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Nunke's problem for abelian p-groups of uncountable length

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Abstract – *Nunke's problem* asks when the torsion product of two abelian p-groups is a direct sum of countable reduced groups. In previous work the author gave a complete answer to this question when the groups involved have countable length. In this paper a complete answer is given in the case of groups of uncountable length, at least in any set-theoretic universe in which $2^{\aleph_1} = \aleph_2$.

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1. An introduction to Nunke's problem

All groups will be abelian p-groups for some fixed prime p. Our notation and terminology will generally follow [2]. For an ordinal α we will assume some familiarity with the theory of p^α-purity, which can be found, for example, in [3]. We will also use some fairly basic set theory for which we refer the reader to [1]. If A and B are groups, then we will denote their torsion product by the convenient, albeit unorthodox, notation $A \bigtriangledown B$.

A subgroup $K \subseteq G$ is said to be p^α*-high* if it is maximal with respect to the property that $K \cap p^\alpha G = 0$, and G is said to be a C_λ*-group* if for every $\alpha < \lambda$, the p^α-high subgroups of G are all direct sums of countable reduced groups (or dsc groups, for short). If $\lambda \leq \omega_1$ is a limit, then G will be a C_λ-group if and only if for every $\alpha < \lambda$ the quotient $G/p^\alpha G$ is dsc group [see, for example, ([5], Theorem 8)].

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There is a strong connection between the torsion product and the class of dsc groups. *Nunke's problem* asks us for a description of when $A \bigtriangledown B$ is a dsc group of length $\lambda \leq \omega_1$. The following was the first important result on Nunke's problem, which has been rephrased using our terminology:

THEOREM 1.1 ([11], Theorem 6). *Suppose A and B are reduced groups, B has length $\lambda \leq \omega_1$ and $p^\lambda A \neq 0$. Then $A \bigtriangledown B$ is a dsc group if and only if A is a C_λ-group and B is a dsc group.*

So the case of interest is where both A and B have length λ. First, it is straightforward to verify that $A \bigtriangledown B$ is a C_λ group of length at least λ if and only if both A and B are C_λ-groups of length at least λ [see, for example, ([8], Proposition 4) with $n = 2$]. So Nunke's problem is really a question about C_λ-groups.

It turns out that the countable case [i.e., $\lambda < \omega_1$] of Nunke's problem behaves very differently from the uncountable one [i.e., $\lambda = \omega_1$]. Let $\mathcal{R}$ be the class of all uncountable regular cardinals and $\mathcal{R}_f$ be the class of all finite subsets of $\mathcal{R}$. Let $0_{\mathcal{R}} = \emptyset \subseteq \mathcal{R}_f$, and if $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{R}_f$, let $\mathcal{X} \cdot \mathcal{Y}$ be the class of all disjoint unions $X \cup Y$ where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. If $\lambda < \omega_1$, then in [9] for every C_λ-group G there was defined an invariant $L_G^\lambda \subseteq \mathcal{R}_f$. This definition was based on transfinite induction using filtrations on subgroups of G. Its relevance to Nunke's problem is summarized in the following:

THEOREM 1.2. *Suppose $\lambda < \omega_1$ is countable and G, A and B are C_λ-groups.*

(a) ([9], *Theorem* 1.6) *G is a dsc group if and only if $L_G^\lambda = 0_{\mathcal{R}}$.*

(b) ([9], *Theorem* 3.10) *$A \bigtriangledown B$ is a dsc group if and only if $L_A^\lambda \cdot L_B^\lambda = 0_{\mathcal{R}}$.*

In other words, the elements of L_G^λ can be viewed as obstructions to G splitting into countable summands, and the invariant behaves well enough with respect to the torsion product to give a complete solution to Nunke's problem in the countable case.

Turning now to the uncountable case $\lambda = \omega_1$, in [7] it was shown that Nunke's problem leads us to consider a set-theoretic statement known as *Kurepa's Hypothesis* (KH). There are several equivalent ways to express KH. For example, it asserts the existence of a tree of height ω_1 having at least ω_2 branches, but whose levels are all countable. Equivalently, KH asserts the existence of a family $\mathcal{K}$ of subsets of ω_1 such that $|\mathcal{K}| \geq \omega_2$, but for every countable $\lambda < \omega_1$, $\{X \cap \lambda \colon X \in \mathcal{K}\}$ is countable. KH is known to be true in the constructible universe, but to be undecidable over ZFC [in fact, KH is a consequence of $\diamondsuit^+$, which is true in V=L].

It is easy to see that a C_{ω_1}-group G is a dsc group if and only if it is p^{ω_1}-projective; i.e., its p^{ω_1}-projective dimension (or p^{ω_1}-p.d. for short) is 0. Let $\mathcal{F}$ be the class of all C_{ω_1}-groups whose p^{ω_1}-p.d. is at most 1. The connection between these ideas stems from the following result:

THEOREM 1.3 ([7], Theorem 13). *The following statements are equivalent.*

(a) $\neg KH$.

(b) *Every p^{ω_1}-bounded C_{ω_1}-group G is in $\mathcal{F}$.*

(c) *For all p^{ω_1}-bounded C_{ω_1}-groups A and B, $A \bigtriangledown B$ is a dsc group.*

These are only a few of a much longer list of algebraic statements that were shown to be equivalent to ¬KH.

In [10] the following variation on Nunke's problem was considered. If A and B are C_{ω_1}-groups, describe exactly when $A \bigtriangledown B \in \mathcal{F}$. If we let $\mathcal{Q} = \mathcal{R} - \{\aleph_1\}$ and $\mathcal{Q}_f = \mathcal{R}_f \cap \mathcal{Q}$, then for every C_{ω_1}-group G an invariant $J_G \subseteq \mathcal{Q}_f$ was defined using techniques similar to the definition of L_G^λ in the countable case. It answered the above variation by establishing the following:

THEOREM 1.4. *Suppose that $\lambda = \omega_1$ is uncountable and that G, A and B are C_{ω_1}-groups.*

(a) ([10], *Theorem* 1.12) $G \in \mathcal{F}$ *if and only if* $J_G = 0_{\mathcal{Q}}$.

(b) ([10], *Theorem* 1.15) $A \bigtriangledown B \in \mathcal{F}$ *if and only if* $J_A \cdot J_B = 0_{\mathcal{Q}}$.

In other words, the elements of J_G can be viewed as obstructions to G being in $\mathcal{F}$, and the invariant behaves well enough with respect to the torsion product to give a complete solution to this variation of Nunke's problem. The purpose of the present paper is to combine the above two threads to answer the following questions.

(A) *Given a C_{ω_1}-group G, J_G tells us when $G \in \mathcal{F}$*; *is there an addition invariant that tells us when a given $G \in \mathcal{F}$ is actually a dsc group*?

(B) *If A and B are C_{ω_1}-groups, can we use the solution of* (*A*) *to answer Nunke's problem in the uncountable case, at least in some reasonable versions of set theory, such as the constructible universe* (*V=L*)?

We will give a complete answer to (A). If $G \in \mathcal{F}$, then we will define an invariant $F_G \subseteq \mathcal{Q}_f$ in a manner similar to the definition of L_G^λ and J_G using transfinite induction on subgroups and filtrations. We will then show that the invariant satisfies the following:

Theorem 1.5. *If $G \in \mathcal{F}$, then G is a dsc group if and only if $F_G = 0_{\mathcal{Q}}$.*

So using Theorems 1.4(a) and 1.5 we have the following:

Corollary 1.6. *A C_{ω_1}-group G is a dsc group if and only if $J_G = 0_{\mathcal{Q}}$ and $F_G = 0_{\mathcal{Q}}$.*

In other words, for a C_{ω_1}-group G, elements of J_G are obstructions to G being in $\mathcal{F}$, and when $G \in \mathcal{F}$, elements of F_G are obstructions to G being a dsc group.

Turning to question (B), if G is a p^{ω_1}-bounded C_{ω_1}-group, we define an invariant $I_G \subseteq \mathcal{R}_f$ in a manner similar to that used in defining L_G^λ, J_G and F_G. The difference between the definitions of these various invariants is in the base case and the type of subgroups we use in constructing our filtrations. If $2^{\aleph_1} = \aleph_2$, or if one of the groups A and B is actually in $\mathcal{F}$, then we get a complete answer to (B).

Theorem 1.7. *Suppose A and B are p^{ω_1}-bounded C_{ω_1}-groups.*

(a) *If $2^{\aleph_1} = \aleph_2$, then $A \triangledown B$ is a dsc if and only if $I_A \cdot I_B = 0_{\mathcal{R}}$.*

(b) *If $B \in \mathcal{F}$, then $A \triangledown B$ is a dsc if and only if $J_A \cdot F_B = 0_{\mathcal{Q}}$.*

In particular, this gives a complete solution to all countable and uncountable cases of Nunke's problem in any set-theoretic universe in which the generalized continuum hypothesis holds, such as the constructible universe. We need this assumption since the invariant I_G behaves like a combination of both J_G and F_G and it is not easy to see exactly how to combine them without some such restriction. In particular, $2^{\aleph_1} = \aleph_2$ will guarantee that the torsion product of any two p^{ω_1}-bounded C_{ω_1}-groups is in $\mathcal{F}$, which is used frequently in our arguments.

Giving a quick outline of the paper, Section 2 is devoted to reviewing some background information, leading to a particular formulation of Shelah's Singular Compactness Theorem which will be useful for our purposes. Section 3 is devoted to proving Theorem 1.5, and Section 4 contains a proof of Theorem 1.7.

2. Preliminaries and singular compactness

We begin with a review of a few well-known properties of p^{ω_1}-purity.

Lemma 2.1. *Suppose G is a group and H is a dsc group of length ω_1.*

(a) ([8], *Theorem* 2) *G is a C_{ω_1}-group if and only if $G \triangledown H$ is a dsc.*

(b) ([8], *Lemma* 1) *A short exact sequence,* $0 \to K \to G \to (G/K) \to 0$, *is* p^{ω_1}*-pure if and only if the corresponding sequence,* $0 \to K \bigtriangledown H \to G \bigtriangledown H \to (G/K) \bigtriangledown H \to 0$, *is splitting exact.*

(c) *If* G *is a* C_{ω_1} *group and* K *is* p^{ω_1}*-pure in* G*, then* K *and* G/K *are* C_{ω_1}*-groups.*

Part (c) is an immediate consequence of (a) and (b). We quote a couple of other simple results that are pertinent to our discussions.

LEMMA 2.2. *Suppose* A*,* B *and* C *are* p^{ω_1}*-bounded* C_{ω_1}*-groups.*

(1) ([4], *Theorem* 6) *If* A *and* B *have cardinality at most* $\aleph_1$*, then* $A \bigtriangledown B$ *is a dsc group.*

(2) ([5], *Theorem* 25) *If* A*,* B *and* C *have cardinality at most* $\aleph_2$*, then* $A \bigtriangledown B \bigtriangledown C$ *is a dsc group.*

We pause to review some set theoretic terminology. If κ is a regular cardinal, then a subset $C \subseteq \kappa$ is a *CUB* if it is closed and unbounded in the order topology on κ. A subset $S \subseteq \kappa$ is said to be *stationary* if $C \cap S \neq \emptyset$ for all CUB subsets $C \subseteq \kappa$. If G is a group of regular cardinality κ, then a *filtration* of G is a smoothly ascending chain of subgroups, $\{X_i\}_{i<\kappa}$, whose union is G, such that $|X_i| < \kappa$ for all $i < \kappa$.

Suppose C is a dsc group and we fix a particular decomposition of C into a collection, $\mathcal{D}$, of countable subgroups. A subgroup K of C will be a $\mathcal{D}$*-summand* if K is the direct sum of a subcollection of these terms. If K is any summand of a dsc group C, then a decomposition of K into countable groups can be extended to a similar decomposition of C, so that K will be a $\mathcal{D}$-summand of C.

Now suppose H is a dsc group of length ω_1 and cardinality $\aleph_1$ (for example, H_{ω_1}, the "generalized Prüfer group" of length ω_1). If G is a C_{ω_1}-group, then using Lemma 2.1(a), fix a decomposition $\mathcal{D}$ of $G \bigtriangledown H$ into countable summands. Let $\mathcal{H}$ be the collection of subgroups $A \subseteq G$ such that $A \bigtriangledown H$ is a $\mathcal{D}$-summand of $G \bigtriangledown H$; we call this the H*-system* for G determined by $\mathcal{D}$. It follows from Lemma 2.1(b) that every element of an H-system is p^{ω_1}-pure in G. The following are straightforward consequences of this definition:

LEMMA 2.3. *Suppose* G *is a* C_{ω_1}*-group with* H*-system* $\mathcal{H}$*.*

(a) $\mathcal{H}$ *is closed under unions of chains*;

(b) *If* $Z \subseteq G$ *satisfies* $|Z| \geq \aleph_1$*, then there is a* $K \in \mathcal{H}$ *containing* Z *such that* $|K| = |Z|$;

(c) *If* $|G| = \kappa > \aleph_1$ *and* κ *is regular, then* G *has a filtration* $\{X_i\}_{i<\kappa} \subseteq \mathcal{H}$*.*

In this work all topological terms will be with respect to the ω_1-topology; for a group G this uses $\{p^\alpha G\}_{\alpha<\omega_1}$ as a neighborhood base of 0.

Suppose G is a p^{ω_1}-bounded C_{ω_1}-group, $\kappa > \aleph_1$ is a cardinal and K is a p^{ω_1}-pure subgroup of G with $|K| = \aleph_1$. We say K is a *κ-Kurepa* subgroup if $\kappa \leq |\overline{K}|$ (where $\overline{K}$ is its closure in the ω_1-topology). In addition, we say K is *$\aleph_1$-Kurepa* if either it is $\aleph_2$-Kurepa, or else it is closed in G but fails to be a dsc. Let ν_G be the least cardinal $\kappa \geq \aleph_1$ such that G does not have a κ-Kurepa subgroup. Let ν be the supremum of ν_G over all p^{ω_1}-bounded C_{ω_1}-groups G; clearly $\nu_G \leq \nu \leq (2^{\aleph_1})^+$. By ([6], Theorem 5), $\kappa < \nu$ if and only if there is a family $\mathcal{K}$ of subsets of ω_1 with $|\mathcal{K}| \geq \kappa$ satisfying KH. We will primarily be concerned with the situation where $\nu \leq \aleph_3$, which will follow if we have $2^{\aleph_1} = \aleph_2$. In particular, in the constructible universe, V=L, both KH and the generalized continuum hypothesis hold, so that $\nu = \aleph_3$.

Suppose G is a p^{ω_1}-bounded C_{ω_1}-group and $\mathcal{H}$ is an H-system for G. Consider the set $\mathcal{C}$ of elements of $\mathcal{H}$ that are closed; we call this the *C-system* for G determined by $\mathcal{H}$. The next statement then follows directly from Lemma 2.3 and some results from [6].

Lemma 2.4. *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group and $\mathcal{C}$ is a C-system for G.*

(a) ([6], *Theorem* 3) *$\mathcal{C}$ is closed under unions of chains*;

(b) ([6], *Theorem* 8) *If $|G| = \kappa \geq \nu_G$ and κ is regular, then G has a filtration $\{X_i\}_{i<\kappa} \subseteq \mathcal{C}$.*

(c) *If $K \subseteq G$ is p^{ω_1}-pure and $\kappa := |K| \geq \nu_G$, then $|\overline{K}| = \kappa$ and there is an $X \in \mathcal{C}$ such that $K \subseteq X$ and $|X| = \kappa$.*

Proof. Regarding (c), if $|\overline{K}| > \kappa$, we could find $\widetilde{G} \in \mathcal{H}$ of cardinality κ^+ containing K such that $|\widetilde{G} \cap \overline{K}| = \kappa^+$. This, however, would contradict (b). To construct our $X \in \mathcal{C}$, find $\{X_m\}_{m<\emptyset} \subseteq \mathcal{H}$, all of cardinality κ, such that $K \subseteq X_0 \subseteq \overline{X_0} \subseteq X_1 \subseteq \overline{X_1} \subseteq X_2 \subseteq \cdots$. By ([6], Theorem 3), $X := \cup_{m<\emptyset} X_m$ is a closed element of $\mathcal{H}$, i.e., $X \in \mathcal{C}$. And clearly $K \subseteq X$ and $|X| = \kappa$. $\square$

Our next objective is to prove a variant on the Singular Compactness Theorem of S. Shelah. It is similar to other versions (see, for example, [1]), but it appears to be somewhat simpler than many.

Suppose γ is a singular cardinal. If G is a p^{ω_1}-bounded C_{ω_1}-group, then a collection $\mathcal{S}$ of p^{ω_1}-pure subgroups will be called a *γ-dsc system* if it satisfies the following:

(1) $0 \in \mathcal{S}$;

(2) every $S \in \mathcal{S}$ is a dsc group with $|S| < \gamma$;

(3) if $Z \subseteq G$ with $\aleph_1 \leq |Z| < \gamma$, then there is an $S \in \mathcal{S}$ such that $Z \subseteq S$ and $|S| = |Z|$;

(4) if $\mathcal{S}' \subseteq \mathcal{S}$ is a chain with union S, then $|S| < \gamma$ implies that $S \in \mathcal{S}$.

If $\aleph_1 \leq \kappa < \gamma$ is a cardinal, let $\mathcal{S}_\kappa = \{S \in \mathcal{S} \colon |S| \leq \kappa\}$. We note one consequence of the above conditions.

(5) If $Z \subseteq G$ with $|Z| \leq \kappa$, then there is an $S \in \mathcal{S}_\kappa$ such that $Z \subseteq S$ and whenever $S' \in \mathcal{S}_\kappa$ with $S \subseteq S'$, then S is a summand of S'.

If (5) failed, then we could construct a smoothly ascending chain

$$Z \subseteq S_0 \subseteq S_1 \subseteq \cdots \subseteq S_i \subseteq S_{i+1} \subseteq \cdots \quad (i < \kappa^+)$$

of elements of $\mathcal{S}_\kappa$ such that for each $i < \kappa^+$, S_i fails to be a summand of S_{i+1}. Now, if we let $S = \cup_{i<\kappa^+} S_i$, then $|S| \leq \kappa^+ < \gamma$. So by (4), $S \in \mathcal{S}$, and in particular, S is a dsc group. Note that $\{S_i\}_{i<\kappa^+}$ will be a filtration of S, so for a CUB subset $C \subseteq \kappa^+$, S_i is a summand of S for each $i \in C$. However, this contradicts that this S_i is not even a summand of S_{i+1}, let alone S.

Often, when we mention condition (5) we will assume that we extend a decomposition of S to a decomposition of S'; in this way S will be a $\mathcal{D}$-summand of S'. More generally, if $S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$ is an ascending sequence of elements of $\mathcal{S}_\kappa$ satisfying (5), then we can successively build up our decompositions so that each S^k is a $\mathcal{D}$-summand of S^{k+1}.

This brings us to our version of the Singular Compactness Theorem.

Theorem 2.5. *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group of cardinality γ, where γ is a singular cardinal. Then G is a dsc group if and only if it has a γ-dsc system.*

Proof. One direction is clear, so assume that $\mathcal{S}$ is a γ-dsc system for G. Let λ be the cofinality of γ and let $\{\kappa_i\}_{i<\gamma}$ be a smoothly increasing sequence of cardinals greater than λ whose limit is γ. We start by defining $S_i^0 = 0$ for all $i < \lambda$ and letting $\{C_i^0\}_{i<\lambda}$ be a smoothly ascending chain of subgroups whose union is G, such that each $|C_i^0| \leq \kappa_i$. Next, for each $i < \lambda$ let $\mathcal{T}_i \subseteq \mathcal{S}_{\kappa_i}$ be a collection of subgroups that satisfy (5) for $\kappa = \kappa_i$.

For some positive integer n suppose we have defined

(1) $\{S_i^{n-1}\}_{i<\lambda}$ where each $S_i^{n-1} \in \mathcal{T}_i$ has a decomposition $\mathcal{D}_i^{n-1}$ into countable subgroups;

(2) a smoothly ascending chain of subgroups $\{C_i^{n-1}\}_{i<\lambda}$ whose union is G such that $S_i^{n-1} \subseteq C_i^{n-1}$ and $|C_i^{n-1}| \leq \kappa_i$ for all $i < \lambda$.

We now define $\{S_i^n\}_{i<\lambda}$ so that for all $i < \lambda$:

(3) S_i^{n-1} is a summand of $S_i^n \in \mathcal{T}_i$ and the decomposition $\mathcal{D}_i^{n-1}$ for S_i^{n-1} extends to a decomposition $\mathcal{D}_i^n$ for S_i^n;

(4) $S_i^{n-1} \subseteq C_i^{n-1} \subseteq P_i^n \subseteq S_i^n$, where P_i^n is a $\mathcal{D}_{i+1}^n$-summand of S_{i+1}^n.

For all $i < \lambda$ let $U_i^n \in \mathcal{T}_i$ such that $|U_i^n| \leq \kappa_i$ and $S_i^{n-1} \subseteq C_i^{n-1} \subseteq U_i^n$; extend the decomposition $\mathcal{D}_i^{n-1}$ on each S_i^{n-1} to a decomposition $\mathcal{E}_i^n$ for each U_i^n. Now $C_i^{n-1} \subseteq C_{i+1}^{n-1} \subseteq U_{i+1}^n$, so there is an $\mathcal{E}_{i+1}^n$-summand P_i^n of U_{i+1}^n containing C_i^{n-1} for which $|P_i^n| \leq \kappa_i$. We then find $S_i^n \in \mathcal{T}_i$ such that $U_i^n + P_i^n \subseteq S_i^n$. If we extend the decomposition $\mathcal{E}_i^n$ of each U_i^n to a decomposition $\mathcal{D}_i^n$ for each S_i^n, then (3) and (4) follow.

Now, given $\{S_i^n\}_{i<\lambda}$ as above, we show how to define a smoothly ascending chain of subgroups $\{C_i^n\}_{i<\lambda}$ with union G so that

(5) $|C_i^n| \leq \kappa_i$ and $S_i^n \subseteq C_i^n$.

For each $i < \lambda$, let our decomposition $\mathcal{D}_i^n$ of S_i^n be written $\bigoplus_{j<\kappa_i} X_{i,j}$. For each $i < \lambda$, define

$$C_i^n = \langle X_{k,j} : k < \lambda,\, j < \min\{\kappa_i, \kappa_k\}\rangle.$$

Clearly $|C_i^n| \leq \kappa_i$, and $\{C_i^n\}_{i<\lambda}$ is a smoothly ascending chain with union G. In addition, $S_i^n = \bigoplus_{j<\kappa_i} X_{i,j} \subseteq C_i^n$.

So by induction, for all $n < \omega$ and $i < \lambda$ we have defined S_i^n, C_i^n and P_i^n. Observe that if we let $S_i = \cup_{n<\omega} S_i^n$, then S_i is a dsc group with a decomposition $\mathcal{D}_i$ determined by its inductive construction. Next, since $S_i^n \subseteq C_i^n \subseteq S_i^{n+1}$, we can conclude that $S_i = \cup_{n<\omega} C_i^n$, so that $\{S_i\}_{i<\lambda}$ is a smoothly increasing chain of subgroups with union G.

Finally, by (4), $S_i^{n-1} \subseteq P_i^n \subseteq S_i^n$ and P_i^n is a $\mathcal{D}_{i+1}^n$ summand of S_{i+1}^n, and so a $\mathcal{D}_{i+1}$ summand of S_{i+1}. Therefore, $S_i = \cup_{n<\omega} P_i^n$ will also be a $\mathcal{D}_{i+1}$-summand of S_{i+1}. This readily implies that G is a dsc group. $\square$

3. When is $G \in \mathcal{F}$ actually a dsc group?

For a C_{ω_1}-group G we now review how $J_G \subseteq \mathcal{Q}_f$ was defined in [10]. Suppose $T \in \mathcal{Q}_f$; if $T = \emptyset$, let $\mu(T) = \aleph_1$, and if $T \neq \emptyset$, let $\mu(T)$ be its greatest element. Let $T' = T - \{\mu(T)\}$, and if $i < \mu(T)$, let $T_i = (T' \cup \{i\}) \cap \mathcal{Q}$ (that is, $T_i = T'$ if $i \notin \mathcal{Q}$ and $T_i = T' \cup \{i\}$ when $i \in \mathcal{Q}$). In particular, if $T \in \mathcal{Q}_f$ is non-empty and $i < \mu(T)$, then $\mu(T_i) < \mu(T)$.

For every C_{ω_1}-group G and every $T \in \mathcal{Q}_f$ we decide whether $T \in J_G$ by induction on $\mu(T) := \kappa$. First the base case:

(J-0) if $\kappa = \aleph_1$ (i.e., $T = \emptyset$), then $T \in J_G$ if and only if $p^{\omega_1} G \neq 0$.

Suppose now that $\kappa \geq \aleph_2$, and that for all $S \in \mathcal{Q}_f$ with $\mu(S) < \kappa$ and for all C_{ω_1}-groups $\widehat{G}$ we have defined when $S \in J_{\widehat{G}}$. Then $T \in J_G$ if and only if one of following conditions holds:

(J-1) $\Upsilon^J_T(G) = \{i < \kappa : T_i \in J_G\}$ is stationary in κ; or

(J-2) G has a p^{ω_1}-pure subgroup A of cardinality κ with a p^{ω_1}-pure filtration $\{X_i\}_{i<\gamma}$ such that $\Lambda^J_T(A) = \{i < \kappa : T_i \in J_{A/X_i}\}$ is stationary in κ.

We include the following example to give a simple illustration of the computation of this invariant:

Proposition 3.1 ([10], Corollary 1.20). *Suppose that G is a p^{ω_1}-bounded C_{ω_1}-group, and $\kappa \in \mathcal{Q}$ is a successor cardinal. Then $T := \{\kappa\} \in J_G$ if and only if G has a κ-Kurepa subgroup.*

Proof. If γ is a cardinal with $\kappa = \gamma^+$, then $C := \{i < \kappa : i > \gamma\}$ is certainly a CUB in κ. Since $T_i = \emptyset$ for all $i \in C$, our definition is simplified considerably.

Now, if $\widetilde{G}$ is any C_{ω_1} group, then $\emptyset \in J_{\widetilde{G}}$ if and only if $p^{\omega_1}\widetilde{G} \neq 0$. In particular, for all $i \in C$, $T_i = \emptyset \notin \Upsilon^J_T(G)$. Therefore, $T \in J_G$ if and only if (J-2) holds for T.

Suppose, then, that G has a κ-Kurepa subgroup, K. Let $A \subseteq G$ be a p^{ω_1}-pure subgroup containing K such that $\kappa = |A| = |A \cap \overline{K}|$. So, if $\{X_i\}_{i<\kappa}$ is a p^{ω_1}-pure filtration of A, then A/X_i is never p^{ω_1}-bounded. In other words, $\Lambda^J_T(A) = \kappa$, so that (J-2) holds and $T \in J_G$.

On the other hand, suppose that G does not have a κ-Kurepa subgroup and $A \subseteq G$ is any p^{ω_1}-pure subgroup of cardinality κ. Certainly, A cannot have a κ-Kurepa subgroup, so by Lemma 2.4(b), we can find a closed p^{ω_1}-pure filtration $\{X_i\}_{i<\kappa}$ of A. Since A/X_i is p^{ω_1}-bounded for every $i < \kappa$, we can conclude that $\Lambda^J_T(A) \cap C = \emptyset$. Since A was arbitrary, (J-2) does not hold. $\square$

A little more intuitively, a statement such as $\{\kappa, \kappa'\} \in J_G$ (where $\aleph_2 \leq \kappa < \kappa'$) indicates that either G has a κ-Kurepa subgroup, or it has some p^{ω_1}-pure subgroup $A \subseteq G$ of cardinality κ' such that when $\{X_i\}_{i<\kappa'}$ is any filtration of A, at least one of the quotients A/X_i has a κ-Kurepa subgroup. More generally, the more elements an element of J_G contains, the more we are looking at filtrations of subgroups that are themselves embedded in filtrations of subgroups, etc. If in this kind of dissection it is inevitable that we encounter a κ-Kurepa subgroup, then by Theorem 1.4(a) we have an obstruction to G being a member of $\mathcal{F}$.

So in computing J_G we are searching deep inside of G for κ-Kurepa subgroups with $\kappa \geq \aleph_2$. If $J_G = 0_{\mathbb{Q}}$, then there are no such embedded subgroups and $G \in \mathcal{F}$. And when this happens we will need an additional invariant to detect the presence of similarly embedded $\aleph_1$-Kurepa subgroups. The elements of this invariant will then be obstructions to $G \in \mathcal{F}$ actually being a dsc group. This will require some preliminary work.

The group M is an *ω_1-elementary S-group* if there is a dsc group H containing M as a p^{ω_1}-pure subgroup such that $H/M \simeq \mathbb{Z}_{p^\infty}$. From now on we will let M be some fixed ω_1-elementary S-group of cardinality ω_1 (it does not matter exactly which one is chosen). If G is any group, then $0 \to G \triangledown M \to G \triangledown H \to G\ [\simeq\ G \triangledown \mathbb{Z}_{p^\infty}] \to 0$ is a p^{ω_1}-projective resolution of G.

The following summarizes some well-known properties of the class $\mathcal{F}$.

Lemma 3.2. *Suppose G, A and B are C_{ω_1}-groups.*

(a) *$G \in \mathcal{F}$ if and only if $G \triangledown M$ is a dsc group.*

(b) *If $G \in \mathcal{F}$, then G is p^{ω_1}-bounded.*

(c) *If $|G| \leq \aleph_1$, then $G \in \mathcal{F}$ if and only if it is p^{ω_1}-bounded.*

(d) *If $B \in \mathcal{F}$, then $A \triangledown B \in \mathcal{F}$.*

(e) *If $G \in \mathcal{F}$ and $K \subseteq G$ is a p^{ω_1}-pure subgroup, then $K \in \mathcal{F}$.*

(f) *If A and B are p^{ω_1}-bounded and of cardinality at most $\aleph_2$, then $A \triangledown B \in \mathcal{F}$.*

(g) ([6], *Theorem* 19 *with* $n = 2$) *If $\nu \leq \aleph_3$ and A and B are p^{ω_1}-bounded, then $A \triangledown B \in \mathcal{F}$.*

Proof. (a): Follows from the above resolution of G. (b): If $G \triangledown M$ is a dsc group, then G must be p^{ω_1}-bounded by Theorem 1.1. (c): One direction is (b), the other follows from Lemma 2.2(a) with $A = G$ and $B = M$. (d): If $B \in \mathcal{F}$, then $B \triangledown M$ is a dsc group so that $A \triangledown B \triangledown M$ is also a dsc group. (e): By (d), $(G \triangledown M)/(K \triangledown M) \simeq (G/K) \triangledown M \in \mathcal{F}$. So, since $G \triangledown M$ is a dsc, so is $K \triangledown M$. (f): Follows from Lemma 2.2(b) with $C = M$. $\square$

If $G \in \mathcal{F}$, then a p^{ω_1}-pure subgroup $A \subseteq G$ will be $\mathcal{F}$-*pure* if $G/A \in \mathcal{F}$. By Lemma 3.2(b) any $\mathcal{F}$-pure subgroup is closed. In addition, if G is a dsc and $A \subseteq G$ is an $\mathcal{F}$-pure subgroup, then since $G/A \in \mathcal{F}$, we can conclude that A is also a dsc group, i.e., an $\mathcal{F}$-pure subgroup of a dsc group is also a dsc group.

Now if $G \in \mathcal{F}$, then $G \triangledown M$ will be a dsc group. So $A \subseteq G$ is $\mathcal{F}$-pure if and only if $A \subseteq G$ is p^{ω_1}-pure and $(G/A) \triangledown M$ is a dsc group if and only if the sequence $0 \to A \triangledown M \to G \triangledown M \to (G/A) \triangledown M \to 0$ splits. [Since M has a summand that is a dsc group of length ω_1, if this latter sequence splits, then A will automatically be a p^{ω_1}-pure subgroup of G.] This characterization clearly implies that $\mathcal{F}$-purity is transitive; i.e., if B is $\mathcal{F}$-pure in A and A is $\mathcal{F}$-pure in G, then B is $\mathcal{F}$-pure in G.

If $G \in \mathcal{F}$, then fix a decomposition $\mathcal{D}$ of $G \triangledown M$ into countable groups. Consider the collection $\mathcal{M}$ of all subgroups $A \subseteq G$ such that $A \triangledown M \subseteq G \triangledown M$ is a $\mathcal{D}$-summand. We call this the F-*system* determined by $\mathcal{D}$. As in Lemma 2.4, we have the following:

Lemma 3.3. *Suppose $G \in \mathcal{F}$ and $\mathcal{M}$ is an F-system for G.*

(a) *$\mathcal{M}$ is closed under unions of chains*;

(b) *If $A \subseteq G$ has cardinality $\kappa \geq \aleph_1$, then there is an $X \in \mathcal{M}$ such that $A \subseteq X$ and $|X| = \kappa$.*

(c) *If $|G| = \kappa \geq \aleph_2$ and κ is regular, then G has a filtration $\{X_i\}_{i<\lambda} \subseteq \mathcal{M}$.*

Note that Lemma 3.3(b) implies that a group $G \in \mathcal{F}$ cannot have a κ-Kurepa subgroup for any $\kappa > \aleph_1$. We also note the following simple idea.

Lemma 3.4. *If $G \in \mathcal{F}$, then G has an $\aleph_1$-Kurepa subgroup if and only if it has an $\mathcal{F}$-pure subgroup A of cardinality $\aleph_1$ that fails to be a dsc group.*

Proof. Since an $\mathcal{F}$-pure subgroup is closed, sufficiency is obvious. Conversely, if $A \subseteq G$ is an $\aleph_1$-Kurepa subgroup, then by Lemma 3.3(b) we can find a $\mathcal{F}$-pure subgroup $X \subseteq G$ containing A with $|X| = \aleph_1$. Since A is closed in X, by Lemma 3.2(c), $X/A \in \mathcal{F}$. Therefore, X also fails to be a dsc group, as desired. □

If $G \in \mathcal{F}$, we define our invariant $F_G \subseteq \mathcal{Q}_f$ by induction on $\mu(T) := \kappa$ as follows:

(F-0) if $\kappa = \aleph_1$ (i.e., $T = \emptyset$), then $T \in F_G$ if and only if G has an $\mathcal{F}$-pure subgroup A of cardinality ω_1 that is not a dsc group; or equivalently, an $\aleph_1$-Kurepa subgroup.

Suppose now that $\kappa \geq \aleph_2$, and that for all $S \in \mathcal{Q}_f$ with $\mu(S) < \kappa$ and for all groups $\widetilde{G} \in \mathcal{F}$ we have defined when $S \in F_{\widetilde{G}}$. Then $T \in F_G$ if and only if one of the following two conditions holds:

(F-1) $\Upsilon^F_T(G) = \{i < \kappa : T_i \in F_G\}$ is stationary in κ; or

(F-2) G has an $\mathcal{F}$-pure subgroup A of cardinality κ with an $\mathcal{F}$-pure filtration $\{X_i\}_{i<\gamma}$ such that $\Lambda^F_T(A) = \{i < \kappa : T_i \in F_{A/X_i}\}$ is stationary in κ.

The following gives some basic properties of this invariant. It is similar to results for L^λ_G and J_G given in earlier works.

Lemma 3.5 (cf. [9], Lemma 1.4). *Suppose $G \in \mathcal{F}$, $T \in \mathcal{Q}_f$ and $\kappa = \mu(T)$.*

(a) *If $T \in F_G$, $S \in \mathcal{Q}_f$ and $T \subseteq S$, then $S \in F_G$.*

(b) *If K is an $\mathcal{F}$-pure subgroup of G, then $F_K \subseteq F_G$.*

(c) *If $G = G_1 \oplus G_2$, then $F_G = F_{G_1} \cup F_{G_2}$.*

(d) *If $T \in F_G$, then there is an $\mathcal{F}$-pure subgroup $K \subseteq G$ such that $|K| \leq \kappa$ and $T \in F_K$.*

(e) *If $T \in F_G$ is minimal under inclusion, then $\mu(T) \leq |G|$.*

(f) *If K is an $\mathcal{F}$-pure subgroup of G and $|K| = |G| = \kappa \geq \aleph_2$, then there are $\mathcal{F}$-pure filtrations of K and G such that $\Lambda^F_T(K) \subseteq \Lambda^F_T(G)$.*

Proof. As in [9], all but (f) follow from a straightforward induction. For example, consider (e). If $\kappa = \aleph_1$ and $T \in F_G$, then it follows immediately from (F-0) that G has cardinality at least $\aleph_1$. Suppose now that $\kappa > \aleph_1$. If (F-1) holds, consider any $i \in \mathcal{E} := \Upsilon^F_T(G)$. Find $S_i \subseteq T_i$ with $S_i \in F_G$ minimal under inclusion. Since T is also minimal under inclusion, we must have $i \in S_i$. Therefore by induction, $|G| \geq \sup\{\mu(S_i)\}_{i\in\mathcal{E}} = \{i\}_{i\in\mathcal{E}} = \kappa$. If (F-2) holds, we know that G has an $\mathcal{F}$-pure subgroup A of cardinality κ, giving the result.

Finally, consider (f). Note that $G/K \in \mathcal{F}$. Let $\{Z_i \subseteq G/K\}_{i<\kappa}$ be defined as follows: If $|G/K| = \kappa$, then let it be an $\mathcal{F}$-pure filtration of G/K; otherwise, let each $Z_i = G/K$. Now, let $\{X_i\}_{i<\kappa}$ be an $\mathcal{F}$-pure filtration of K and $\{Y_i\}_{i<\kappa}$ be an $\mathcal{F}$-pure filtration of G. It is clear that the set of $i < \kappa$ such that $X_i = Y_i \cap K$ and $Z_i = [Y_i + K]/K$ is a CUB in κ. Restricting our filtration to the members of this CUB, the result easily follows from (b). □

We will have use for the following elementary property:

Lemma 3.6. *If A and B are p^{ω_1}-bounded C_{ω_1}-groups, then $A \triangledown B$ does not have a κ-Kurepa subgroup for any cardinal κ.*

Proof. If K is any p^{ω_1}-pure subgroup of $A \bigtriangledown B$ of cardinality $\aleph_1$, then we can find p^{ω_1}-pure subgroups $X \subseteq A$ and $Y \subseteq B$ of cardinality $\aleph_1$ such that $K \subseteq X \bigtriangledown Y$. It easily follows that $X \bigtriangledown Y$ is closed in $X \bigtriangledown B$, which is closed in $A \bigtriangledown B$. This implies that $|\bar{K}| = \aleph_1$; so $A \bigtriangledown B$ does not have κ-Kurepa subgroups for any $\kappa \geq \aleph_2$. So assume $\kappa = \aleph_1$. Lemma 2.2(a) implies that $X \bigtriangledown Y$ is a dsc group. If K were closed in $A \bigtriangledown B$, then by Lemma 3.2(c), $(X \bigtriangledown Y)/K \in \mathcal{F}$. This would imply that K would be a dsc group. Therefore, $A \bigtriangledown B$ has no $\aleph_1$-Kurepa subgroups. □

Before we state and prove the main result of this section, we mention a useful variation on Fodor's Lemma. Suppose γ is a regular cardinal and $\mathcal{V} \subseteq \gamma$ is a stationary subset. A function $f: \mathcal{V} \to \mathcal{Q}_f$ such that $f(i) \subseteq i$ for all $i \in \mathcal{V}$ will be called *regressive.*

Lemma 3.7 ([9], Lemma 1.5). *Suppose $\gamma \in \mathcal{Q}$ and $\mathcal{V} \subseteq \gamma$ is a stationary subset. If $f: \mathcal{V} \to \mathcal{Q}_f$ is a regressive function, then there is a stationary subset $\mathcal{V}' \subseteq \mathcal{V}$ such that $f(i) = f(j)$ for all $i, j \in \mathcal{V}'$.*

So we have arrived at the solution to the question in the title of this section and one of the main results of the paper.

Proof of Theorem 1.5. Suppose first that G is a dsc group. We show $T \notin F_G$ for each $T \in \mathcal{Q}_f$ by induction on $\kappa = \mu(T)$. First, if $\kappa = \aleph_1$, then $T = \emptyset$ and we are in the base case (F-0). However, we observed that any $\mathcal{F}$-pure subgroup of a dsc group is actually another dsc group, so that $T \notin F_G$. Now, if $\kappa > \aleph_1$, then we show (F-1) cannot hold. By induction, for all $i < \kappa$, $\mu(T_i) < \kappa$, so that $T_i \notin F_G$. Therefore, $\Upsilon_T^F(G) = \emptyset$ and (F-1) fails. Finally, suppose $A \subseteq G$ is as in (F-2). There is clearly a decomposition $G = G_1 \oplus G_2$, where $A \subseteq G_1$ and $|G_1| = \kappa$. It follows from Lemma 3.5(f) that $\Lambda_T^I(A) \subseteq \Lambda_T^I(G_1)$. But G_1 has a filtration by summands, so by induction, $\Lambda_T^I(G_1)$ is empty. So (F-2) fails, and this completes one direction of the argument.

Conversely, we show by induction on $\gamma := |G|$ that $F_G = 0_{\mathcal{Q}}$ implies that G is a dsc group. Consider first the case where $\gamma = \aleph_1$. Since $\emptyset \notin F_G$, G cannot have an $\mathcal{F}$-pure subgroup of cardinality $\aleph_1$ which is not a dsc group. In particular, G itself must be a dsc group.

So, suppose the result holds for all groups in $\mathcal{F}$ of cardinality strictly less than $\gamma > \aleph_1$. Suppose K is an $\mathcal{F}$-pure subgroup of G with $|K| < \gamma$. By Lemma 3.2(e), $K \in \mathcal{F}$, and by Corollary 3.5(b), $F_K \subseteq F_G = 0_{\mathcal{Q}}$; and by induction, this implies that K is a dsc group. In particular, every group in an F-system for G of strictly smaller cardinality than γ will be a dsc. We now divide the argument into two cases.

Suppose first that γ is singular. If $\mathcal{M}$ is an F-system for G, then clearly $\mathcal{S} = \{S \in \mathcal{M} \colon |S| < \gamma\}$ is a γ-dsc system for G. So by Theorem 2.5, G is a dsc group, as desired.

Suppose now that γ is regular and $\{X_i\}_{i<\gamma} \subseteq \mathcal{M}$ is a filtration of G; so each X_i is a dsc group. Let $\mathcal{U} \subseteq \gamma$ be the collection of all $i < \gamma$ such that X_i is a summand of X_j whenever $i \leq j < \gamma$. If $\mathcal{U}$ contains a CUB subset $\mathcal{E}$, then replacing our filtration by restricting to the elements of $\mathcal{E}$, we may assume that each X_i is a summand of X_{i+1}. This easily implies that G itself is a dsc group, as required.

We show that if $\mathcal{U}$ fails to contain a CUB, then we are led to a contradiction. Let $\mathcal{V} = \gamma - \mathcal{U}$; so we are assuming that $\mathcal{V}$ is stationary. Again replacing our filtration by restricting our attention to the terms in some CUB, we may assume that for every $i \in \mathcal{V}$, that X_i fails to be a summand of X_{i+1}. In addition, since $\mathcal{W} := \{i < \gamma \colon |X_i| = |i|\}$ is a CUB, replacing $\mathcal{V}$ by the stationary set $\mathcal{V} \cap \mathcal{W}$, we may assume that $|X_i| = |i|$ for all $i \in \mathcal{V}$.

If $i \in \mathcal{V}$, then there is a decomposition $X_{i+1} = Y \oplus Z$, where $X_i \subseteq Y$ and $|Y| = |X_i| = |i| < \gamma$. Now $Y/X_i \in \mathcal{F}$, but since X_i is not a summand of Y, Y/X_i will not be a dsc group. It follows from induction and Lemma 3.5(e) that there is a $T^i \in F_{Y/X_i} \subseteq F_{X_{i+1}/X_i} \subseteq F_{G/X_i}$ such that $\mu(T^i) \leq |i| \leq i$.

The mapping $i \mapsto T^i - \{i\}$ will be regressive, so by Lemma 3.7, there is a stationary subset $\mathcal{V}' \subseteq \mathcal{V}$ on which this assignment is constant. If $\widehat{T}$ is this constant value, then we let $T = \widehat{T} \cup \{\gamma\}$. It follows that $\mathcal{V}' \subseteq \Lambda^F_T(G)$ so that by (F-2), $T \in F_G$. This means that $F_G \neq 0_{\mathcal{Q}}$ and this contradiction completes the proof. $\square$

4. Nunke's problem when $2^{\aleph_1} = \aleph_2$

We would like to somehow combine J_G and F_G into a single invariant to address Nunke's problem. Our construction needs to detect κ-Kurepa subgroups, both for $\kappa \geq \aleph_2$, as does J_G, but also for $\kappa = \aleph_1$, as does F_G in the case where $G \in \mathcal{F}$.

Recall $\mathcal{R} = \{\aleph_1\} \cup \mathcal{Q}$. Given a p^{ω_1}-bounded C_{ω_1}-group G, we define a collection of *non-empty* finite subsets of $\mathcal{R}$, $I_G \subseteq \mathcal{R}_f$, by induction on $\mu(T) := \kappa$ as follows:

(I-0) If $\kappa < \nu$ and η is the least element of T, then $T \in I_G$ if and only if G has an η-Kurepa subgroup.

Suppose now that $\kappa \geq \nu$, and that for all $\emptyset \neq S \in \mathcal{R}_f$ with $\mu(S) < \kappa$ and for all p^{ω_1}-bounded C_{ω_1}-groups $\widehat{G}$ we have defined when $S \in I_{\widehat{G}}$. Then $T \in I_G$ if and only if one of the following two conditions holds:

(I-1) $\Upsilon^I_T(G) = \{i < \kappa : T_i \in I_G\}$ is stationary in κ; or

(I-2) G has a closed p^{ω_1}-pure subgroup A of cardinality κ with a closed p^{ω_1}-pure filtration $\{X_i\}_{i<\gamma}$ such that $\Lambda^I_T(A) = \{i < \kappa : T_i \in I_{A/X_i}\}$ is stationary in κ.

The following summary of the basic properties of I_G is proved in an almost identical manner to the corresponding result for F_G (that is, Lemma 3.5).

Lemma 4.1. *Suppose G is a p^{ω_1}-bounded C_{ω_1}-group, $T \in \mathcal{R}_f$ and $\kappa = \mu(T)$.*

(a) *If $T \in I_G$, $S \in \mathcal{R}_f$ and $T \subseteq S$, then $S \in I_G$.*

(b) *If K is a closed p^{ω_1}-pure subgroup of G, then $I_K \subseteq I_G$.*

(c) *If $G = G_1 \oplus G_2$, then $I_G = I_{G_1} \cup I_{G_2}$.*

(d) *If $T \in I_G$ and $\kappa \geq \nu$, then there is a closed p^{ω_1}-pure subgroup $K \subseteq G$ such that $|K| \leq \kappa$ and $T \in I_K$.*

(e) *If $T \in I_G$ is minimal under inclusion, then $\mu(T) \leq |G|$.*

(f) *If K is a closed p^{ω_1}-pure subgroup of G and $|K| = |G| = \kappa \geq \nu$, then there are closed p^{ω_1}-pure filtrations of K and G such that $\Lambda^I_T(K) \subseteq \Lambda^I_T(G)$.*

It also follows easily from induction that if G is a p^{ω_1}-bounded C_{ω_1}-group and $T \in I_G$ has least element η, then $\eta < \nu$ (since this holds for the base case).

We now turn to our main objective, addressing the uncountable case of Nunke's problem. The following result will be the base cases of our inductions:

Lemma 4.2. *If A and B are p^{ω_1}-bounded C_{ω_1}-groups of cardinality at most $\aleph_2$, then $A \bigtriangledown B \in \mathcal{F}$ and the following are equivalent*:

(a) *$A \bigtriangledown B$ is not a dsc*;

(b) $\{\aleph_2\} \in F_{A \bigtriangledown B}$;

(c) *one of A, B has an $\aleph_1$-Kurepa subgroup and the other has an $\aleph_2$-Kurepa subgroup*;

(d) $\{\aleph_1, \aleph_2\} \in I_A \cdot I_B$.

Proof. By Lemma 3.2(f) we know that $A \bigtriangledown B \in \mathcal{F}$. Clearly, (b) implies (a) by Theorem 1.5. Assuming (a), by Lemma 3.5(e) there is $T \in F_{A \bigtriangledown B}$ with $\mu(T) \leq \aleph_2$. By Lemma 3.6, $\emptyset \notin F_{A \bigtriangledown B}$. Therefore, $T = \{\aleph_2\}$, so that (b) follows. The equivalence of (c) and (d) is due to the definitions of I_A, I_B and the fact that every element of these invariants is non-empty. Finally, the equivalence of (a) and (c) is due to the $n = 2$ case of ([6], Theorem 15). $\square$

If $T \in \mathcal{Q}_f$, let $\widetilde{T} = \{\aleph_1\} \cup T \in \mathcal{R}_f$. This brings us to half our main result:

THEOREM 4.3. *Suppose A and B are p^{ω_1}-bounded C_{ω_1}-groups.*

(a) *If $\nu \leq \aleph_3$, then $A \bigtriangledown B \in \mathcal{F}$ and $I_A \cdot I_B \subseteq \widetilde{F}_{A \bigtriangledown B} := \{\widetilde{T} : T \in F_{A \bigtriangledown B}\}$.*

(b) *If $B \in \mathcal{F}$, then $A \bigtriangledown B \in \mathcal{F}$ and $J_A \cdot F_B \subseteq F_{A \bigtriangledown B}$.*

PROOF. Observe that in case (a), if $S \in I_A \cdot I_B$, then since the least elements of the sets in I_A and I_B are at most $\aleph_2$, we can conclude that $\{\aleph_1, \aleph_2\} \subseteq S$, and in particular, that $S = \widetilde{T}$ for some unique $T \in \mathcal{Q}_f$.

We begin with a crucial observation. We will often consider p^{ω_1}-pure subgroup $X \subseteq A$ and $Y \subseteq B$. We have containments $X \bigtriangledown Y \subseteq X \bigtriangledown B \subseteq A \bigtriangledown B$ with quotients $Q_1 := X \bigtriangledown (B/Y)$ and $Q_2 := (A/X) \bigtriangledown B$. In the context of part (a) we will further suppose X and Y are closed, i.e., A/X and B/Y are p^{ω_1}-bounded. Therefore by Lemma 3.2(g), $Q_1, Q_2 \in \mathcal{F}$. In the context of part (b) we will further suppose that Y is F-pure in B, i.e., $B/Y \in \mathcal{F}$. Since both B and B/Y are both in $\mathcal{F}$, by Lemma 3.2(d) we will have $Q_1, Q_2 \in \mathcal{F}$. In either case, it follows that $X \bigtriangledown Y$ is $\mathcal{F}$-pure in $A \bigtriangledown B$. In particular, if $X = Y = 0$, we can conclude that $A \bigtriangledown B \simeq (A \bigtriangledown B)/(0 \bigtriangledown 0) \in \mathcal{F}$.

By induction on $\kappa := \mu(T)$, for each $T \in \mathcal{Q}_f$ we show that if (a) $\widetilde{T} \in I_A \cdot I_B$; or (b) $T \in J_A \cdot F_B$, then $T \in F_{A \bigtriangledown B}$. For our base case, suppose $\kappa = \mu(T) \leq \aleph_2$.

(a): Since every element of I_A and I_B is non-empty, $\widetilde{T} \in I_A \cdot I_B$ exactly if $\widetilde{T} = \{\aleph_1, \aleph_2\} \in I_A \cdot I_B$. We can find closed p^{ω_1}-pure subgroups $X \subseteq A$ and $Y \subseteq B$ of cardinality at most $\aleph_2$ such that $\{\aleph_1, \aleph_2\} \in I_X \cdot I_Y$. And since $X \bigtriangledown Y$ is $\mathcal{F}$-pure in $A \bigtriangledown B$, by Lemma 4.2, $T = \{\aleph_2\} \in F_{X \bigtriangledown Y} \subseteq F_{A \bigtriangledown B}$.

(b): Since A is p^{ω_1}-bounded, every element of J_A is non-empty. So if $T \in J_A \cdot F_B$ and $\kappa \leq \aleph_2$, then we must have $\{\aleph_2\} \in J_A$ and $\emptyset \in F_B$. By Lemma 3.5(d) and Proposition 3.1, this means that A has a p^{ω_1}-pure subgroup X of cardinality $\aleph_2$ that has an $\aleph_2$-Kurepa subgroup; and B has an $\mathcal{F}$-pure subgroup Y of cardinality $\aleph_1$, that is not a dsc group. Since $X \bigtriangledown Y$ is an $\mathcal{F}$-pure subgroup of $A \bigtriangledown B$, by Lemma 4.2 we can conclude that $T = \{\aleph_2\} \in F_{X \bigtriangledown Y} \subseteq F_{A \bigtriangledown B}$.

So we may assume $\kappa > \aleph_2$. For the remainder of the proof we will concentrate on part (a), indicating parenthetically the very minor changes needed to establish part (b). Suppose $\widetilde{T} \in I_A \cdot I_B$. It follows that $\widetilde{T}$ will be a disjoint union of the non-empty sets $U \in I_A$ and $V \in I_B$. Suppose first that $\kappa \in U$. If U satisfies (I-1), then $\Upsilon^I_U(A)$ is stationary. However, if $i \in \Upsilon^I_U(A)$ and $\mu(V) < i < \kappa$, then by induction $\widetilde{T}_i = U_i \cup V \in I_A \cdot I_B$ implies that $T_i \in F_{A \bigtriangledown B}$. In other words, $\Upsilon^I_U(A) \subseteq \Upsilon^F_T(A \bigtriangledown B)$, and since the left is stationary, so is the right; i.e., $T \in F_{A \bigtriangledown B}$. The case where $\kappa \in V$ is identical. [(b): We assume T is the disjoint union of $U \in J_A$ and $V \in F_B$. If $\kappa \in U$ and U satisfies (J-1), it follows that $\Upsilon^J_U(A) \subseteq \Upsilon^F_T(A \triangledown B)$ and if $\kappa \in V$ and V satisfies (F-1), it follows that $\Upsilon^F_U(B) \subseteq \Upsilon^F_T(A \triangledown B)$.]

Now, if $\kappa \in U$ and $U \in I_A$ satisfies (I-2), then we can find a closed p^{ω_1}-pure subgroup $X \subseteq A$ such that $|X| = \kappa$ and $\Lambda^I_U(X)$ is stationary. By Lemma 3.5(d) we can find a closed p^{ω_1}-pure subgroup $Y \subseteq B$ such that $V \in I_Y$ and $|Y| \leq \mu(V) < \kappa$. If $\{Z_i\}_{i<\kappa}$ is a closed p^{ω_1}-pure filtration of X, then $\{Z_i \triangledown Y\}_{i<\kappa}$ will be an $\mathcal{F}$-pure filtration of $X \triangledown Y$. So if $i \in \Lambda^I_U(X)$ with $i > \mu(V)$, then by induction $T_i = U_i \cup V \in F_{(X/Z_i)\triangledown Y} = F_{(X\triangledown Y)/(Z_i\triangledown Y)}$. This means that $\Lambda^I_U(X) \subseteq \Lambda^F_T(X \triangledown Y)$, so that $T \in F_{X\triangledown Y} \subseteq F_{A\triangledown B}$. By symmetry, this also works when $\kappa \in V$ and $V \in I_B$ satisfies (I-2). [(b): Again, supposing T is the disjoint union $U \cup V$, if $\kappa \in U$ and $U \in J_A$ satisfies (J-2), then we can find a p^{ω_1}-pure subgroup $X \subseteq A$ and an $\mathcal{F}$-pure subgroup $Y \subseteq B$ such that $|Y| < |X| = \kappa$, $U \in I_X$, $V \in I_Y$. This time using a p^{ω_1}-pure filtration $\{Z_i\}_{i<\kappa}$ of X shows that $\Lambda^J_U(X) \subseteq \Lambda^F_T(X \triangledown Y)$, so that $T \in F_{X\triangledown Y} \subseteq F_{A\triangledown B}$. Similarly, if $\kappa \in V$ and $V \in F_B$ satisfies (F-2), in the above, $|X| < |Y| = \kappa$. If $\{Z_i\}_{i<\kappa}$ is an $\mathcal{F}$-pure filtration of Y, we can conclude that $\Lambda^F_V(Y) \subseteq \Lambda^F_T(X \triangledown Y)$, so that $T \in F_{X\triangledown Y} \subseteq F_{A\triangledown B}$.] $\square$

We now conclude the proof of the main result of this paper. For simplicity, in the statement of Theorem 1.7(a) we included $2^{\aleph_1} = \aleph_2$ as a hypothesis; what we really need is that $\nu \leq \aleph_3$, as in Theorem 4.3(a).

Proof of Theorem 1.7. Suppose first that $A \triangledown B$ is a dsc. In part (a), by Theorems 1.5 and 4.3(a) we have $I_A \cdot I_B \subseteq \widetilde{F}_{A\triangledown B} = \widetilde{0}_{\mathcal{Q}} = 0_{\mathcal{R}}$; and in part (b), by Theorem 4.3(b) we have $J_A \cdot F_B \subseteq F_{A\triangledown B} = 0_{\mathcal{Q}}$.

To prove the converse, in either case, since $A \triangledown B \in \mathcal{F}$, we need to show that $F_{A\triangledown B} = 0_{\mathcal{Q}}$. In (a) we are assuming $I_A \cdot I_B = 0_{\mathcal{Q}}$ and in (b) we are assuming that $J_A \cdot F_B = 0_{\mathcal{Q}}$. As above, these two arguments are similar and we will concentrate on (a), indicating parenthetically the very minor changes needed to address (b).

We prove the following using induction on $\kappa := \mu(T)$. If $T \in F_{A\triangledown B}$, then there is an $S \in \mathcal{Q}_f$ such that $\widetilde{S} \in I_A \cdot I_B$ and $\mu(S) \leq \kappa$. [(b): $S \in J_A \cdot F_B$.]

Suppose first that $\kappa = \aleph_1$, i.e., $T = \emptyset$. By Lemma 3.6, $T = \emptyset \notin F_{A\triangledown B}$, so the implication is vacuously true.

Suppose next that $\mu(T) = \aleph_2$, i.e., $T = \{\aleph_2\}$. Using Lemmas 3.5(d) and 2.4(c) we can find closed p^{ω_1}-pure subgroups $X \subseteq A$ and $Y \subseteq B$ such that $|X \triangledown Y| \leq \kappa$ and $T \in F_{X\triangledown Y}$. So by Lemma 4.2, $\widetilde{T} = \{\aleph_1, \aleph_2\} \in I_X \cdot I_Y \subseteq I_A \cdot I_B$, as required. [(b): Here $X \subseteq A$ would be p^{ω_1}-pure and $Y \subseteq B$ would be $\mathcal{F}$-pure. We would still have $T = \{\aleph_2\} \in F_{X\triangledown Y}$. Since $Y \in \mathcal{F}$ cannot have an $\aleph_2$-Kurepa subgroup, by Lemma 4.2 we could conclude that X has an $\aleph_2$-Kurepa subgroup and Y an $\aleph_1$-Kurepa subgroup. So again, by Proposition 3.1, $\widetilde{T} = \{\aleph_2\} \cup \{\aleph_1\} \in J_X \cdot F_Y \subseteq J_A \cdot F_B$, as required.]

Now suppose $\kappa > \aleph_2$, so $T \in F_{A \triangledown B}$ must satisfy (F-1) or (F-2). Suppose first that T satisfies (F-1), i.e., $\Upsilon^F_T(A \triangledown B)$ is stationary. For any $i \in \Upsilon^F_T(A \triangledown B)$, by induction $T_i \in F_{A \triangledown B}$ implies that there is an element $\widetilde{S} \in I_A \cdot I_B$ with $\mu(S) \leq \mu(T_i) < \kappa$. [(b): Here $S \in J_A \cdot F_B$.]

Finally suppose T satisfies (F-2). We can find closed p^{ω_1}-pure subgroups $A' \subseteq A$, $B' \subseteq B$ such that $|A' \triangledown B'| = \kappa$ and $T \in F_{A' \triangledown B'}$. Replacing A by A' and B by B', from Lemma 3.5(f) we can conclude that $\mathcal{E} := \Lambda^F_T(A \triangledown B)$ is stationary. Define $\{X_i\}_{i<\kappa}$ as follows. If $|A| = \kappa$, let it be a closed p^{ω_1}-pure filtration of A; otherwise, let each $X_i = A$. And if $|B| = \kappa$, let $\{Y_i\}_{i<\kappa}$ be a closed p^{ω_1}-pure filtration of B; otherwise let each $Y_i = B$. It follows that $\{X_i \triangledown Y_i\}_{i<\kappa}$ is an $\mathcal{F}$-pure filtration of $A \triangledown B$. [(b): We assume that A' and each X_i is p^{ω_1}-pure in A; and B' and each Y_i is $\mathcal{F}$-pure in B.]

For each $i < \kappa$ there is a commutative diagram with $\mathcal{F}$-pure rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_i \triangledown Y_i & \longrightarrow & X_i \triangledown B & \longrightarrow & X_i \triangledown (B/Y_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A \triangledown Y_i & \longrightarrow & A \triangledown B & \longrightarrow & A \triangledown (B/Y_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (A/X_i) \triangledown Y_i & \longrightarrow & (A/X_i) \triangledown B & \longrightarrow & (A/X_i) \triangledown (B/Y_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

So the following push-out of both the right column and bottom row is $\mathcal{F}$-pure:

$$
\begin{aligned}
0 \longrightarrow (A \triangledown B)/(X_i \triangledown Y_i) &\longrightarrow ([A/X_i] \triangledown B) \oplus (A \triangledown [B/Y_i]) \\
&\longrightarrow [A/X_i] \triangledown [B/Y_i] \longrightarrow 0.
\end{aligned}
$$

For any $i \in \mathcal{E}$, $T_i \in F_{(A \triangledown B)/(X_i \triangledown Y_i)}$. And by Lemma 3.5(c) and the above sequence we can conclude that

$$T_i \in F_{[A/X_i] \triangledown B} \cup F_{A \triangledown [B/Y_i]}.$$

Since there are only two choices, there is a stationary subset $\mathcal{E}' \subseteq \mathcal{E}$ such that either $T_i \in F_{[A/X_i] \triangledown B}$ for all $i \in \mathcal{E}'$, or $T_i \in F_{A \triangledown [B/Y_i]}$ for all $i \in \mathcal{E}'$. Assume

$T_i \in F_{[A/X_i]\triangledown B}$ for all $i \in \mathcal{E}'$ (the other case, again, is symmetric). By induction we can conclude that whenever $i \in \mathcal{E}'$, there is a disjoint union $\widetilde{S}^i = U^i \cup V^i$, where $U^i \in I_{A/X_i}$ and $V^i \in I_B$, with $\mu(S^i) \leq \mu(T_i) \leq i < \kappa$. [(b): We will have $S^i = U^i \cup V^i$, with our two possibilities being $U^i \in J_{A/X_i}$ and $V^i \in F_B$; or else $U^i \in J_A$ and $V^i \in F_{B/Y_i}$ - in either case with $\mu(S^i) \leq \mu(T_i) \leq i < \kappa$.]

The map $i \mapsto U^i - \{i\}$ is regressive, so by Lemma 3.7 there is a stationary subset $\mathcal{E}'' \subseteq \mathcal{E}'$ such that for all $i, j \in \mathcal{E}''$, $U^i - \{i\} = U^j - \{j\} := \widehat{U}$. If we define $U = \widehat{U} \cup \{\kappa\}$, it follows that $\mathcal{E}'' \subseteq \Lambda^I_U(A)$, so that $U \in I_A$. If we choose any $i \in \mathcal{E}''$ with $\mu(\widehat{U}) < i < \kappa$, then it follows that $\widetilde{S} := U \cup V^i \in I_A \cdot I_B$ and $\mu(S) = \kappa$, completing the argument. [(b): Again, replace I by the appropriate J or F.] □

5. Examples and further discussion

We start with a straightforward observation.

Proposition 5.1. *If K is a closed p^{ω_1}-pure subgroup of the dsc group G, then $I_K = 0_{\mathcal{R}}$.*

Proof. If we can show $I_G = 0_{\mathcal{R}}$, then this follows from Lemma 4.1(b). As in the first paragraph of Theorem 1.5, we show $T \notin F_G$ for each $T \in \mathcal{R}_f$ by induction on $\kappa = \mu(T)$. First, if $\kappa < \nu$, then we are in the base case (I-0). As was noted previously, a dsc group cannot have a κ-Kurepa subgroup for any $\kappa < \nu$, so $T \notin I_G$.

Now, if $\kappa \geq \nu$, then the fact that (I-1) cannot hold follows easily by induction. Finally, suppose $A \subseteq G$ is as in (I-2). There is clearly a decomposition $G = G_1 \oplus G_2$, where $A \subseteq G_1$ and $|G_1| = \kappa$. It follows from Lemma 4.1(f) that $\Lambda^I_T(G_1)$ is stationary. But G_1 has a filtration by summands, so by induction, $\Lambda^I_T(G_1)$ is empty. This contradiction completes the argument. □

If $\nu = \aleph_3$, then there is a p^{ω_1}-pure short exact sequence $0 \to G \to H \to H/G \to 0$ where H is a dsc group of cardinality $\aleph_2$ and H/G is a p^{ω_1}-bounded C_{ω_1}-group with p^{ω_1}-p.d. equaling 2. So G is a closed p^{ω_1}-pure subgroup of H, but not an $\mathcal{F}$-pure subgroup. It follows that this G is not a dsc group, but by Proposition 5.1, we know that $I_G = 0_{\mathcal{R}}$. In other words, the invariant I_G does not necessarily tell us when an individual p^{ω_1}-bounded C_{ω_1}-group is a dsc group; this is in contrast to Theorems 1.2(a) and 1.4(a). In addition, for every p^{ω_1}-bounded C_{ω_1}-group A we will have $I_A \cdot I_G = I_A \cdot 0_{\mathcal{Q}} = 0_{\mathcal{Q}}$; so by Theorem 1.7(a), $A \triangledown G$ is always a dsc group even though G itself is not [this fact was previously observed in ([6], Theorem 20)].

Staying with this $G \subseteq H$, note that $G \in \mathcal{F}$, and since it fails to be a dsc group, $F_G \neq 0_{\mathcal{Q}}$. Since $I_G = 0_{\mathcal{Q}}$, we know that G cannot have an $\aleph_1$-Kurepa subgroup. Therefore, we can conclude that $\emptyset \notin F_G$. If $T \in F_G$ is minimal under inclusion, then by Lemma 3.5(e) we must have $\mu(T) \leq \aleph_2$; so $T = \{\aleph_2\}$ is the only minimal set and $F_G = \{S \in \mathcal{Q}_f : \aleph_2 \in S\}$. Notice that this shows that when $G \subseteq H$ is $\mathcal{F}$-pure we can conclude that $F_G \subseteq F_H$, but this does not necessarily hold if $G \subseteq H$ is only assumed to be a closed p^{ω_1}-pure subgroup.

We now show that a naive generalization of Theorem 1.7(a) for $\nu = \aleph_4$ does not hold. Suppose A is a p^{ω_1}-bounded C_{ω_1}-group of cardinality $\aleph_3$ with an $\aleph_3$-Kurepa subgroup. Let $B = A \triangledown A$. It follows from ([6], Theorem 15 with $n = 3$) that $A \triangledown B = A \triangledown A \triangledown A$ is not a dsc. On the other hand, if $T \in I_B$ is minimal under inclusion, then $\mu(T) \leq \aleph_3 < \nu = \aleph_4$. However, by Lemma 3.6, B does not have a κ-Kurepa subgroup for any κ; so no such T exists. Therefore, $I_A \cdot I_B = I_A \cdot 0_{\mathcal{R}} = 0_{\mathcal{R}}$. So in any possible generalization of Theorem 1.7(a) for $\nu \geq \aleph_4$ it will be necessary to amend the definition of I_G.

We next observe that Theorem 1.7(b) holds even when A is p^{ω_1}-unbounded. In this case, then $J_A = 1_{\mathcal{Q}} := \mathcal{Q}_f$. So by Theorems 1.1 and 1.5, if $B \in \mathcal{F}$, then $A \triangledown B$ is a dsc group if and only if B is a dsc group if and only if $J_A \cdot F_B = 1_{\mathcal{Q}} \cdot F_B = F_B = 0_{\mathcal{Q}}$. In fact, Theorem 1.5 is simply one case of Theorem 1.7(b), where $A = \mathbb{Z}_{p^\infty}$ and $B = G$, so that $A \triangledown B \simeq G \in \mathcal{F}$ and $J_A \cdot F_B = F_G$.

In Theorems 4.3 and 1.7 we would clearly prefer the following hold: (a) $I_A \cdot I_B = \widetilde{F}_{A \triangledown B}$ (for $\nu \leq \aleph_3$); and (b) $J_A \cdot F_B = F_{A \triangledown B}$. In both cases Theorem 4.3 gives one containment, the question is the reverse containment. The problems come in the presence of regular cardinals κ that are *weakly Mahlo*, i.e., $\mathcal{S} = \{\gamma \in \mathcal{Q} : \gamma < \kappa\}$ is stationary in κ. As in ([9], Theorem 2.3), these inclusions will in fact be equalities when we restrict to the class of non-weakly Mahlo regular cardinals.

Finally, when $2^{\aleph_1} = \aleph_2$ we have stated that Theorem 1.7(a) solves Nunke's problem. It would perhaps be more accurate to say that the result shifts the problem from a question regarding the torsion product, to a question of computing invariants whose values, even in easy cases, depend upon undecidable statements from set theory. Still, it gives a way to analyse the problem by considering each group as a separate entity without any reference to their torsion product.

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