Rend. Sem. Mat. Univ. Padova, Vol. 140 (2018), 123–128
DOI 10.4171/RSMUP/5

A brief journey through extensions of rational groups

Stefan Friedenberg (*) – Paul Wolf (**)

Abstract – Let A and B be rational groups, i.e. torsion-free groups of rank-1 and thus subgroups of the rational numbers. This paper gives a short overview of the structure of $\mathrm{Ext}(A, B)$ especially considering some interesting classes of torsion-free pairs.

Mathematics Subject Classification (2010). 20K15, 20K35.

Keywords. Abelian group, extension, rational group, torsion-free.

1. Introduction

Throughout this paper the phrase extension of rational groups means extension of a rational group by a rank-1 group.

For the convenience of the reader, we give a short summary of the concept of types: For any element $a \neq 0$ of a group A the height sequence $(h_p)_{p \in \mathbb{P}}$ is defined by $h_p = n$ if there is a non-negative integer n with $a \in p^n A \setminus p^{n+1} A$ and $h_p = \infty$ if no such n exists. The set of height sequences has a partial ordering given by $\alpha = (\alpha_p) \leq (\beta_p) = \beta$ if $\alpha_p \leq \beta_p$ for each $p \in \mathbb{P}$. It forms a lattice by defining $\sup\{\alpha, \beta\} = (\max\{\alpha_p, \beta_p\})$ and $\inf\{\alpha, \beta\} = (\min\{\alpha_p, \beta_p\})$.

Two height sequences (α_p) and (β_p) are said to be equivalent if they only differ in finitely many entries and if $\alpha_p \neq \beta_p$, both have to be finite. The arising equivalence classes are called types and build a lattice induced by the lattice structure of the height sequences, where $[(\alpha_p)] \leq [(\beta_p)]$ if and only if $\alpha_p \leq \beta_p$ for all but finitely many primes $p \in \mathbb{P}$ and if $\alpha_p \not\leq \beta_p$, then α_p is an integer.

(*) *Indirizzo dell'A.*: University of Stralsund, Zur Schwedenschanze 15, 18435 Stralsund, Germany
E-mail: stefan.friedenberg@hochschule-stralsund.de

(**) *Indirizzo dell'A.*: University of Stralsund, Zur Schwedenschanze 15, 18435 Stralsund, Germany
E-mail: paul.wolf@hochschule-stralsund.de

It is easy to see that in a rank-1 group A all elements have equivalent height sequences. Hence the lattice of isomorphism classes of rank-1 groups is isomorphic to the lattice of types, which was shown by Reinhold Baer in 1935. Due to this fact it is obvious to identify a rank-1 group A by its type $\mathrm{tp}(A)$. For simplicity, we write $\mathrm{tp}(A) = (\alpha_p)$ without explicitly indicating that this is an equivalence class.

Furthermore we can define an addition of types: if $\mathrm{tp}(A) = (\alpha_p)$ and $\mathrm{tp}(B) = (\beta_p)$, then we put $\mathrm{tp}(A) + \mathrm{tp}(B) = (\alpha_p + \beta_p)$. In particular, this is the type of the group $A \otimes B$.

Recall the definition of the *nucleus* of a group A, which was originally given by Phil Schultz:

Definition 1.1. For any group A we call

$$\mathrm{Nuc}(A) := \left\langle \frac{1}{p^\omega} \;\middle|\; p \in \mathbb{P} \text{ with } (\cdot p) \in \mathrm{Aut}(A) \right\rangle \leq \mathbb{Q}$$

the nucleus of A denoted by A_0.

In other words, A_0 is the largest subring of $\mathbb{Q}$ such that A is still an A_0-module. Thus for any group A we have $\mathrm{tp}(A_0) = (\alpha_p)$ with $\alpha_p = \infty$ if A is p-divisible and $\alpha_p = 0$ otherwise. Hence $\mathrm{tp}(A_0)$ is an idempotent type. In particular $\mathrm{tp}(A_0) \leq \mathrm{tp}(A)$ applies for any rational group A.

One of the very valuable properties of the functor Ext in the category of Abelian groups is the fact that given a torsion-free Abelian group A the group $\mathrm{Ext}(A, B)$ is divisible for any Abelian group B. Hence its structure is very much determined and $\mathrm{Ext}(A, B)$ must be of the form

$$\mathrm{Ext}(A, B) = \bigoplus_{r_0} \mathbb{Q} \oplus \bigoplus_{p} \Big[\bigoplus_{r_p} \mathbb{Z}_{p^\infty} \Big]$$

for some uniquely determined cardinals r_0 and r_p which are called the *torsion-free rank* and the *p-rank* of $\mathrm{Ext}(A, B)$, respectively. In [2] it was shown what values for these cardinals are possible in general. We will now apply these results on extensions of rank-1 groups.

2. The structure of Ext by comparing types

At first we consider the case $\mathrm{tp}(A) \leq \mathrm{tp}(B)$. By [3, Theorem 2.1.4] we know that $\mathrm{Ext}(A, B)$ is torsion-free if and only if the following applies:

$$\mathrm{OT}((A \otimes B_0)/D) \leq \mathrm{IT}(B),$$

with D being the divisible subgroup of $A \otimes B_0$ for any torsion-free groups A and B of finite rank and $\mathrm{OT}(B) \neq \mathrm{tp}(\mathbb{Q})$.

Theorem 2.1. *For any rational groups A and B the following statements are equivalent*:

(1) $\mathrm{Ext}(A, B)$ *is torsion-free*;

(2) $\mathrm{tp}(A) \leq \mathrm{tp}(B)$ *or* $A \otimes B_0 = \mathbb{Q}$.

Proof. First let be $\mathrm{tp}(A) \leq \mathrm{tp}(B)$. Since inner type, outer type and the type of any rational group are all equal, $\mathrm{Ext}(A, B)$ is torsion-free by a result of Pat Goeters, see [4, Proposition 1.7]. If otherwise $A \otimes B_0 = \mathbb{Q}$, then we conclude that $\mathrm{Ext}(A, B) \cong \mathrm{Ext}(A \otimes B_0, B) \cong \mathrm{Ext}(\mathbb{Q}, B)$ is torsion-free since $\mathbb{Q}$ is divisible. See [2, Lemma 2.6] for the first isomorphism.

Now let $\mathrm{Ext}(A, B)$ be torsion-free. If $\mathrm{tp}(B) = \mathrm{tp}(\mathbb{Q})$, then trivially $\mathrm{tp}(A) \leq \mathrm{tp}(B)$ because $\mathrm{tp}(\mathbb{Q})$ is the maximal element in the lattice of types. So assume $\mathrm{tp}(B) \neq \mathrm{tp}(\mathbb{Q})$ and we have to consider $\mathrm{tp}((A \otimes B_0)/D)$. Either $A \otimes B_0 = \mathbb{Q}$ or $A \otimes B_0$ has no divisible subgroup since it is a rank-1 group. Thus $\mathrm{tp}(A) \leq \mathrm{tp}(A \otimes B_0) = \mathrm{OT}((A \otimes B_0)/D) \leq \mathrm{tp}(B)$. □

In particular, the group of self-extensions $\mathrm{Ext}(A, A)$ is torsion-free for any rational group A.

One of the main results of [2] says that $r_0(\mathrm{Ext}(A, B)) = 0$ if and only if $\mathrm{Ext}(A, B) = 0$, or $r_0 = 2^{\aleph_0}$. Thus a not-vanishing torsion-free extension of rational groups is of the form

$$\mathrm{Ext}(A, B) = \bigoplus_{2^{\aleph_0}} \mathbb{Q}.$$

Assuming the stricter condition $\mathrm{tp}(A) \leq \mathrm{tp}(B_0)$ it is possible to point out when Ext vanishes for rational groups A and B. By [2] this happens if and only if $A \otimes B_0$ is a free B_0-module. In this case we receive:

Theorem 2.2. *For any rational groups A and B the following are equivalent*:

(1) $\mathrm{Ext}(A, B) = 0$;

(2) $\mathrm{tp}(A) \leq \mathrm{tp}(B_0)$.

Proof. So let be $\mathrm{Ext}(A, B) = 0$. Thus $A \otimes B_0 = B_0$ since it is a free B_0-module of rank-1. Hence $\mathrm{tp}(A \otimes B_0) = \mathrm{tp}(A) + \mathrm{tp}(B_0) = \mathrm{tp}(B_0)$ which is equivalent to $\mathrm{tp}(A) \leq \mathrm{tp}(B_0)$. □

Following Pat Goeters we define the *support* of a group A as

$$\operatorname{supp}(A) = \{p \in \mathbb{P} \mid pA \neq A\},$$

that is the set of all primes not dividing A. Trivially, $\operatorname{supp}(A) \subseteq \operatorname{supp}(B)$ if $\operatorname{tp}(A) > \operatorname{tp}(B)$ because for a rational group $A = (\alpha_p)$ the support of A is given by $\operatorname{supp}(A) = \{p \in \mathbb{P} \mid \alpha_p \neq \infty\}$

Theorem 2.3. *For any rational groups A and B the following are equivalent*:

(1) $r_p(\operatorname{Ext}(A, B)) = 1$ *for any* $p \in \operatorname{supp}(A) \cap \operatorname{supp}(B)$;

(2) $\operatorname{tp}(A) > \operatorname{tp}(B)$ *or the types are incomparable.*

Proof. Assume (2) holds. Due to Warfiled it is well-known that the p-rank of $\operatorname{Ext}(A, B)$ can be calculated by $r_p(\operatorname{Ext}(A, B)) = r_p(A) \cdot r_p(B) - r_p(\operatorname{Hom}(A, B))$ for finite rank Abelian groups A and B, where $r_p(A) = \dim_{\mathbb{Z}/p\mathbb{Z}}(A/pA)$ if A is torsion-free. But there are no homomorphisms $\varphi\colon A \to B$ except the trivial one and hence $\operatorname{Hom}(A, B) = 0$ if and only if $\operatorname{tp}(A) > \operatorname{tp}(B)$ or the types are incomparable. Therefore we conclude $r_p(\operatorname{Ext}(A, B)) = r_p(A) \cdot r_p(B)$ and thus $r_p(\operatorname{Ext}(A, B)) = 1$ if both A and B are not p-divisible.

If we assume the negation of (2), $\operatorname{Ext}(A, B)$ is torsion-free by 2.1 and thus $r_p(\operatorname{Ext}(A, B)) = 0$. Hence the assertion holds. □

So any not torsion-free extension of rational groups is of the form

$$\operatorname{Ext}(A, B) = \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{p} \mathbb{Z}_{p^\infty},$$

with $p \in \operatorname{supp}(A) \cap \operatorname{supp}(B)$.

3. Torsion-free pairs

In analogy to Luigi Salces cotorsion pairs we call a pair $(\mathcal{A}, \mathcal{B})$ of classes of groups a *torsion-free pair* if $\operatorname{Ext}(A, B)$ is torsion-free for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and the classes $\mathcal{A}$ and $\mathcal{B}$ are closed with respect to this property. This means X has to be an element of $\mathcal{B}$ if $\operatorname{Ext}(A, X)$ is torsion-free for all $A \in \mathcal{A}$ as well as $X \in \mathcal{A}$ if $\operatorname{Ext}(X, B)$ is torsion-free for all $B \in \mathcal{B}$. Like in [5] we can define a partial order on the class of torsion-free pairs by putting $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$ if $\mathcal{B} \subseteq \mathcal{B}'$ or, equivalently $\mathcal{A}' \subseteq \mathcal{A}$. Then the torsion-free pairs become a complete lattice by setting

$$\bigwedge_{i \in I}(\mathcal{A}_i, \mathcal{B}_i) = \Big({}^*\Big(\bigcap_{i\in I} \mathcal{B}_i\Big), \bigcap_{i\in I} \mathcal{B}_i\Big) \quad \text{and} \quad \bigvee_{i \in I}(\mathcal{A}_i, \mathcal{B}_i) = \Big(\bigcap_{i\in I} \mathcal{A}_i, \Big(\bigcap_{i\in I} \mathcal{A}_i\Big)^*\Big)$$

for a family $\{(\mathcal{A}_i, \mathcal{B}_i)\}_{i\in I}$ of torsion-free pairs. We define

(1) $\mathcal{A}^* := \{X \mid \operatorname{Ext}(A, X)\ is$ torsion-free for all $A \in \mathcal{A}\}$,

(2) $^*\mathcal{B} := \{X \mid \operatorname{Ext}(X, B)\ is$ torsion-free for all $B \in \mathcal{B}\}$,

and call $(^*(\mathcal{A}^*), \mathcal{A}^*)$ the *torsion-free pair co-generated by* $\mathcal{A}$ and $(^*\mathcal{B}, (^*\mathcal{B})^*)$ the *torsion-free pair generated by* $\mathcal{B}$.

One of the main results of [3] is the following theorem.

Theorem 3.1. *The lattice of types is anti-isomorphic to the lattice of all rational generated* $(\mathfrak{Tffr}, \mathfrak{Tffr})$*-torsion-free pairs, which mean torsion-free pairs restricted on torsion-free groups of finite rank.*

For the proof and more general results we recommend to have a look at [3].

Since our main purpose in this section is to shed some light on the extensions of rational groups, we replace the restriction on torsion-free groups of finite rank by rational groups, the so-called $(\mathfrak{R}, \mathfrak{R})$-torsion-free pairs. Unfortunately, 3.1 does not hold for these rational torsion-free pairs.

Theorem 3.2. *There exist rational groups A and B such that* $\operatorname{tp}(A) < \operatorname{tp}(B)$ *but* $^*A =^* B$.

Proof. Take $B = \mathbb{Q}$. Then $\operatorname{Ext}(A, \mathbb{Q}) = 0$ for any group A and thus $^*\mathbb{Q} \cap \mathfrak{R} = \mathfrak{R}$. Now consider the group $\mathbb{Q}_p$ of all rational numbers with denominator prime to p. There is only one group which has a type greater than $\operatorname{tp}(\mathbb{Q}_p)$, namely $\mathbb{Q}$. Furthermore, any group of uncomparable type has to be p-divisible. So if X is an arbitrary rank-1 group, either $\operatorname{tp}(X) \le \operatorname{tp}(\mathbb{Q}_p)$ or $X \otimes \mathbb{Q}_p = \mathbb{Q}$ which implies that also $^*\mathbb{Q}_p \cap \mathfrak{R} = \mathfrak{R}$. □

It turns out that 3.1 holds if we restrict on rational groups $\neq \mathbb{Q}$:

Theorem 3.3. *The lattice of types is anti-isomorphic to the lattice of all rational generated* $(\mathfrak{R} \setminus \{\mathbb{Q}\}, \mathfrak{R} \setminus \{\mathbb{Q}\})$*-torsion-free pairs.*

Proof. Let be $\operatorname{tp}(A) \le \operatorname{tp}(B)$. If $X \in^* A$ we know by 2.1 that $\operatorname{tp}(X) \le \operatorname{tp}(A)$ or $X \otimes A_0 = \mathbb{Q}$. But then also $\operatorname{tp}(X) \le \operatorname{tp}(B)$ or $X \otimes B_0 = \mathbb{Q}$ which implies that $\operatorname{Ext}(X, B)$ is also torsion-free and thus $^*A \subseteq {}^*B$.

Now consider the strict inequality $\operatorname{tp}(A) < \operatorname{tp}(B)$ which implies that $A \otimes B_0 = \mathbb{Q}$ is only possible if $B = \mathbb{Q}$. Since this is excluded, $A \otimes B_0$ cannot be divisible, so $B \otimes A_0 \neq \mathbb{Q}$ as well. Hence there has to be a prime p such that A and B are not p-divisible and thus $\operatorname{Ext}(B, A)$ is not torsion-free. Indeed, $\operatorname{Ext}(B, B)$ is torsion-free. So we conclude $^*A \subsetneq^* B$. □

Putting 3.1 and 3.3 together we obtain:

THEOREM 3.4. *The lattices of all rational generated* $(\mathfrak{T}\mathfrak{f}\mathfrak{f}\mathfrak{r}, \mathfrak{T}\mathfrak{f}\mathfrak{f}\mathfrak{r})$*-torsion-free pairs and* $(\mathfrak{R} \setminus \{\mathbb{Q}\}, \mathfrak{R} \setminus \{\mathbb{Q}\})$*-torsion-free pairs are isomorphic.*

REFERENCES

[1] D. M. ARNOLD, *Finite rank torsion free abelian groups and rings,*, Lecture Notes in Mathematics, 931. Springer-Verlag, Berlin etc., 1982.

[2] S. FRIEDENBERG – L. STÜNGMANN, *Extensions in the class of countable torsion-free Abelian groups,* Acta Math. Hungar. 140 (2013), no. 4, pp. 316–328.

[3] S. FRIEDENBERG, *Torsion-free extensions of torsion-free abelian groups of finite rank,* Ph.D thesis, University of Duisburg-Essen, Duisburg, 2009.

[4] H. P. GOETERS, *When is* $\operatorname{Ext}(A, B)$ *torsion-free? and related problems,* Comm. Algebra 16 (1988), no. 8, pp. 1605–1619.

[5] L. Salce, *Cotorsion theories for abelian groups,* in *Symposia Mathematica,* Vol. XXIII (Roma, 1977), Academic Press, London and New York, 1979, pp. 11–32.

Manoscritto pervenuto in redazione il 6 marzo 2017.