativity degree of finite P -groups

First of all, we recall the notion of P -group, according to [13]. Let p be a prime, $n \geq 2$ be a cardinal number and G be a group. We say that G belongs to the class $P(n, p)$ if it is either elementary abelian of order p^n , or a semidirect product of an elementary abelian normal subgroup M of order p^{n-1} by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on M . The group G is called a P -group if $G \in P(n, p)$ for some prime p and some cardinal number $n \geq 2$. It is well known that the class $P(n, 2)$ consists only of the elementary abelian group of order 2^n . Also, for $p > 2$ the class $P(n, p)$ contains the elementary abelian group of order p^n and, for every prime divisor q of $p - 1$, exactly one non-abelian P -group with elements of order q . Moreover, the order

of this group is $p^{n-1}q$ if n is finite. The most important property of the groups in a class $P(n, p)$ is that they are all lattice-isomorphic (see Theorem 2.2.3 of [13]).

In the following, we will focus on finite non-abelian P -groups. So, assume that $p > 2$ and $n \in \mathbb{N}$ are fixed, and take a divisor q of $p - 1$. The non-abelian group of order $p^{n-1}q$ in the class $P(n, p)$ will be denoted by $G_{n,p}$. By Remarks 2.2.1 of [13], it is of type

$$G_{n,p} = M \langle x \rangle,$$

where $M \cong \mathbb{Z}_p^{n-1}$ (i.e. the direct product of $n - 1$ copies of \mathbb{Z}_p), $o(x) = q$ and there exists an integer r such that $x^{-1}yx = y^r$, for all $y \in M$. Notice that we have

$$N(G_{n,p}) = L(M) \cup \{G_{n,p}\}.$$

The set $L_1(G_{n,p})$ has been described in [15]: it consists of the trivial subgroup 1, of the subgroups of order p in M and of the subgroups of type $\langle yx \rangle$ with $y \in M$. Then

$$|L_1(G_{n,p})| = 1 + \frac{p^{n-1} - 1}{p - 1} + p^{n-1} = 2 + p + p^2 + \cdots + p^{n-1}.$$

On the other hand, we have

$$C_1(H) = L_1(G_{n,p}), \text{ for all } H \leq M,$$

and

$$C_1(\langle yx \rangle) = L_1(M) \cup \{\langle yx \rangle\}, \text{ for all } y \in M.$$

In this way, an explicit value of $\text{csd}(G_{n,p})$ is obtained by using (2).

THEOREM 3.1.1. *The cyclic subgroup commutativity degree of the P -group $G_{n,p}$ is given by the following equality:*

$$\text{csd}(G_{n,p}) = \frac{1}{(2+p+p^2+\cdots+p^{n-1})^2} \left(\frac{(2+p+p^2+\cdots+p^{n-2})}{(2+p+p^2+\cdots+p^{n-1})} + p^{n-1}(3+p+p^2+\cdots+p^{n-2}) \right).$$

We observe that for $p = 3$, $q = 2$ and $n = 2$ we have $G_{2,3} \cong S_3$, and hence $\text{csd}(S_3) = 19/25$ can be also computed by the above formula. The following consequence of Theorem 3.1.1 is immediate, too.

COROLLARY 3.1.2. $\lim_{n \rightarrow \infty} \text{csd}(G_{n,p}) = (2p - 1)/p^2.$

3.2 – The cyclic subgroup commutativity degree of finite dihedral groups

The dihedral group D_{2m} ($m \geq 2$) is the symmetry group of a regular polygon with m sides and it has the order $2m$. The most convenient abstract description of D_{2m} is obtained by using its generators: a rotation x of order m and a reflection y of order 2. Under these notations, we have

$$D_{2m} = \langle x, y \mid x^m = y^2 = 1, yxy = x^{-1} \rangle.$$

It is well known that for every divisor r of m , D_{2m} possesses a subgroup isomorphic to \mathbb{Z}_r , namely $H_0^r = \langle x^{\frac{m}{r}} \rangle$, and $\frac{m}{r}$ subgroups isomorphic to D_{2r} , namely $H_i^r = \langle x^{\frac{m}{r}}, x^{i-1}y \rangle, i = 1, 2, \dots, \frac{m}{r}$. The normal subgroups of D_{2m} are

$$N(D_{2m}) = \begin{cases} L(H_0^m) \cup \{G\}, & m \equiv 1 \pmod{2}, \\ L(H_0^m) \cup \{G, H_1^{\frac{m}{2}}, H_2^{\frac{m}{2}}\}, & m \equiv 0 \pmod{2}, \end{cases}$$

while the cyclic subgroups of D_{2m} are

$$L_1(D_{2m}) = L(H_0^m) \cup \{H_i^1 \mid i = 1, 2, \dots, m\}.$$

It follows that

$$|L_1(D_{2m})| = \tau(m) + m,$$

where $\tau(m)$ denotes the number of divisors of m . Clearly, we have

$$|C_1(H)| = \tau(m) + m, \text{ for all } H \in L(H_0^m).$$

On the other hand, it is easy to see that

$$C_1(H_i^1) = \begin{cases} L(H_0^m) \cup \{H_i^1\}, & m \equiv 1 \pmod{2}, \\ L(H_0^m) \cup \{H_i^1, H_{i+\frac{m}{2}}^1\}, & m \equiv 0 \pmod{2}, \end{cases}$$

and therefore

$$|C_1(H_i^1)| = \begin{cases} \tau(m) + 1, & m \equiv 1 \pmod{2}, \\ \tau(m) + 2, & m \equiv 0 \pmod{2}, \end{cases}$$

for all $i = 1, 2, \dots, m$. Then (2) leads to the following result.

THEOREM 3.2.1. *The cyclic subgroup commutativity degree of the dihedral group D_{2m} is given by the following equality:*

$$\text{csd}(D_{2m}) = \begin{cases} \frac{\tau(m)(\tau(m) + m) + m(\tau(m) + 1)}{(\tau(m) + m)^2}, & m \equiv 1 \pmod{2} \\ \frac{\tau(m)(\tau(m) + m) + m(\tau(m) + 2)}{(\tau(m) + m)^2}, & m \equiv 0 \pmod{2}. \end{cases}$$

The cyclic subgroup commutativity degree of the dihedral group D_{2^n} is obtained directly from Theorem 3.2.1.

COROLLARY 3.2.2. *We have*

$$\text{csd}(D_{2^n}) = \frac{n^2 + (n+1)2^n}{(n+2^{n-1})^2}$$

and in particular

$$\text{csd}(D_8) = \frac{41}{49}.$$

We are also able to compute the limit of $\text{csd}(D_{2^n})$ when $n \rightarrow \infty$.

COROLLARY 3.2.3. $\lim_{n \rightarrow \infty} \text{csd}(D_{2^n}) = 0$.

3.3 – The subgroup commutativity degree of finite p -groups possessing a cyclic maximal subgroup

Let p be a prime, $n \geq 3$ be an integer and denote by \mathcal{G} the class consisting of all finite p -groups of order p^n having a maximal subgroup which is cyclic. Obviously, \mathcal{G} contains finite abelian p -groups of type $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$ whose cyclic subgroup commutativity degree is 1, but some finite non-abelian p -groups belong to \mathcal{G} , too. They are exhaustively described in [14, II, Theorem 4.1]: a non-abelian group is contained in \mathcal{G} if and only if it is isomorphic to the modular p -group

$$M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{p^{n-2}+1} \rangle$$

when p is odd, or to one of the following groups:

- $M(2^n)$ ($n \geq 4$),
- the dihedral group D_{2^n} ,
- the generalized quaternion group

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, yxy^{-1} = x^{2^{n-1}-1} \rangle,$$

- the quasi-dihedral group

$$S_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{2^{n-2}-1} \rangle \quad (n \geq 4)$$

when $p = 2$.

In the following the cyclic subgroup commutativity degrees of the above p -groups will be determined. As we observed in Section 2, we have

$$\text{csd}(M(p^n)) = 1.$$

Because $\text{csd}(D_{2^n})$ has been obtained in §3.2, we need to focus only on computing $\text{csd}(Q_{2^n})$ and $\text{csd}(S_{2^n})$.

THEOREM 3.3.1. *The cyclic subgroup commutativity degree of the generalized quaternion group Q_{2^n} is*

$$\text{csd}(Q_{2^n}) = \frac{n^2 + (n + 1)2^{n-1}}{(n + 2^{n-2})^2}.$$

In particular, we have

$$\text{csd}(Q_{16}) = \frac{7}{8}.$$

PROOF. Under the above notation, it is easy to see that $L_1(Q_{2^n})$ consists of all subgroups contained in $\langle x \rangle$ and of all subgroups of type $\langle x^k y \rangle$, $k = 0, 1, \dots, 2^{n-2} - 1$. Moreover, we have

$$|C_1(H)| = |L_1(Q_{2^n})| = n + 2^{n-2}, \quad \text{for all } H \leq \langle x \rangle.$$

We also remark that

$$\langle x^{k_1} y \rangle \langle x^{k_2} y \rangle = \langle x^{k_2} y \rangle \langle x^{k_1} y \rangle \iff k_1 = k_2 \text{ or } |k_1 - k_2| = 2^{n-3}.$$

This leads to

$$|C_1(\langle x^k y \rangle)| = n + 2, \quad \text{for all } k = 0, 1, \dots, 2^{n-2} - 1.$$

One obtains

$$\begin{aligned} \text{csd}(Q_{2^n}) &= \frac{1}{(n + 2^{n-2})^2} \sum_{H \in L_1(Q_{2^n})} |C_1(H)| \\ &= \frac{1}{(n + 2^{n-2})^2} \left[\sum_{\substack{H \in L_1(Q_{2^n}) \\ H \leq \langle x \rangle}} |C_1(H)| + \sum_{k=0}^{2^{n-2}-1} |C_1(\langle x^k y \rangle)| \right] \\ &= \frac{1}{(n + 2^{n-2})^2} [n(n + 2^{n-2}) + (n + 2)2^{n-2}] \\ &= \frac{n^2 + (n + 1)2^{n-1}}{(n + 2^{n-2})^2}, \end{aligned}$$

as desired. □

COROLLARY 3.3.2. $\lim_{n \rightarrow \infty} \text{csd}(Q_{2^n}) = 0$.

The same type of reasoning will be used to calculate $\text{csd}(S_{2^n})$.

THEOREM 3.3.3. *The cyclic subgroup commutativity degree of the quasi-dihedral group S_{2^n} is*

$$\text{csd}(S_{2^n}) = \frac{n^2 + 3n \cdot 2^{n-2} + 5 \cdot 2^{n-3}}{(n + 3 \cdot 2^{n-3})^2}.$$

In particular, we have

$$\text{csd}(S_{16}) = \frac{37}{50}.$$

PROOF. It is a simple exercise to check that the poset $L_1(S_{2^n})$ of cyclic subgroups of S_{2^n} consists of

$$L(\langle x \rangle) \cup \{\langle x^{2^k} y \rangle \mid k = 0, 1, \dots, 2^{n-2} - 1\} \cup \{\langle x^{2^{k+1}} y \rangle \mid k = 0, 1, \dots, 2^{n-3} - 1\}.$$

Again, we have

$$|C_1(H)| = |L_1(S_{2^n})| = n + 3 \cdot 2^{n-3}, \quad \text{for all } H \leq \langle x \rangle.$$

In order to study the commutativity of the other two types of subgroups of S_{2^n} , the following remarks are essential:

- $\langle x^{2k_1} y \rangle \langle x^{2k_2} y \rangle = \langle x^{2k_2} y \rangle \langle x^{2k_1} y \rangle \iff 2^{n-3} \mid k_1 - k_2$;
- $\langle x^{2k_1+1} y \rangle \langle x^{2k_2+1} y \rangle = \langle x^{2k_2+1} y \rangle \langle x^{2k_1+1} y \rangle \iff 2^{n-3} \mid k_1 - k_2$
 $\iff k_1 = k_2$;
- $\langle x^{2k_1} y \rangle \langle x^{2k_2+1} y \rangle \neq \langle x^{2k_2+1} y \rangle \langle x^{2k_1} y \rangle$, for all k_1, k_2 .

We infer that

$$|C_1(\langle x^{2^k} y \rangle)| = n + 2, \quad \text{for all } k = 0, 1, \dots, 2^{n-2} - 1$$

and

$$|C_1(\langle x^{2^{k+1}} y \rangle)| = n + 1, \quad \text{for all } k = 0, 1, \dots, 2^{n-3} - 1.$$

Hence

$$\begin{aligned}
 \text{csd}(S_{2^n}) &= \frac{1}{(n + 3 \cdot 2^{n-3})^2} \sum_{H \in L_1(S_{2^n})} |C_1(H)| \\
 &= \frac{1}{(n + 3 \cdot 2^{n-3})^2} \left[\sum_{\substack{H \in L_1(S_{2^n}) \\ H \leq \langle x \rangle}} |C_1(H)| + \sum_{k=0}^{2^{n-2}-1} |C_1(\langle x^{2^k} y \rangle)| \right. \\
 &\quad \left. + \sum_{k=0}^{2^{n-3}-1} |C_1(\langle x^{2^{k+1}} y \rangle)| \right] \\
 &= \frac{1}{(n + 3 \cdot 2^{n-3})^2} [n(n + 3 \cdot 2^{n-3}) + (n + 2)2^{n-2} + (n + 1)2^{n-3}] \\
 &= \frac{n^2 + 3n \cdot 2^{n-2} + 5 \cdot 2^{n-3}}{(n + 3 \cdot 2^{n-3})^2},
 \end{aligned}$$

completing the proof. □

COROLLARY 3.1. $\lim_{n \rightarrow \infty} \text{csd}(S_{2^n}) = 0$.

4. A criterion for a finite group to be Iwasawa

A famous result by Gustafson [3] concerning the commutativity degree states that if $d(G) > 5/8$ then G is abelian, and we have $d(G) = 5/8$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In this section a similar problem is studied for the cyclic subgroup commutativity degree, namely: *is there a constant $c \in (0, 1)$ such that if $\text{csd}(G) > c$ then G is Iwasawa?*

The answer to this problem is negative, as shows the following theorem.

THEOREM 4.1. *The cyclic subgroup commutativity degree of the non-Iwasawa group $\mathbb{Z}_{2^n} \times Q_8$, $n \geq 2$, tends to 1 when n tends to infinity.*

PROOF. Let $a = (a_1, a_2) \in \mathbb{Z}_{2^n} \times Q_8$. Then $o(a) = 2^k$ if and only if either $o(a_1) = 2^k$ and $o(a_2) \leq 2^k$ or $o(a_1) < 2^k$ and $o(a_2) = 2^k$. We infer that $\mathbb{Z}_{2^n} \times Q_8$ has one element of order 1, 3 elements of order 2, 28 elements of order 4, and 2^{k+2} elements of order 2^k , for all $k = 3, 4, \dots, n$. These generate one cyclic subgroup of order 1, 3 cyclic subgroups of order 2, 14 cyclic subgroups of order 4, and 8 cyclic subgroups of order 2^k , for all $k = 3, 4, \dots, n$. Consequently,

$$|L_1(\mathbb{Z}_{2^n} \times Q_8)| = 1 + 3 + 14 + 8(n - 2) = 8n + 2.$$

Then

$$\begin{aligned} \text{csd}(\mathbb{Z}_{2^n} \times Q_8) &= \frac{1}{(8n+2)^2} |\{(H, K) \in L_1(\mathbb{Z}_{2^n} \times Q_8)^2 \mid HK = KH\}| \\ &= 1 - \frac{1}{(8n+2)^2} |\{(H, K) \in L_1(\mathbb{Z}_{2^n} \times Q_8)^2 \mid HK \neq KH\}|. \end{aligned}$$

One the other hand, by Theorem 2.15 of [1] we know that $\mathbb{Z}_{2^n} \times Q_8$ has $24(n+2)$ pairs of subgroups which do not permute. This implies that

$$\text{csd}(\mathbb{Z}_{2^n} \times Q_8) \geq 1 - \frac{24(n+2)}{(8n+2)^2}$$

and so $\lim_{n \rightarrow \infty} \text{csd}(\mathbb{Z}_{2^n} \times Q_8) = 1$, completing the proof. \square

COROLLARY 4.2. *There is no constant $c \in (0, 1)$ such that if $\text{csd}(G) > c$ then G is Iwasawa.*

However, we can get a positive answer to the above problem if we replace the condition “ $\text{csd}(G) > c$ ” by the stronger condition “ $\text{csd}^*(G) > c$,” where

$$\text{csd}^*(G) = \min\{\text{csd}(S) \mid S \text{ section of } G\}.$$

This was suggested by the fact that a p -group is modular if and only if each of its sections of order p^3 is. Moreover, if a p -group is not modular then it contains a section isomorphic to D_8 or to $E(p^3)$, the non-abelian group of order p^3 and exponent p for $p > 2$ (see Lemma 2.3.3 of [13]).

LEMMA 4.3. *Let G be a finite p -group such that $\text{csd}^*(G) > 41/49$. Then G is modular, and consequently an Iwasawa group.*

PROOF. Assume that G is not modular. Then there is a section S of G such that $S \cong D_8$ or $S \cong E(p^3)$ for $p > 2$. We can easily check that

$$\text{csd}(E(p^3)) = \frac{p^3 + 5p^2 + 4p + 4}{(p^2 + p + 2)^2} < \frac{41}{49} = \text{csd}(D_8).$$

Therefore $\text{csd}(S) \leq 41/49$, contradicting our assumption. \square

LEMMA 4.4. *Let G be a finite group such that $\text{csd}^*(G) > 19/25$. Then G is nilpotent.*

PROOF. We will show by induction on $|G|$ that if G is not nilpotent then $\text{csd}^*(G) \leq 19/25$, i.e. there is a section S of G with $\text{csd}(S) \leq 19/25$. For $|G| = 6$ we have $G \cong S_3$ and the desired conclusion follows by taking $S = G$. Assume now that it is true for all non-nilpotent groups of order $< |G|$. We distinguish the following two cases.

If G contains a proper non-nilpotent subgroup H , then H has a section S with $\text{csd}(S) \leq 19/25$ by the inductive hypothesis and we are done since S is also a section of G .

If all proper subgroups of G are nilpotent, then G is a Schmidt group. By [12] (see also [10]) it follows that G is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p -subgroup P and a cyclic Sylow q -subgroup Q , and hence G is a semidirect product of P by Q . Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(G)$;
- $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;
- $|P/P'| = p^r$, where r is the order of p modulo q ;
- if P is abelian, then P is an elementary abelian p -group of order p^r and P is a minimal normal subgroup of G ;
- if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We infer that $S = G/Z(G)$ is also a Schmidt group of order $p^r q$ which can be written as a semidirect product of an elementary abelian p -group P_1 of order p^r by a cyclic group Q_1 of order q (note that S_3 and A_4 are examples of such groups). Then $L_1(S) = L_1(P_1) \cup \{Q_1^x \mid x \in S\}$ and

$$|L_1(S)| = \frac{p^r - 1}{p - 1} + 1 + p^r = \frac{p^{r+1} + p - 2}{p - 1}.$$

One obtains:

(a)
$$\text{csd}(S) = \frac{5p + 4}{(p + 2)^2} \quad \text{for } r = 1$$

and

(b)
$$\text{csd}(S) = \frac{p^{2r} + 3p^{r+2} - 4p^{r+1} - p^r + p^2 - 4p + 4}{(p^{r+1} + p - 2)^2} \quad \text{for } r \geq 2.$$

In both cases (a) and (b) we can easily check that

$$\text{csd}(S) \leq \frac{19}{25},$$

as desired. □

We are now able to prove the main result of this section.

THEOREM 4.5. *Let G be a finite group such that $\text{csd}^*(G) > 41/49$. Then G is an Iwasawa group. Moreover, we have $\text{csd}^*(G) = 41/49$ if and only if $G \cong X \times Y$, where X is a 2-group with $\text{csd}^*(X) = 41/49$ and Y is an Iwasawa group of odd order.*

PROOF. Since $\text{csd}^*(G) > 41/49 > 19/25$, Lemma 4.4 implies that G is nilpotent. Then it can be written as

$$(4) \quad G = \prod_{i=1}^k G_i,$$

where G_i is a Sylow p_i -subgroup of G , $i = 1, 2, \dots, k$. For each i we have

$$\text{csd}^*(G_i) \geq \text{csd}^*(G) > \frac{41}{49},$$

and therefore G_i is an Iwasawa group by Lemma 4.3. Consequently, G is also an Iwasawa group.

Suppose now that $\text{csd}^*(G) = 41/49$. Then G is nilpotent by Lemma 4.4, and therefore it has a direct decomposition of type (4), where we can assume $p_1 < p_2 < \dots < p_k$. Remark that $p_1 = 2$. Indeed, if $p_1 > 2$ then all p_i 's are odd, which implies that G_i cannot have sections isomorphic with D_8 , for all $i = 1, 2, \dots, k$. On the other hand, G_i cannot also have sections isomorphic with $E(p_i^3)$ because $\text{csd}^*(G_i) \geq \text{csd}^*(G) = 41/49$. Thus G_i is Iwasawa, for all $i = 1, 2, \dots, k$, and the same thing can be said about G , a contradiction. Hence $p_1 = 2$ and we are done by taking

$$X = G_1 \quad \text{and} \quad Y = \prod_{i=2}^k G_i.$$

Conversely, since X and Y are of coprime orders, every section S of $G \cong X \times Y$ is of type $S \cong S_X \times S_Y$, where S_X and S_Y are sections of X and Y , respectively. Then

$$\text{csd}(S) = \text{csd}(S_X) \text{csd}(S_Y) = \text{csd}(S_X)$$

because Y is Iwasawa. This shows that

$$\text{csd}^*(G) = \text{csd}^*(X) = \frac{41}{49},$$

completing the proof. □

We end this section by noting that the problem of finding the structure of 2-groups X with $\text{csd}^*(X) = 41/49$ remains open.

5. Conclusions and further research

Similarly with our previous concepts of *subgroup commutativity degree*, *normality degree* or *cyclicity degree* of a finite group, the *cyclic subgroup commutativity degree* can also constitute a significant aspect of probabilistic finite group theory. Clearly, the study started in this paper can successfully be extended to other classes of finite groups and all problems on $\text{sd}(G)$, $\text{ndeg}(G)$, $\text{cdeg}(G)$ (see e.g. [16]–[20]) can be investigated for $\text{csd}(G)$, too. On the other hand, the connections between the above concepts seem to be very interesting. These will surely constitute the subject of some further research.

Finally, we formulate several specific open problems on cyclic subgroup commutativity degrees.

PROBLEM 5.1. Compute explicitly the cyclic subgroup commutativity degree of $\text{ZM}(m, n, r)$ (see [21]), or, more generally, the cyclic subgroup commutativity degree of an arbitrary metacyclic group.

PROBLEM 5.2. Let G be a finite group. Study the properties of the map

$$\text{csd}: L(G) \longrightarrow [0, 1], \quad H \longmapsto \text{csd}(H).$$

Is it true that for every $H, K \in L(G)$, we have $H \subseteq K \implies \text{csd}(H) \geq \text{csd}(K)$?

PROBLEM 5.3. For many finite groups G , the commutativity of $x, y \in G$ is strongly connected with the commutativity of $\langle x \rangle, \langle y \rangle \in L_1(G)$. Can be extended this to a connection between $\text{d}(G)$ and $\text{csd}(G)$?

PROBLEM 5.4. Do there exist finite groups G such that $\text{csd}(G) = \text{sd}(G) \neq 1$?

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