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A result on a singular Cauchy problem with a radial point revisited using microdifferential calculus

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Abstract – We study the ramified Cauchy problem for a linear PDE with a radial point using the theory of microdifferential operators.

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1. Introduction

The theory of microdifferential operators, developed since [11] and exposed in [12], contains a number of powerful analytic results stated in algebraic language and relevant to the study of the ramified Cauchy problem for linear PDEs in the complex domain. Results on existence of ramified analytic solutions of PDEs, due to Leray, Hamada, Wagschal, and their school, can be reproduced by algebraic techniques, e.g. [12, §III.2.2].

In the present paper we use the theory of microdifferential operators to prove the local existence of ramified analytic solutions for a typical equation (1) below with a radial, or degenerate, point. The solutions are representable as infinite series involving hypergeometric functions. This result is stated as Theorem 2.5 below.

Let $a_\nu(x)$ be analytic functions of $x \in \mathbb{C}^n$ and $\nu \in \mathbb{Z}^n$ be a multi-index. With a precise definition given in Section 3 below, for an expression of the form

$$\sum_{-\infty<|\nu|<m} a_\nu(x)\frac{\partial^{|\nu|}}{\partial x^\nu}$$

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to define a microdifferential operator it is required that $a_\nu(x)$ satisfy a factorial growth estimate $|a_\nu(x)| < C|-\nu|!$ for some contant C. This growth estimate allows one to define an action of microdifferential operators on holomorphic functions, [1, §2.1], and makes them relevant for the study of PDEs in complex analytic settings.

Very roughly, the algebraic approach to a ramified Cauchy problem uses the following tools. Given a differential operator P with analytic coefficients on a complex analytic manifold X, locally on T^*X one seeks to transform irreducible components of the characteristic variety $\operatorname{char} P$ to the normal form which is a conormal bundle to the hypersurface $\{x_1 = 0\} \subset \mathbb{C}^n$. On the level of microdifferential operators, such a purely geometric transformation gives rise to a quantized contact transformation, see [12, §I.5]. The difficult and general analytic result is that a quantized cntact transformation preserves the factorial growth conditions on the coefficients of a microdifferential operator. After the quantized contact transformation and some additional work, the differential equation $P\phi = 0$ transforms into the equation $x_1\tilde{\phi} = 0$, where $\tilde{\phi}$ belongs to an appropriate space of generalized functions formalized as a module over a ring of microdifferential operators, [12, §I.4]. The theory of quantized contact transformations allows one to transform a solution $\tilde{\phi}$ into a ramified analytic solution ϕ of the original equation.

An alternative to the reduction to the normal form is the use of sophisticated majorant series, e.g. [14].

However, the characteristic variety $\operatorname{char} P$ can have what is called degenerate, or radial, points, see Section 4.1 below. In a neighborhood of a radial point $\operatorname{char} P$ is not equivalent to the conormal bundle of $\{x_1 = 0\} \subset \mathbb{C}^n$, so different normal forms become relevant. In the smooth settings, PDEs with a radial points have been studied since Guillemin and Schäffer [4].

The starting point of our investigation is the paper by Urabe [13] where he studied the ramified complex analytic solutions of the following typical PDE with a radial point in two complex variables:

$$(1)\quad L\Phi(x,t) := (D_t^2 - (x+bt^2)D_x^2 - a(t,x)D_t - c(t,x)D_x - d(t,x))\Phi(x,t) = 0.$$

The plan of the present paper is as follows. In Section 2 we follow [13] to give the notation, assumptions, and state the Cauchy problem for the equation (1). To discuss (1), we work with the cotangent bundle $T^*\mathbb{C}^2$ whose standard coordinates are denoted by $(t, x; \tau, \xi)$. We treat the operator L from (1), and its main part P_c, see (3), as elements of the ring of microdifferential operators $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}$; the relevant definitions and results are given in Section 3. In the Section 4 we transform our equation to a more convenient normal form by means of a quantized contact transformation. In Section 5 we use the result of [9] to prove that operators P_c and L are conjugate inside $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}$, thus making precise the idea that L is a perturbation of P_c. In Section 6 we show that expansions related to the ansatz (8) form modules $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}U_\alpha^c$ and $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}V_\alpha^c$ over the ring $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}$ and are in fact quotient modules of $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}/\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}P_c$. This gives morphisms of $\mathcal{E}$-modules from $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}/\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}L$ to $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}U_\alpha^c$ and $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}V_\alpha^c$, i.e. a family of solutions of the PDE $L\varphi = 0$. In Section 7 we form a linear combination of these solutions to satisfy the initial conditions of the Cauchy problem. Our main result, Theorem 2.5 below, is a modification of [13, Theorem on p. 3]; the differences are discussed in Section 8.

The point of our paper is to show that algebraic techniques can give an alternative, more transparent treatment of this equation, by referring it to normal forms of [7] and [9]. We demonstrate that the division theorem in microdifferential calculus can replace some of the difficult majorant series estimates in [13] if one views expansions (8) as elements of an appropriate $\mathcal{E}$-module. While we are primarily interested in illustration and development of the techniques, we note that the radial points have been attracting attention in several scattering theory papers, e.g. [5]. Some algebraic discussion of this equation is present in [16].

2. Statement of the problem, notation, and the main result

We are solving the Cauchy problem for the PDE (1), where $a(t,x), c(t,x), d(t,x)$ are analytic functions on a neighborhood $(0,0) \in \Omega \subset \mathbb{C}^2$ and $b \in \mathbb{C}$ is a constant. We are interested in solutions in a class of ramified analytic functions. Ramifications are expected along the union of two characteristic curves (notice a prefactor $1/8$ corrected from Urabe's $1/4$)

$$K = \left\{x - \frac{1}{8}(1 \pm D)t^2 = 0\right\}, \tag{2}$$

where

$$D := \sqrt{1+16b}, \quad \operatorname{Re} D \geq 0 \text{ or } D \in i\mathbb{R}_+.$$

Assumption 2.1 ([13, Condition (c)]). $c(0,x) = c$ where c is a constant, i.e. $c(t,x) = c + t\tilde{c}(t,x)$.

In the precise sense described in Proposition 5.1, the operator L is a perturbation of

$$P_c = D_t^2 - (x + bt^2)D_x^2 - cD_x. \tag{3}$$

The Cauchy problem will be stated as follows:

$$Lu(t,x)=0 \quad \text{with initial data} \begin{cases} u(0,x)=x^\alpha f(x)/\Gamma(\alpha+1); \\ u_t(0,x)=x^\alpha g(x)/\Gamma(\alpha+1), \end{cases} \tag{4}$$

where $f(x), g(x)$ are holomorphic functions in $\Omega\cap\{t=0\}$.

Assumption 2.2. $\alpha\notin\mathbb{Z}$.

The following functions will be used as elementary building blocks for the solution of our differential equation

(5)

$$U^c_\alpha(t,x)=\frac{\left(x-\frac{1}{8}(1+D)t^2\right)^\alpha}{\Gamma(\alpha+1)} \times {}_2F_1\Bigl(-\alpha,\frac{1}{4}\Bigl(1+\frac{1}{D}\Bigr)+\frac{\alpha+c-1}{D};\frac{1}{2};-\frac{Dt^2}{4\left(x-\frac{1}{8}(1+D)t^2\right)}\Bigr);$$

$$V^c_\alpha(t,x)=\frac{t\left(x-\frac{1}{8}(1+D)t^2\right)^\alpha}{\Gamma(\alpha+1)} \times {}_2F_1\Bigl(-\alpha,\frac{3}{4}\Bigl(1+\frac{1}{D}\Bigr)+\frac{\alpha+c-1}{D};\frac{3}{2};-\frac{Dt^2}{4\left(x-\frac{1}{8}(1+D)t^2\right)}\Bigr),$$

where ${}_2F_1$ denotes the Gauss hypergeometric functions.

Their choice is motivated by the following properties:

$$\begin{aligned} &P_cU^c_\alpha=0, \quad U^c_\alpha(0,x)=x^\alpha/\Gamma(\alpha+1), \quad D_tU^c_\alpha(0,x)=0; \\ &P_cV^c_\alpha=0, \quad V^c_\alpha(0,x)=0, \quad D_tV^c_\alpha(0,x)=x^\alpha/\Gamma(\alpha+1). \end{aligned} \tag{6}$$

Our expressions for U^c_α, V^c_α already contain the correction entailed by (2) and were double-checked numerically with Wolfram Mathematica. Refer to [4, §6] for a systematic way of obtaining such hypergeometric solutions.

The following estimate follows from the proof of [13, Proposition (E.U), p. 23], especially the bottom of [13, p. 22], and the Cauchy integral formula, with a slight change in the notation ω_λ.

Proposition 2.3 (Urabe). *Assume* [13, Condition (α,b,c)]:

$$\alpha+\frac{1}{4}\pm\frac{1}{D}\Bigl(\alpha-\frac{3}{4}+c\Bigr)\notin\mathbb{Z} \quad \textit{and} \quad \alpha+\frac{3}{4}\pm\frac{1}{D}\Bigl(\alpha-\frac{1}{4}+c\Bigr)\notin\mathbb{Z}. \tag{7}$$

Let

$$\omega_\lambda = \left\{(t,x) : x \neq \frac{1}{8}(1 \pm D)t^2, |x| < \lambda, |t^2| < \lambda\right\} \subset \Omega,$$

and $\tilde{\omega}_\lambda$ be its universal covering space. There exist positive constants T_1, T_2, $B(\alpha, c)$, such that for any compact subset $\widetilde{K} \subset \tilde{\omega}_\lambda$ that wraps around the characteristic curves $\{x = \frac{1}{8}(1 \pm D)t^2\}$ at most β times,

$$\begin{aligned}|U^{c-k+1}_{\alpha+r-1}|,\ |\partial_t U^{c-k+1}_{\alpha+r-1}|,\ |V^{c-k+1}_{\alpha+r-1}|, |\partial_t V^{c-k+1}_{\alpha+r-1}|& \\ \leq B(\alpha, c)^{\beta(r-k)} T_1^r T_2^k \frac{1}{r!}\lambda^r \quad &\text{on } \widetilde{K}.\end{aligned}$$

We make the following assumption which will play the role in Proposition 5.1:

Assumption 2.4. Assume $b \neq 0$ is such that at least one of the numbers $(1 \pm \sqrt{1+16b})/(2b)$ falls outside of $(-\infty, -\frac{1}{2}] \cup \{-\frac{1}{4}\} \cup [0, +\infty)$.

The main result of the present paper is:

Theorem 2.5. *Under assumptions 2.1, 2.2, 2.4, and equation (7), for every $\beta \in \mathbb{N}_0$, there is a sufficiently small neighborhood ω (depending on β) of the origin in $\mathbb{C}^2$ where the Cauchy problem (4) has a holomorphic solution defined on any compact subset of $\omega - K$ (see (2)) that wraps around K at most β times. The solutions is expressed by*

$$\begin{aligned}(8)\quad u(t,x) = \sum_{r=0}^{\infty}\sum_{k=0}^{r}\{&u_{r,k}(t,x)U^{c-k+1}_{\alpha+r-1}(t,x) + g_{r,k}(t,x)D_t U^{c-k+1}_{\alpha+r-1}(t,x) \\ &+ v_{r,k}(t,x)V^{c-k+1}_{\alpha+r-1}(t,x) + h_{r,k}(t,x)D_t V^{c-k+1}_{\alpha+r-1}(t,x)\},\end{aligned}$$

where $u_{r,k}$, $g_{r,k}$, $v_{r,k}$, and $h_{r,k}$ are holomorphic functions on ω and U^c_α, V^c_α were defined in (5).

Moreover, there are constants a_r, b_r and functions $u_\ell(t,x)$, $g_\ell(t,x)$, $v_\ell(t,x)$, $h_\ell(t,x)$ such that $u_{r,k} = a_r u_{r-k}$, $g_{r,k} = a_r g_{r-k}$, $v_{r,k} = b_r v_{r-k}$, $h_{r,k} = b_r h_{r-k}$.

3. Preliminaries from the microdifferential calculus

Recall from [12, §I.1] that a microdifferential operator (MDO) of order m on an open subset $U \subset T^*\mathbb{C}^n$ is defined by a formal series $P = \sum_{-\infty<j\leq m} p_j$, where p_j is a section of $\mathcal{O}_{T^*X}(j)$ on U, with an additional requirement that for any compact set $K \subset U$ there is an $\varepsilon > 0$ such that $\sum_{j\geq 0} |p_{-j}|_K \varepsilon^j/j! < \infty$, where $|p_j|_K = \sup_K |p_j|$. The spaces of such operators are denoted $\mathcal{E}_{\mathbb{C}^n}(U)(m)$, they form a sheaf $\mathcal{E}_{\mathbb{C}^n}(m)$, and $\mathcal{E}_{\mathbb{C}^n} := \bigcup_{m\in\mathbb{Z}} \mathcal{E}_{\mathbb{C}^n}(m)$. The noncommutative product is defined by [12, (I.1.2.4)].

The spaces of MDOs are topologized by means of Boutet de Monvel-Krée quasi-norms

$$N_m(P,K,t) = \sum_{\substack{k\geq 0\\ \alpha,\beta\in\mathbb{N}_0^n}} \frac{2(2n)^{-k}k!}{(|\alpha|+k)!(|\beta|+k)!}|D_x^\alpha D_\xi^\beta p_{m-k}|_K t^{2k+|\alpha|+|\beta|}. \tag{9}$$

A formal sum $P = \sum_{-\infty<j\leq m} p_j$ belongs to $\mathcal{E}_{\mathbb{C}^n}(U)(m)$ iff for any compact set K in U there exists $t>0$ such that $N_m(P,K,t)<\infty$.

Generalizing [12, Definition I.2.1.1], for a compact set $K \subset T^*\mathbb{C}^n$ denote by $X_m(K,t)$ the vector space of formal sums $P = \sum_{-\infty<j\leq m} p_j$ such that $N_m(P,K,t)$ is finite, and endow this space with the norm $N_m(\cdot,K,t)$.

The rest of this Section 3 consists of a few preparatory facts about MDOs for which we do not know a reference.

3.1 – *Characterization of* $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)}$.

Consider $\mathbb{C}^2$ with coordinates (x_1,x_2) and $T^*\mathbb{C}_2$ with symplectic coordinates $(x_1,x_2;\xi_1,\xi_2)$. Let $\mathbf{x}=(0,0;0,1)$.

Given a neighborhood $\mathbf{x}\in U\subset T^*\mathbb{C}^2$ and a formal sum $P = \sum_{-\infty<j\leq m} p_j$, $p_j \in \mathcal{O}_{T^*X}(U)(j)$, by [12, Remark I.1.2.2], P can also be written as a formal sum

$$\sum_{\substack{-m\leq j<\infty,\\ n\geq 0}} a_{n,j}(x_1,x_2)(\xi_1/\xi_2)^n\xi_2^{-j},$$

where $a_{n,j}$ are analytic functions near the origin in $\mathbb{C}^2$, such that all the series $\varphi_j(x_1,x_2,s) := \sum_{j\geq 0} a_{n,j}(x_1,x_2)s^n$ converge in a neighborhood of origin in $\mathbb{C}^3$ independent of j.

Proposition 3.1. *A formal sum* $\sum_{j,n} a(x_1,x_2)(\xi_1/\xi_2)^n\xi_2^{-j}$ *defines an element of* $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ *iff there is a neighborhood* U *of the origin in* $\mathbb{C}^2$ *and constants* C,θ,ρ *such that*

$$|a_{n,j}(x_1,x_2)| < C\theta^n\rho^j j!. \tag{10}$$

Proof. Suppose $\sum_{j,n} a(x_1,x_2)(\xi_1/\xi_2)^n\xi_2^{-j}$ defines an MDO in $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$. Then there is an $\varepsilon>0$ and a compact neighborhood K of $\mathbf{x}$ such that

$$\sum_j \Big|\sum_n a_{n,j}(x_1,x_2)(\xi_1/\xi_2)^n\xi_2^{-j}\Big|_K \varepsilon^j/j! < \infty; \tag{11}$$

in particular, the summands in (11) are bounded by a constant C, hence, with $\rho = 1/\varepsilon$,

$$\left| \sum_n a_{n,j}(x_1, x_2)(\xi_1/\xi_2)^n \xi_2^{-j} \right|_K < Cj!\rho^j. \tag{12}$$

Shrink K so that it has the form, for some $\eta > 0$,

$$K = \{|\xi_1/\xi_2| \le \eta;\ |\xi_2 - 1| \le \eta;\ |x_1| \le \eta;\ |x_2| \le \eta\}.$$

For such a K, (12) becomes

$$\left| \sum_n a_{n,j}(x_1, x_2)(\xi_1/\xi_2)^n \right|_K (1-\eta)^{-j} < Cj!\rho^j, \tag{13}$$

and the extra factor on the left can be dropped by weakening the inequality

$$\left| \sum_n a_{n,j}(x_1, x_2)(\xi_1/\xi_2)^n \right|_K < Cj!\rho^j.$$

Denote $(\xi_1/\xi_2) = s$ and get

$$\max_{|t|\le\eta} \left| \sum_n a_{n,j}(x_1, x_2)s^n \right| < Cj!\rho^j, \quad \text{for all } (x_1, x_2) \text{ such that } |x_1, x_2| < \eta.$$

By Cauchy's inequality this implies

$$|a_{n,j}(x_1, x_2)| < C\eta^{-n}\rho^j j!, \quad \text{for all } (x_1, x_2) \text{ such that } |x_1, x_2| < \eta;$$

putting $\theta = 1/\eta$, we get (10).

The proof of the converse is similar and left to the reader. □

3.2 – *Approximation of an MDO by finite sums*

We continue to use the notation of the previous subsection.

THEOREM 3.2. *Let* $P = \sum_{n\ge 0; j\ge -m} a_{n,j}(x_1, x_2)(D_{x_1}/D_{x_2})^m D_{x_2}^{-j}$ *be an MDO in a neighborhood of* $\mathbf{x}$. *Order the pairs* (j, n) *as* (j_k, n_k), $k \in \mathbb{N}$, *and define* $P_N = \sum_{k\ge N} a_{n_k, j_k}(x_1, x_2)(D_{x_1}/D_{x_2})^{n_k} D_x^{-j_k}$. *Then there is a* $t > 0$ *and a compact neighborhood* K' *of* $\mathbf{x}$, *both depending on* P, *such that*

$$N_m(P_N; K', t) \longrightarrow 0 \quad \textit{as } N \to \infty.$$

Proof. Let $P = \sum_{j\geq -m} p_{-j}$ such that for some $\varepsilon > 0$ and some compact neighborhood K of $\mathbf{x}$

$$\sum_j \left|p_{-j}\right|_K \varepsilon^j / j! < \infty.$$

Then P defines an MDO on any open set contained in K.

Suppose further that $p_{-j} = \sum_{n\geq 0} p_{-j,n}$ and each of these sums converges uniformly and absolutely on K. Then the double sum $\sum_{j,n} p_{-j,n}\varepsilon^j/j!$ converges uniformly and absolutely on K (in particular, the sum is independent of the order of summation, i.e. of the linear ordering (j_k, n_k) of the pairs of indices). In the situation of the theorem $p_{-j,n} = a_{n,j}(x_1, x_2)(\xi_1/\xi_2)^n \xi_2^{-j}$.

Denote by P_N the MDO with the total symbol $\sum_{k=N}^{\infty} p_{-j_k,n_k}$ defined on any open subset of K. We are going to show that there exists a compact neighborhood K' of $\mathbf{x}$ such that $N_m(P_N, K', t) \to 0$ for sufficiently small ts.

Denote by Q any one of the operators P_N and let $\sum_{j\geq -m} q_{-j}$ be its total symbol. Imitating the proof of [3, Chapter 4, Lemma 3.2], let us take a polydisc Δ of radius δ centered at $\mathbf{x}$ such that $2\Delta \subset K$. It follows that $|q_{m-j}|_{2\Delta} \leq A_1 j! C_1^j$ for some constants A_1, C_1. Cauchy's inequality gives $|\partial_x^\alpha \partial_\xi^\beta q_{m-j}|_\Delta \leq (\alpha!)(\beta!)\delta^{-|\alpha|-|\beta|}|q_{m-j}|_{2\Delta}$ and hence

$$N_m(Q, \Delta, t) \leq \sum_{k,\alpha,\beta} \frac{2\cdot 4^{-k}(k!)(\alpha!)(\beta!)}{(k+|\alpha|)!(k+|\beta|)!} k! \frac{|q_{m-k}|_K}{k!} \delta^{-|\alpha|-|\beta|} t^{2k+|\alpha|+|\beta|}.$$

Using $(\alpha!)k! \leq (k+|\alpha|)!$ and $(\beta!)k! \leq (k+|\beta|!)$, we have

$$N_m(Q, \Delta, t) \leq \sum_{k,\alpha,\beta} 2\cdot 4^{-k} \frac{|q_{m-k}|_K}{k!} \delta^{-|\alpha|-|\beta|} t^{2k+|\alpha|+|\beta|};$$

assuming $t < 1/(2\delta)$ and summing over α and β, we obtain

$$N_m(Q, \Delta, t) \leq 8 \sum_k 4^{-k} \frac{|p_{-k}|_K}{k!} t^{2k}.$$

Assuming $t < \sqrt{4\varepsilon}$ we have

$$N_m(Q; \Delta, t) \leq 8 \sum_k \frac{|q_{m-k}|_K}{k!} \varepsilon^k. \tag{14}$$

From this it is clear that $N_m(P_N, \Delta, t) \to 0$ as $N \to \infty$, and the theorem follows with $K' = \Delta$. $\square$

3.3 – *Division theorem with a continuity supplement*

The proof of [12, Theorem I.2.2.1] implies, in fact, the following more detailed statement. We highlight the part not included in the book's formulation:

THEOREM 3.3 (Späth). *Let P be an MDO defined in a neighborhood of $(x^0, \xi^0) \in T^*X$. Assume that $\frac{\partial^j}{\partial \xi_1^j}(\sigma(P))$ is zero at (x_0, ξ_0) for $0 \le j < p$ and different from zero for $j = p$. Then for any section Q of $\mathcal{E}_X$* ***of order m*** *defined in a neighborhood of (x^0, ξ^0) there exist unique S and R such that*

$$Q = SP + R, \quad \operatorname{ord} R \le \operatorname{ord} Q, \quad \operatorname{ad}_{x_1}^{p}(R) = 0.$$

Moreover, there is a compact neighborhood K of (x_0, ξ_0) and a positive t such that $Q \mapsto R$ defines a continuous map $X_m(K, t) \to X_m(K, t)$.

NOTATION 3.4. By $Q \mod \cdot P$ we denote the operator R from the above theorem.

4. A quantized contact transformation relating $P_c D_{x_2}^{-1}$ and $\beta_1 y_1 D_{y_1} + \beta_2 y_2 D_{y_2} + \beta_3$

4.1 – *Classification of radial points*

We begin by giving some mathematical context for the operator P_c.

In at least three sources, [4], [9], [7], plus [8], the following question is discussed. Suppose given a linear (micro-, pseudo-) differential operator with homogeneous involutive characteristic variety $V \subset T^*\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), let $\alpha = \sum p_i dx_i$ be the canonical 1-form on $T^*\mathbb{K}^n$. Consider those smooth points $\mathbf{x} \in V$ outside of the zero section, called *degenerate* or *radial* points, where $\alpha|V$ vanishes. These points are interesting because in their neighborhood the operator P is not microlocally equivalent to $\partial/\partial x_1$, i.e. results such as [12, corollaries A.4.5 and I.6.2.3] do not extend to the case of radial points. So what *are* the microlocal normal forms for V and P near radial point?

A much-simplified result from [7], [8], and [9] says:

THEOREM 4.1 (Lychagin and Oshima). *If $\lambda_1, \dots, \lambda_{n-1}; \mu_1, \dots, \mu_{n-1}$ are complex numbers such that the sets $\{\lambda_1, \dots, \lambda_{n-1}\} \ne \{\mu_1, \dots, \mu_{n-1}\}$, then the equations*

$$\sum_{i=1}^{n-1} \lambda_i x_i \xi_i + x_n \xi_n = 0 \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i x_i \xi_i + x_n \xi_n = 0 \tag{15}$$

define non-equivalent germs at the point $\mathbf{x} = (0, \dots 0; 0, .., 0, 1)$ of homogeneous involutive submanifolds in $T^*\mathbb{C}^n$ for which $\mathbf{x}$ is a radial point.

Equivalence is understood with respect to the group of germs of homogeneous symplectic biholomorphisms preserving $\mathbf{x}$.

It follows from the theorem that the corresponding microdifferential operators

$$\sum_{i=1}^{n-1} \lambda_i x_i D_{x_i} + x_n D_{x_n}, \sum_{i=1}^{n-1} \mu_i x_i D_{x_i} + x_n D_{x_n} \in \mathcal{E}_{\mathbb{C}^n,\mathbf{x}}$$

are pairwise microlocally inequivalent.

However, not every codimension one involutive submanifold with a radial point is equivalent to (15). In fact, the classification of radial points is parallel and similar in difficulty to the Birkhoff normal form, compare [8, especially §§1.5–1.8] and [15, §19]. The result of Theorem 4.1 comes from linearized theory of Hamiltonians near a stationary point; the possibility of resonances ($\mathbb{Z}$-linear relations among eigenvalues of the linear part of the Hamiltonian) makes an exhaustive list of normal forms unfeasible in either case.

A different system of normal forms of an operator whose characteristic manifold has a radial point at $(t, x; \tau, \xi) = (0, 0; 0, 1)$ is presented in [4, (1.3) and (1.7)], and our P_c is one of these normal forms. The content of the next subsection 4.2 is to explicitly transform the operator P_c, which is in the normal form from the point of view of [4], into another operator whose principal symbol is in the normal form from the point of view of [7] and [9].

4.2 – *Construction of the transformation*

A quantized contact transformation [12, §I.5] is an isomorphism of $\mathcal{E}_X(U)$ onto $\mathcal{E}_Y(V)$, where U, V are some open sets in T^*X, T^*Y, respectively, where X, Y are complex manifolds. Such a transformation is often found staring from a homogeneous contact transformation between U and V, hence the name.

In this subsection we will re-denote our coordinates $(t, x; \tau, \xi)$ as $(x_1, x_2; \xi_1, \xi_2)$. Then

$$P_c = D_{x_1}^2 - (x_2 + b x_1^2) D_{x_2}^2 - c D_{x_2}.$$

As before $\mathbf{x} = (0, 0; 0, 1)$.

Let us perform a transformation of the ring $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$:

$$\begin{aligned} x_1 &= z_1 - C D_{z_1} D_{z_2}^{-1}, \quad & D_{x_1} &= D_{z_1}; \\ x_2 &= z_2 + \tfrac{C}{2} D_{z_1}^2 D_{z_2}^{-2}, \quad & D_{x_2} &= D_{z_2}, \end{aligned} \tag{16}$$

where C is a constant.

We see, using [12, (I.1.1.9)], that the commutation relations among operators are respected. The analytic part of the proof of the fact that (16) defines an automorphism of $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ goes through [12, §I.5] applied to the ideal

$$\mathcal{I} := \mathcal{E}_{\mathbb{C}^2\times\mathbb{C}^2}\Big(x_1 - z_1 + C\zeta_1/\zeta_2,\ x_2 - z_2 - \frac{C}{2}\zeta_1^2/\zeta_2^2,\ \xi_1 + \zeta_1,\ \xi_2 + \zeta_2\Big). \tag{17}$$

Formulas [12, (I.5.1.2)] are obvious directly and [Sch,(I.5.1.3)] follows by uniqueness in the division theorem, so we do not need to worry about calculating the symbol ideal of $\mathcal{I}$. Notice the opposite sign in front of ζ's in (17) compared to (16) taken to counteract the order reversal in [12, (I.5.1.4)].

We will omit the notation for the automorphism and will simply write P_c in the new coordinates:

$$\begin{aligned} P_c &= D_{z_1}^2 - \Big\{z_2 + \frac{C}{2}D_{z_1}^2 D_{z_2}^{-2} + b(z_1 - CD_{z_1}D_{z_2}^{-1})^2\Big\}D_{z_2}^2 - cD_{z_2} \\ &= \Big(1 - \frac{C}{2} - bC^2\Big)D_{z_1}^2 - z_2 D_{z_2}^2 - bz_1^2 D_{z_2}^2 + 2Cz_1 D_{z_1}D_{z_2} - cD_{z_2}. \end{aligned}$$

Setting to zero the coefficient in front of $D_{z_1}^2$, we $C = (-1 \pm \sqrt{1+16b})/(4b)$, the sign chosen in such a way that $-2C$ satisfies Assumption 2.4.

We now perform one more change of variables defining an automorphism of $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$; this one is simply induced by a biholomorphism of the (y_1, y_2)-plane:

$$\begin{aligned} z_1 &= y_1, & D_{z_1} &= D_{y_1} + Ky_1 D_{y_2}, \\ z_2 &= y_2 - \frac{K}{2}y_1^2, & D_{z_2} &= D_{y_2}. \end{aligned}$$

Then

$$\begin{aligned} P_c &= -\Big(y_2 - \frac{K}{2}y_1^2\Big)D_{y_2}^2 - by_1^2 D_{y_2}^2 + 2Cy_1(D_{y_1} + Ky_1 D_{y_2})D_{y_2} - cD_{y_2} \\ &= \Big(\frac{K}{2} - b + 2CK\Big)y_1^2 D_{y_2}^2 - y_2 D_{y_2}^2 + 2Cy_1 D_{y_1}D_{y_2} - cD_{y_2} \end{aligned}$$

and equate to zero the coefficient of $y_1^2 D_{y_2}^2$. $K = b/(\frac{1}{2} + 2C)$, the denominator equals $\frac{1}{2} + \frac{-1\pm\sqrt{1+16b}}{2b}$; the assumption 2.4 contains the condition $C \neq -1/4$ in order to avoid division by zero at this step.

In summary, we have found a quantized contact transformation transforming P_c into $-y_2 D_{y_2}^2 + 2Cy_1 D_{y_1}D_{y_2} - cD_{y_2}$, or, $P_c D_x^{-1}$ into $-y_2 D_{y_2} + 2Cy_1 D_{y_1} - c$.

5. Operators L and P_c are conjugate

Proposition 5.1. *Under Assumption 2.4, there is an operator* $\mathcal{U} \in \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ *of order* 0*, with principal symbol invertible near* $\mathbf{x}$*, such that*

$$\mathcal{U}LD_x^{-1} = P_c D_x^{-1}\mathcal{U}.$$

Proof. From Assumption 2.1, we see that $(L - P_c)D_x^{-1}$ is an operator of order 0, with principal symbol vanishing at $\mathbf{x}$. Therefore also in terms of $y_1, y_2, D_{y_1}, D_{y_2}$ from Section 4.2, the operator L can be written as

$$L = -y_2 D_{y_2}^2 + 2Cy_1 D_{y_1} D_{y_2} - cD_{y_2} - AD_{y_2},$$

where $A \in \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ of order 0, with principal symbol vanishing at $\mathbf{x}$.

The proposition therefore reduces to finding $\mathcal{U}$ conjugating

$$y_2 D_{y_2} - 2Cy_1 D_{y_1} + c + A$$

into $y_2 D_{y_2} - 2Cy_1 D_{y_1} + c$, and this is accomplished the result below. □

From [9, Theorem 3.2], we extract the following particular case.

Proposition 5.2. (Oshima) *Suppose that*

$$Q(y_1, y_2, D_{y_1}, D_{y_2}) = \beta y_1 D_{y_1} + y_2 D_{y_2} + c$$

and

$$P(y_1, y_2, D_{y_1}, D_{y_2}) = \beta y_1 D_{y_1} + y_2 D_{y_2} + c + A$$

belong to $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ *where* A *is an MDO of order* ≤ 0 *with* $\sigma_0(A)$ *vanishing at* $\mathbf{x}$*. Assume moreover*

$$0 \notin \textit{convex hull}\{1, \beta, 1 - \beta\}. \tag{18}$$

Then near $\mathbf{x}$ *there is an invertible MDO* $\mathcal{U}$ *of order* 0*, with principal symbol invertible near* $\mathbf{x}$*, such that* $P\mathcal{U} = Q\mathcal{U}$.

A simple drawing shows that (18) is equivalent to

$$\beta \notin (-\infty, 0] \cup [1, \infty) \subset \mathbb{R} \tag{19}$$

which follows from Assumption 2.4.

REMARK 5.3. In interpreting [9, Theorem 3.2] we prioritize conditions which seem to be actually used in its proof on [9, p. 77] over the literal meaning of the assumption [9, A.3.4]; [9, A.3.4] as stated would be trivially true for any β. Namely, let (in our situation and notation) $u_{-k}(y, \eta)$ be the $(-k)$-th homogeneous component of the total symbol of $\mathcal{U}$ and let $\mathcal{U}_{-k}$ be the MDO with total symbol u_{-k}. Then $[\mathcal{U}_{-k}, Q_1]$ from [9, (3.7)] is $[\mathcal{U}_{-k}, \beta y_1 D_{y_1} + y_2 D_{y_2}]$ and has the symbol

$$-\beta y_1 \frac{\partial u_{-k}}{\partial y_1} - y_2 \frac{\partial u_{-k}}{\partial y_2} + \beta \frac{\partial u_{-k}}{\partial \eta_1} + \eta_2 \frac{\partial u_{-k}}{\partial \eta_2}. \tag{20}$$

As u_{-k} has to be homogeneous with respect to η's of degree $-k$, we can (cf. [6, (3.8)]) rewrite (20) as

$$-\beta y_1 \frac{\partial u_{-k}}{\partial y_1} - y_2 \frac{\partial u_{-k}}{\partial y_2} + (\beta - 1)\eta_1 \frac{\partial u_{-k}}{\partial \eta_1} - k u_{-k}. \tag{21}$$

Inserting this into [9, (3.7)], we impose (18) in order to satisfy [9, A.1.3 and A.1.4] and apply [9, Theorem 1.1].

REMARK 5.4. The purpose of assumption (18) and ultimately of Assumption 2.4 is, as explained in the previous remark, to avoid too close approximations of β and $1 - \beta$ by negative rationals. The condition (18) is most likely not the minimal assumption, but the need of some kind of Diophantine property can be traced back to counterexamples [10, Examples 8 and 9, p. 87]. The paper [2] works with similar Poincaré conditions. On the other hand, we admit that we do not fully understand the discussion of Poincaré conditions in [13, p. 8]; we prefer however to postpone the discussion of this issue until our more basic concerns posed in Section 8 are resolved.

Let us rewrite the result of the Proposition 5.1 in terms of an isomorphism of $\mathcal{E}$-modules. We have $L = \mathcal{U}^{-1} P_c D_x^{-1} \mathcal{U} D_x$ and notice that $\mathcal{U}^{-1} P_c \mathcal{D}_x$ is an MDO of order 0. Hence we can construct an (iso)morphism of $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ modules

$$\mathbf{i}\colon \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} L \longrightarrow \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} P_c, \quad [1] \longmapsto [D_x^{-1} \mathcal{U}^{-1} D_x]. \tag{22}$$

6. Modules $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)} U_\alpha^c$, $\mathcal{E}_{\mathbb{C}^2,(0,0;0,1)} V_\alpha^c$

Let K be the ramification locus (2), let ω_δ, $\tilde{\omega}_\delta$ be as in Proposition 2.3. Define $\mathcal{M}$ to be the set of ramified analytic functions on some $\tilde{\omega}_\delta$ representable as infinite sums, absolutely and uniformly convergent on compact sets – a.u.c.c.s. for short:

$$\sum_{j \geq -N} (u_j(x,t) U_{\alpha+j}^{c-j} + g_j(t,x) D_t U_{\alpha+j}^{c-j}), \tag{23}$$

with $N \in \mathbb{Z}$ and satisfying the condition

> there exist $\eta > 0$ and $\delta > 0$ such that
> $|u_j(x,t)|, |g_j(x,t)| < \eta^j j!$ on ω_δ for all $j > 0$.

This is of course equivalent to

> there exist $\eta > 0$ and $\delta > 0$ such that
> $|u_j(x,t)|, |g_j(x,t)| < \eta^{j+N}(j+N)!$ on ω_δ for all $j > -N$.

By Cauchy's integral inequality on some, possibly smaller, ω_δ, there exists $\theta > 0$ such that

$$u_j(x,t) = \sum_{k \geq 0} u_{j,k}(t) x^k, \quad |u_{j,k}(t)| \leq \eta^{j+N} \theta^k (j+N)! \tag{24a}$$

and

$$g_j(x,t) = \sum_{k \geq 0} g_{j,k}(t) x^k, \quad |g_{j,k}(t)| \leq \eta^{j+N} \theta^k (j+N)!. \tag{24b}$$

By Proposition 2.3, any sum (23) represents a ramified analytic function on some $\tilde{\omega}_\delta$.

Clearly, $\mathcal{M}$ is closed under multiplication by analytic functions of (x,t). We will now define action of D_x^{-1}, D_x, and D_t on $\mathcal{M}$ that will make $\mathcal{M}$ into a $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$-module. The action will be denoted by $\circ$ to distinguish it, at this stage, from usual operations on functions and from the product in $\mathcal{E}$.

Every element $\varphi \in \mathcal{M}$ has the following property: for every $n \in \mathbb{N}_0$, $(\partial^n \varphi / \partial t^n)|_{t=0}$ is of the form $x^\alpha f_n(x)$, where f_n is meromorphic at the origin. The only function independent of x, ramified only along K with this property is the zero function. Hence, any function of the form (23) has a most one antiderivative with respect to x of the form (23). Let us explicitly construct such an antiderivative.

Let $(z)_{\underline{k}}$ denote the descending factorial, $(z)_{\underline{k}} = z(z-1)\dots(z-k+1)$.

We begin by defining

$$D_x^{-1} \circ x^m U_{\alpha+j}^{c-j} := \sum_{k=0}^{m} (-1)^k (m)_{\underline{k}} x^{m-k} U_{\alpha+j+(1+k)}^{c-j-1-k} \tag{25}$$

which is inspired by [12, equation (I.1.1.9)] and [13, Proposition 2.(3)]. One checks by hand that the RHS is indeed an antiderivative of $x^m U_{\alpha+j}^{c-j}$.

We can define the following as an a.u.c.c.s. series on a neighborhood of the origin; convergence will be shown a few lines below:

$$D_x^{-1} \circ \Big\{ \sum_{\substack{j \geq -N \\ m \geq 0}} u_{jm}(t) x^m U_{\alpha+j}^{c-j} \Big\}$$

$$:= \sum_{\substack{j \geq -N \\ m \geq 0}} \sum_{k=0}^{m} (-1)^k u_{jm}(t) (m)_{\underline{k}} x^{m-k} U_{\alpha+j+(1+k)}^{c-j-1-k}$$

(set $m = \ell + k$)

$$= \sum_{\substack{j \geq -N \\ k,\ell \geq 0}} (-1)^k u_{j,k+\ell}(t)(k+\ell)_{\underline{k}} x^{\ell} U_{\alpha+j+(1+k)}^{c-j-1-k}$$

(set $j + k = s - N$)

$$= \sum_{s,\ell \geq 0} \sum_{k=0}^{s} (-1)^k u_{s-k-N,k+\ell}(t)(k+\ell)_{\underline{k}} x^{\ell} U_{\alpha+s-N+1}^{c-s+N-1}.$$

Let us now estimate the coefficient of $x^{\ell} U_{\alpha+s-N+1}^{c-s+N-1}$:

$$\Big| \sum_{k=0}^{s} (-1)^k u_{s-k-N,k+\ell}(t)(k+\ell)_{\underline{k}} \Big|$$

$$\leq \sum_{k=0}^{s} |u_{s-k-N,k+\ell}(t)|(k+\ell)_{\underline{k}}$$

(by (24))

$$\leq \sum_{k=0}^{s} \eta^{s-k}(s-k)!\theta^{k+\ell}(k+\ell)_{\underline{k}}$$

$$= \sum_{k=0}^{s} \theta^{k+\ell}\eta^{s-k}(s-k)!(k+\ell)!/\ell!$$

$$\leq \sum_{k=0}^{s} \eta^{s-k}(s-k)!(2\theta)^{k+\ell}k!$$

$$\leq s!(2\theta)^{\ell}\eta^{s} \sum_{k=0}^{s} \eta^{-k}(2\theta)^{k}$$

$$\leq s!(2\theta)^{\ell}(\eta+2\theta)^{s}, \tag{26}$$

hence the required convergence statement.

Analogously, define

$$D_x^{-1}\circ\Big\{\sum_{\substack{j\geq -N\\ m\geq 0}} g_{jm}(t)x^m D_t U_{\alpha+j}^{c-j}\Big\}$$
$$:=\sum_{\substack{j\geq -N\\ m\geq 0}}\sum_{k=0}^{m}(-1)^k g_{jm}(t)(m)_{\underline{k}}x^{m-k}D_t U_{\alpha+j+(1+k)}^{c-j-1-k}.$$

As differentiation commutes with u.a.c.c.s. series of analytic function, termwise differentiation shows that $D_x^{-1}\circ\varphi$ is indeed an antiderivative of $\varphi\in\mathcal{M}$.

Define further

$$\begin{aligned} D_x\circ u_j(x,t)U_{\alpha+j}^{c-j} &= \frac{\partial u_j}{\partial x}U_{\alpha+j}^{c-j}+u_jU_{\alpha+j-1}^{c-j+1},\\ D_x\circ g_j(x,t)\partial_t U_{\alpha+j}^{c-j} &= \frac{\partial g_j}{\partial x}\partial_t U_{\alpha+j}^{c-j}+g_j\partial_t U_{\alpha+j-1}^{c-j+1}\end{aligned}\tag{27}$$

(inspired by the Leibniz rule and [13, Proposition 2.(3)]) and

(28)
$$\begin{aligned} D_t\circ a(x,t)U_{\alpha+j}^{c-j} &= \frac{\partial a(x,t)}{\partial t}U_{\alpha+j}^{c-j}+a(x,t)\partial_t U_{\alpha+j}^{c-j},\\ D_t\circ a(x,t)\partial_t U_{\alpha+j}^{c-j} &= \frac{\partial a(x,t)}{\partial t}\partial_t U_{\alpha+j}^{c-j}\\ &\quad + a(x,t)[(x+bt^2)\partial_x^2 U_{\alpha+j}^{c-j}+(c-j)\partial_x U_{\alpha+j}^{c-j}]\\ &= \frac{\partial a(x,t)}{\partial t}\partial_t U_{\alpha+j}^{c-j}\\ &\quad + a(x,t)(x+bt^2)U_{\alpha+j-2}^{c-j+2}+a(x,t)(c-j)U_{\alpha+j-1}^{c-j+1}\end{aligned}$$

(inspired by the Leibniz rule and the property $P_{c-j}U_{\alpha+j}^{c-j}=0$).

Notice that estimates in (26) are the same as would be used in a low-tech by-hand proof of the fact that $P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}\implies D_x^{-1}\circ P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$. Similarly but easier, imitating the proofs of $P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}\implies D_x\circ P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ $P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}\implies D_t\circ P\in\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$, one proves that the actions of D_x and D_t extend to the whole $\mathcal{M}$ and are in fact the usual differentiations of functions with respect to x and t, respectively.

With the actions of D_x^{-1}, D_x, and D_t thus unambiguously defined on $\mathcal{M}$, we can make the following observation.

For any $\varphi \in \mathcal{M}$, there is a MDO $P \in \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$, with $\deg_{D_t} P \leq 1$, such that $\varphi = P \circ U^c_\alpha$ (in the sense that if $P = \sum_{-\infty<j\leq N} P_j$, P_j is the homogeneous part of order j, then $\varphi = \sum_{-\infty<j\leq N} P_j U^c_\alpha$ and the sum is a.u.c.c.s.

To recapitulate, at this point $P \circ \varphi$ has been defined a) for $P = a(x,t)$ or $P = D_x^{-1}$ or $P = D_x$ or $P = D_t$ and any $\varphi \in \mathcal{M}$, as well as b) for any $P \in \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$, with $\deg_{D_t} P \leq 1$ and $\varphi = U^c_\alpha$.

Further, from the formulas (25), (27), (28), and using notation 3.4 we see that

$$D_x^{-1} \circ (PU^c_\alpha) = (D_x^{-1} P)U^c_\alpha;\quad D_x \circ (PU^c_\alpha) = (D_x P)U^c_\alpha;$$

$$D_t \circ (PU^c_\alpha) = (D_t P \mod \cdot (D_t^2 - (x+bt^2)D_x^2 - cD_x))U^c_\alpha,$$

where on the left hand side we have an action of an operator on $\mathcal{M}$, and on the right we have a composition of operators in $\mathcal{E}$.

The issue that we have to deal with is that a given analytic function φ might have a non-unique representation as PU^c_α.

Definition-Proposition 6.1. *$\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ acts on $\mathcal{M}$ as follows*:

$$Q \cdot PU^c_\alpha := (QP \mod \cdot (D_t^2 - (x+bt^2)D_x^2 - cD_x))U^c_\alpha$$

where $\deg_{D_t} P \leq 1$.

We stress that quite a non-trivial analytic statement, following from the division theorem, has just been used: namely, that

$$QP \mod \cdot (D_t^2 - (x+bt^2)D_x^2 - cD_x)$$

is an MDO.

Proof. The only non-obvious thing is to show that if $PU^c_\alpha = P'U^c_\alpha$ as functions, then $Q \cdot PU^c_\alpha = Q \cdot P'U^c_\alpha$ as functions.

We will approximate Q by finite sums Q_N of monomials $a(x,t)D_x^j D_t^k$ as in theorem 3.2; because $D_x^{\pm 1}$, D_t are unambiguously defined on the level of functions in $\mathcal{M}$, $Q_N \cdot PU^c_\alpha = Q_N \cdot P'U^c_\alpha$. Let

$$R_N := Q_N P \mod \cdot (D_t^2 - (x+bt^2)D_x^2 - cD_x)$$

and

$$R'_N = Q_N P' \mod \cdot (D_t^2 - (x+bt^2)D_x^2 - cD_x).$$

It remains to show that as a function $(R_N - R'_N)U^c_\alpha \to 0$ u.c.s. That follows from the lemma below. □

Lemma 6.2. *Suppose $P_n \in \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$ are such that for some value $t > 0$ and a compact neighborhood K of $\mathbf{x}$, $N_m(P_n, K, t) \to 0$. Then $P_n U_\alpha^c \to 0$ u.c.s in some neighborhood of the origin.*

Idea of the proof. Since $N_m(P_n, K, t)$, defined by the sum (9) tends to zero as $n \to \infty$, so does the subsum $\sum_{k\geq 0} \frac{2\cdot 4^{-k}}{k!} |p_{m-k}|_K t^{2k}$. From this the lemma follows in a routine way. □

The construction of $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} V_\alpha^c$ is analogous.

7. Solution of the Cauchy problem

As $P_c U_\alpha^c = P_c V_\alpha^c = 0$, we have the quotient morphisms of $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$-modules $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} P_c \to \mathcal{E}_{\mathbb{C}^2,\mathbf{x}} U_\alpha^c$ and $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} P_c \to \mathcal{E}_{\mathbb{C}^2,\mathbf{x}} V_\alpha^c$. Pre-composing them with $\mathbf{i}$ from (22), we get two morphisms

$$\mu_U \colon \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} L \longrightarrow \mathcal{E}_{\mathbb{C}^2,\mathbf{x}} U_\alpha^c \quad \text{and} \quad \mu_V \colon \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}} L \longrightarrow \mathcal{E}_{\mathbb{C}^2,\mathbf{x}} V_\alpha^c,$$

where the image of [1] is $\mathcal{P} U_\alpha^c$, $\mathcal{P} V_\alpha^c$, with $\mathcal{P}$ which is the same MDO in both cases, of order 0, with $\sigma_0(\mathcal{P})$ invertible at $\mathbf{x}$. The operator $\mathcal{P}$ depends on c but not on α. It can be assumed that $\deg_{D_t} \mathcal{P} \leq 1$, and hence $\mathcal{P} = \mathcal{P}_0(t, x, D_x) + \mathcal{P}_1(t, x, D_x) D_t$, where $\mathcal{P}_1$ is of order ≤ -1.

The morphisms μ_U, μ_V yield two solutions of the equation $L\varphi = 0$ in the given classes of ramified analytic functions:

$$\varphi_{\tilde{\alpha}}(x, t) = \mathcal{P} \cdot U_{\tilde{\alpha}}^c = \mathcal{P}_0 U_{\tilde{\alpha}}^c + \mathcal{P}_1 \partial_t U_{\tilde{\alpha}}^c \in \mathcal{E} U_{\tilde{\alpha}}^c$$

and

$$\psi_{\tilde{\alpha}}(x, t) = \mathcal{P} \cdot V_{\tilde{\alpha}}^c = \mathcal{P}_0 V_{\tilde{\alpha}}^c + \mathcal{P}_1 \partial_t V_{\tilde{\alpha}}^c \in \mathcal{E} V_{\tilde{\alpha}}^c$$

for any $\tilde{\alpha} \in \alpha + \mathbb{Z}_{\geq -1}$ if α satisfies (7).

We will now form an appropriate linear combination of $\varphi_{\tilde{\alpha}}$'s and $\psi_{\tilde{\alpha}}$'s that satisfies initial conditions (4).

We have

$$\begin{aligned}
\partial_t \psi_{\tilde{\alpha}}(t, x) &= (D_t \mathcal{P}_0) V_{\tilde{\alpha}}^c + (D_t \mathcal{P}_1) \partial_t V_{\tilde{\alpha}}^c \\
&= [D_t, \mathcal{P}_0] V_{\tilde{\alpha}}^c + \mathcal{P}_0 \partial_t V_{\tilde{\alpha}}^c + [D_t, \mathcal{P}_1] \partial_t V_{\tilde{\alpha}}^c + \mathcal{P}_1 \partial_t^2 V_{\tilde{\alpha}}^c \\
&= \{[D_t, \mathcal{P}_0] + \mathcal{P}_1((x + bt^2) D_x^2 + c D_x)\} V_{\tilde{\alpha}}^c + \{\mathcal{P}_0 + [D_t, \mathcal{P}_1]\} \partial_t V_{\tilde{\alpha}}^c
\end{aligned}$$

and similarly for $\partial_t \varphi_{\tilde{\alpha}}$.

Let $\mathcal{Q}_0(x, D_x) = \mathcal{P}_0|_{t=0}$ and $\mathcal{Q}_1(x, D_x) = \{\mathcal{P}_0 + [D_t, \mathcal{P}_1]\}|_{t=0}$; as $\mathcal{P}_1$ is of order ≤ -1, so $\mathcal{Q}_0, \mathcal{Q}_1$ are still invertible MDOs of order ≤ 0.

We have, using (6),

$$\varphi_{\tilde\alpha}(0,x) = \mathcal{Q}_0 \frac{x^{\tilde\alpha}}{\Gamma(\tilde\alpha+1)},$$

$$\partial_t\varphi_{\tilde\alpha}(0,x) = \{[D_t,\mathcal{P}_0]|_{t=0} + \mathcal{P}_1|_{t=0}(xD_x^2 + cD_x)\}\frac{x^{\alpha}}{\Gamma(\alpha+1)},$$

$$\psi_{\tilde\alpha}(0,x) = \mathcal{P}_1|_{t=0} \frac{x^{\tilde\alpha}}{\Gamma(\tilde\alpha+1)},$$

$$\partial_t\psi_{\tilde\alpha}(0,x) = \mathcal{Q}_1 \frac{x^{\tilde\alpha}}{\Gamma(\tilde\alpha+1)}.$$

Denote $\mathcal{P}_1|_{t=0} = \mathcal{S}D_x^{-1}$, where $\mathcal{S}(x,D_x)$ is an MDO of order 0, and note that $\mathcal{S}D^{-1}(xD_x^2+cD_x) = \mathcal{S}(xD_x+c-1) = x(\mathcal{S}D_x) + [\mathcal{S},x]D_x + \mathcal{S}(c-1)$. Here $[\mathcal{S},x]$ is of order ≤ -1.

In order to satisfy the initial condition of the Cauchy problem, let us find constants a_n, b_n, $n \ge -1$, such that

$$\begin{aligned}
&\sum_{n\ge -1} a_n\varphi_{\alpha+n}(0,x) + \sum_{n\ge -1} b_n\psi_{\alpha+n}(0,x)\\
&\quad= \mathcal{Q}_0 \sum_{n\ge -1}\frac{a_n x^{\alpha+n}}{\Gamma(\alpha+n+1)} + \mathcal{P}_1|_{t=0}\sum_{n\ge -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\\
&\quad= f(x)x^{\alpha};
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\sum_{n\ge -1} a_n\partial_t\varphi_{\alpha+n}(0,x) + \sum_{n\ge -1} b_n\partial_t\psi_{\alpha+n}(0,x)\\
&\quad= \{[D_t,\mathcal{P}_0]|_{t=0} + \mathcal{S}(xD_x+c-1)\}\sum_{n\ge -1}\frac{a_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\\
&\qquad+ \mathcal{Q}_1\sum_{n\ge -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\\
&\quad= x^{\alpha}g(x).
\end{aligned} \tag{30}$$

Equation (29) can be rewritten as

$$\sum_{n\ge -1}\frac{a_n x^{\alpha+n}}{\Gamma(\alpha+n+1)} + Q_0^{-1}\mathcal{P}_1|_{t=0}\sum_{n\ge -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)} = \mathcal{Q}_0^{-1} f(x)x^{\alpha}.$$

The operator $Q_0^{-1}\mathcal{P}_1|_{t=0}$ shall be denoted $D_x^{-1}\mathcal{R}$, where $\mathcal{R}(x, D_x)$ is an MDO of order 0:

$$\sum_{n\geq -1}\frac{a_n x^{\alpha+n}}{\Gamma(\alpha+n+1)} = \mathcal{Q}_0^{-1} f(x)x^\alpha - D_x^{-1}\mathcal{R}\sum_{n\geq -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}. \tag{31}$$

Inserting (31) into (30),

$$\begin{aligned}&\{[D_t,\mathcal{P}_0]|_{t=0} + \mathcal{S}(xD_x + c - 1)\}\Big(\mathcal{Q}_0^{-1} f(x)x^\alpha - D_x^{-1}\mathcal{R}\sum_{n\geq -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\Big)\\ &\quad + \mathcal{Q}_1\sum_{n\geq -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)} = x^\alpha g(x),\end{aligned}$$

or

$$\begin{aligned}&\{\mathcal{Q}_1 + [D_t,\mathcal{P}_0]|_{t=0}D_x^{-1}\mathcal{R}\\ &\quad + \{x(\mathcal{S}D_x) + [\mathcal{S}, x]D_x + \mathcal{S}(c-1)\}D_x^{-1}\mathcal{R}\}\sum_{n\geq -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\\ &\quad = x^\alpha g(x) - \{[D_t,\mathcal{P}_0]|_{t=0} + \mathcal{S}(xD_x + c - 1)\}\mathcal{Q}_0^{-1} f(x)x^\alpha.\end{aligned}$$

Rewrite the left-hand side:

$$\begin{aligned}&\Big\{\underbrace{\mathcal{Q}_1}_{\text{invertible}} + \underbrace{[D_t,\mathcal{P}_0]|_{t=0}D_x^{-1}\mathcal{R}}_{\text{ord}\leq -1}\\ &\quad + x\underbrace{\mathcal{S}\mathcal{R}}_{\text{ord }0} + \underbrace{[\mathcal{S}, x]}_{\text{ord}\leq -1}\mathcal{R} + (c-1)\underbrace{\mathcal{S}D_x^{-1}\mathcal{R}}_{\text{ord}\leq -1}\Big\}\sum_{n\geq -1}\frac{b_n x^{\alpha+n}}{\Gamma(\alpha+n+1)}\\ &\quad = x^\alpha g(x) - \{[D_t,\mathcal{P}_0]|_{t=0} + \mathcal{S}(xD_x + c - 1)\}\mathcal{Q}_0^{-1} f(x)x^\alpha.\end{aligned}$$

The operator on the LHS can be inverted, and the second operator on the RHS is of order ≤ 1, hence indeed we find a series $\sum_{n\geq -1}\frac{b_n}{\Gamma(\alpha+n+1}x^{\alpha+n}$ where the coefficients $b_n/\Gamma(\alpha+n+1) \leq q^n$ for some constant $q > 0$. The series $\sum_{n\geq -1}\frac{b_n}{\Gamma(\alpha+n+1}x^{\alpha+n}$ with the same property is obtained from (31).

Finally, we study the convergence of the expressions

$$\sum_{n\geq -1} a_n\mathcal{P}U^c_{\alpha+n}, \quad \text{and} \quad \sum_{n\geq -1} b_n\mathcal{P}V^c_{\alpha+n}.$$

The former can be written as

$$\sum_{n\geq -1}\sum_{j\geq 0} a_n u_j(t,x)U^{c-j}_{\alpha+n+j}(t,x) + \sum_{n\geq -1}\sum_{j\geq 0} b_n g_j(t,x)\partial_t U^{c-j}_{\alpha+n+j}(t,x). \tag{32}$$

We have, using (10) and Proposition 2.3

$$
\begin{aligned}
&|a_n u_j(t,x) U^{c-j}_{\alpha+n+j}(t,x)| \\
&\quad \le \Gamma(\alpha+n+1)q^n \cdot C\rho^j j! \cdot B(\alpha,c)^{\beta n} T_1^{n+j} T_2^j \frac{1}{(n+j)!}\lambda^{n+j},
\end{aligned}
$$

so by choosing λ small enough we can estimate the first sum of (32) by a product of convergent geometric series. Similarly for three other sums.

This finishes the proof of the main result Theorem 2.5.

8. Discussion of Urabe's argument

In the main result of Urabe, the existence of analytic solutions of the Cauchy problem (4) is claimed on $\tilde{\omega}_\lambda$ – the full universal cover of some neighborhood of the origin minus the ramification locus. In our Theorem 2.5, the claimed domain of existence is smaller: the more turns around the ramification locus you want to allow, the smaller should be the value of λ. With that in mind, we would like to state what we do not understand in Urabe's paper and where a gap in his work may be.

Take $P_c = L$ and the Cauchy problem given by

$$
u(0,x) = f(x)x^\alpha/\Gamma(\alpha+1) = \sum a_n x^{\alpha+n}(\alpha+1)_{\overline{n-1}}/\Gamma(n+1), \quad \partial_t u(0,x) = 0,
$$

where $(\alpha)_{\overline{n}} = \alpha(\alpha+1)\dots(\alpha+n-1)$ is the ascending factorial. The most natural candidate for a solution is

$$
\sum_{n\ge 0} a_n(\alpha+1)_{\overline{n-1}} U^c_{\alpha+n}. \tag{33}
$$

But, unless [13, Proposition (E.U.)] can be dramatically improved, on the whole of $\tilde{\omega}_\lambda$ the function $U^c_{\alpha+n}$ are bounded by a geometric series, and we have a factorial divergence of (33).

Maybe the procedure in [13, §3] leads to a different solution of the initial value problem? Let us take Urabe's u from [13, Theorem on p. 3] and compute $Lu = P_c u$ as on [13, p. 5, bottom]. Let us take $v_{r,k} = h_{r,k} = 0$ (i.e. disregard terms containing $V^{c-k}_{\alpha+r}$). We understand that only the Leibniz rule an the following three properties of U^c_α are used in this calculation

$$
(D_t^2-(x+bt^2)D_x^2-cD_x)U^c_\alpha = 0, \quad (tD_t+2xD_x-2\alpha)U^c_\alpha, \quad D_x U^c_\alpha = U^{c+1}_{\alpha-1}. \tag{34}
$$

However, we get a slightly different expression, the discrepancy coming from the summands at the lower boundary of summation, namely

$$\begin{aligned}P_c u = \sum_{r=0}^{\infty}\sum_{k=0}^{r}&(bt^2\Lambda + 2bt)g_{r,k}U^{c-k+2}_{\alpha+r-2}\\&+[-(M+k-1)u_{r,k}+(\alpha+r+c-k-1)\Lambda g_{r,k}\\&+(P_c-2(\alpha+r-1)D_x)u_{r,k}]U^{c-k+1}_{\alpha+r-1}\\&+\Big(-\Big(\frac{t}{2}\Lambda+M+k\Big)g_{r,k}+\Lambda u_{r,k}\Big)D_tU^{c-k+1}_{\alpha+r-1}\\&+\Big[\Big(P_c-2\Big(\alpha+r-\frac{1}{2}\Big)D_x\Big)g_{r,k}\Big]D_tU^{c-k}_{\alpha+r}.\end{aligned}$$

This leads us to think that one has to start the recursive procedure by equating to zero the coefficients of $U^{c-k+2}_{\alpha+r-2}$ and $D_tU^{c-k+1}_{\alpha+r-1}$, which gives us different conditions than the $k=0$ case on Urabe's p. 7.

Finally, maybe this indexing issue can be repaired preserving the main idea? We do not think so. Indeed, the equations on $u_{r,k}$, $g_{r,k}$ decouple accordingly to the value $r-k$, and as no properties of functions U^c_α are used beyond (34), what is actually studied are the solutions of the equation $P_c u = 0$ in the $\mathcal{E}$-module $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}(P_c,(tD_t+2xD_x-2\alpha))$. In terms of the y-coordinates from Section 4.2, this is the same as looking for $\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}$-module homomorphisms

(35)
$$\begin{aligned}&\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}(D_{y_2}(-y_2D_{y_2}+2Cy_1D_{y_1}-c+1))\\&\quad\longrightarrow \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}(D_{y_2}(-y_2D_{y_2}+2Cy_1D_{y_1}-c+1), y_1D_{y_1}+2y_2D_{y_2}-2\alpha)\end{aligned}$$

because it turns out that $tD_t+2xD_x = y_1D_{y_1}+2y_2D_{y_2}$. Once we rewrite (35) as

(36)
$$\begin{aligned}&\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}(-y_2D_{y_2}+2Cy_1D_{y_1}-c+1)\\&\quad\longrightarrow \mathcal{E}_{\mathbb{C}^2,\mathbf{x}}/\mathcal{E}_{\mathbb{C}^2,\mathbf{x}}(-y_2D_{y_2}+2Cy_1D_{y_1}-c+1,\ y_1D_{y_1}+2y_2D_{y_2}-2\alpha),\end{aligned}$$

it is clear that for an irrational value of C there is only a 1-dimensional space of such morphisms defined by $[1]\to[\lambda]$, $\lambda\in\mathbb{C}$. So, generically, no additional unexpected solutions can be constructed by such a method.

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