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On the uniqueness of the factorization of power digraphs modulo n

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Abstract – For each pair of integers $n = \prod_{i=1}^{r} p_i^{e_i}$ and $k \geq 2$, a digraph $G(n,k)$ is one with vertex set $\{0, 1, \dots, n-1\}$ and for which there exists a directed edge from x to y if $x^k \equiv y \pmod{n}$. Using the Chinese Remainder Theorem, the digraph $G(n,k)$ can be written as a direct product of digraphs $G(p_i^{e_i}, k)$ for all i such that $1 \leq i \leq r$. A fundamental constituent $G_P^*(n,k)$, where $P \subseteq Q = \{p_1, p_2, \dots, p_r\}$, is a subdigraph of $G(n,k)$ induced on the set of vertices which are multiples of $\prod_{p_i \in P} p_i$ and are relatively prime to all primes $p_j \in Q \setminus P$. In this paper, we investigate the uniqueness of the factorization of trees attached to cycle vertices of the type 0, 1, and $(1,0)$, and in general, the uniqueness of $G(n,k)$. Moreover, we provide a necessary and sufficient condition for the isomorphism of the fundamental constituents $G_P^*(n,k_1)$ and $G_P^*(n,k_2)$ of $G(n,k_1)$ and $G(n,k_2)$ respectively for $k_1 \neq k_2$.

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Contents

1. Introduction

The digraphs $G(n,k)$ were first studied extensively by L. Somer and M. Křížek in [13], [15], [12] and [14], and also later by other authors in [19], [5], and [7], based on ideas of S. Bryant [1], G. Chassé [3], T.D. Rogers [10], and L. Szalay [16]. There have also been extensions of these ideas to more general structures (see [6], [8], [9], [17], and [18]).

Every component of $G(n,k)$ has a cycle of length t, and attached to each cycle vertex c is a tree, denoted by $T(n,k,c)$, whose root is c and the non-cycle vertices b are such that $b^{k^i} \equiv c \pmod{n}$ for some positive integer i, but $b^{k^{i-1}}$ is not congruent modulo n to any cycle vertex in $G(n,k)$. Cycles of length 1 are called fixed points. The trees attached to all cycle vertices in a component of $G(n,k)$ are isomorphic.

Let

$$n = \prod_{i=1}^{r} p_i^{e_i},$$

where the p_i's are distinct primes, and consider the direct product $G(p_1^{e_1},k) \times G(p_2^{e_2},k) \times \cdots \times G(p_r^{e_r},k)$. By the Chinese Remainder Theorem, there is a natural isomorphism between $G(n,k)$ and $G(p_1^{e_1},k) \times G(p_2^{e_2},k) \times \cdots \times G(p_r^{e_r},k)$ written as

$$G(n,k) \cong G(p_1^{e_1},k) \times G(p_2^{e_2},k) \times \cdots \times G(p_r^{e_r},k),$$

and we say that the digraph $G(n,k)$ can be factorized into a direct product of digraphs $G(p_i^{e_i},k)$ for all i such that $1 \le i \le r$.

Let $G_1(n,k)$ denote the subdigraph of $G(n,k)$ induced on the set of vertices that are relatively prime to n, and $G_2(n,k)$ denote the subdigraph of $G(n,k)$ induced on the set of vertices not relatively prime to n. L. Somer and M. Křížek [15] proved that every fundamental constituent $G_P^*(n,k)$ of $G(n,k)$ can be written as

$$G^*_P(n,k) \cong G_1(n_1,k) \times T(n_2,k,0), \tag{1.1}$$

where $n = n_1 n_2$ with $\gcd(n_1,n_2) = 1$, and $p_i \mid n_2$ if and only if $p_i \in P$. Then it becomes clear that each fundamental constituent $G^*_P(n,k)$, where $\emptyset \neq P \subsetneq Q = \{p_1, p_2, \ldots, p_r\}$, contains a tree attached to a cycle vertex of the form (1,0), and the fundamental constituents $G^*_Q(n,k)$ and $G^*_\emptyset(n,k)$ contain the trees $T(n,k,0)$ and $T(n,k,1)$, respectively. Since

$$G(n,k) = \bigcup_{P \subseteq Q} G^*_P(n,k), \tag{1.2}$$

where the union is disjoint, then using the fact (proved in [15]) that the trees attached to all cycle vertices in a fundamental constituent of $G(n,k)$ are isomorphic, we can conclude that any tree in $G(n,k)$ is isomorphic to one of $T(n,k,(a_1,a_2,\ldots,a_r))$ where $a_i = 0,1$ for all i. Thus it follows that $G(n,k)$ has exactly $2^{\omega(n)}$ fundamental constituents, where $\omega(n)$ denotes the number of primes dividing n.

A natural question that arises from the factorization of $G(n,k)$ is its uniqueness, in the sense that, if $G(n,k_1) \cong G(n,k_2)$ for $k_1 \neq k_2$, then does $G(p_i^{e_i},k_1) \cong G(p_i^{e_i},k_2)$ holds for all i? We give an answer to this question in Theorem 5.8. Our strategy is to first establish the uniqueness of the factorization of trees attached to the cycle vertices 1 and 0 in $G(n,k)$, these are proved in Corollary 4.3 and Theorem 4.6 , respectively, and finally we show that $G(p_i^{e_i},k_1)$ and $G(p_i^{e_i},k_2)$ have the same cycle structure. The factorization of a tree attached to cycle vertices of the type $(1,0)$ is not unique in general, although it becomes unique under a certain condition. This is proved in Theorem 4.7 and the example preceding it.

A question was asked in [7] regarding the conditions for the isomorphism of $G(p,k_1)$ and $G(p,k_2)$ for all primes p and $k_1 \neq k_2$, and it was answered completely in a paper by G. Deng and P. Yuan [4]. In Theorem 5.4 of this paper, we prove a somewhat generalization of this question in that we determine a necessary and sufficient condition for the isomorphism of the fundamental constituents $G^*_P(n,k_1)$ and $G^*_P(n,k_2)$ of $G(n,k_1)$ and $G(n,k_2)$ respectively for $k_1 \neq k_2$. It was proved in [15] that the trees attached to all cycle vertices in a fundamental constituent of $G(n,k)$ are isomorphic. We extend this result and in Theorem 4.8 we prove a necessary and sufficient condition for the isomorphism of trees attached to all cycle vertices in two distinct fundamental constituents of $G(n,k)$.

Throughout the rest of this paper, we take $n = \prod_{i=1}^r p_i^{e_i}$ to be the prime factorization of $n > 1$, and $k > 1$ an integer.

2. Cycles and Trees in $G(n,k)$

Definition 2.1. The Carmichael function of a positive integer n, denoted by $\lambda(n)$, is defined as the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$ for every integer a relatively prime to n.

Lemma 2.2. *Let n be a positive integer, and ϕ denote the standard Euler's totient function. Then*

$$\lambda(2^k) = \phi(2^k) \quad \text{for } k = 0,1,2, \qquad \lambda(2^k) = \frac{1}{2}\phi(2^k) \quad \text{for } k \geq 3,$$
$$\lambda(p^k) = \phi(p^k) \quad \text{for any odd prime } p \text{ and } k \geq 1,$$
$$\lambda\Big(\prod_{i=1}^{r} p_i^{e_i}\Big) = \operatorname{lcm}[\lambda(p_1^{e_1}), \lambda(p_2^{e_2}), \ldots, \lambda(p_r^{e_r})],$$

where $p_1, p_2, \ldots, p_r$ are distinct primes and $e_i \geq 1$ for all $i = 1,2,\ldots,r$.

For more on the Carmichael function see [2].

The *indegree* of a vertex a in $G(n,k)$, denoted by $\operatorname{indeg}_k^n(a)$, is the number of solutions of the congruence $x^k \equiv a \pmod{n}$.

Lemma 2.3 ([19]). *Let a be a vertex of positive indegree in $G_1(n,k)$. Then*

$$\operatorname{indeg}_k^n(a) = \prod_{i=1}^{r} \varepsilon_i \gcd(\lambda(p_i^{e_i}), k),$$

where $\varepsilon_i = 2$ if $2 \mid k$ and $8 \mid p_i^{e_i}$, and $\varepsilon_i = 1$ otherwise.

Lemma 2.4 ([19]). *Let $a = (a_1, a_2, \ldots, a_r)$ be a vertex in $G(n,k) \cong G(p_1^{e_1}, k) \times G(p_2^{e_2}, k) \times \cdots \times G(p_r^{e_r}, k)$. Then*

$$\operatorname{indeg}_k^n(a) = \prod_{i=1}^{r} \mathrm{indeg}_{\mathrm{k}}^{\mathrm{p_i^{e_i}}}(\mathrm{a_i}).$$

Lemma 2.5 ([13] and [14]). *Let p be a prime and $e > 1$ an integer. Suppose that $b \neq 0$ is a vertex of positive indegree in $G_2(p^e, k)$ where $p^\alpha \parallel b$. Then $\alpha = kr$ for some integer $r \geq 1$, and*

$$\operatorname{indeg}_k^{p^e}(b) = \delta p^{(k-1)r} \gcd(\lambda(p^{e-\alpha}), k),$$

where $\delta = 2$ if $p = 2$ and $e - \alpha \geq 3$, and $\delta = 1$ otherwise. Moreover,

$$\operatorname{indeg}_k^{p^e}(0) = p^{e - \lceil \frac{e}{k} \rceil}.$$

LEMMA 2.6. *Let p be a prime and $e > 1$ an integer. Let $0 \neq a = p^r c$, where $p \nmid c$, be a vertex in $G_2(p^e, k)$. Then $\mathrm{indeg}_k^{p^e}(a) > 0$ if and only if $k \mid r$ and $\mathrm{indeg}_k^{p^{e-r}}(c) > 0$.*

PROOF. If $\mathrm{indeg}_k^{p^e}(a) > 0$, it is seen from Lemma 2.5 that $r = kt$ for some positive integer t. Also, there exists a positive integer s such that $p^r s^k \equiv (p^t s)^k \equiv p^r c \pmod{p^e}$ which implies that $s^k \equiv c \pmod{p^{e-r}}$.

The converse is clear. □

Let $\lambda(n) = uv$, where u is the largest divisor of $\lambda(n)$ relatively prime to k.

LEMMA 2.7 ([19]). *There is a cycle of length t in $G_1(n, k)$ if and only if $t = \mathrm{ord}_d(k)$ for some divisor d of u.*

Notation. Let $\mathcal{A}(G(n, k))$ denote the set of all cycle lengths in $G(n, k)$.

LEMMA 2.8 ([14]). *Let $A_t(G(n, k))$ denotes the number of t-cycles in $G(n, k)$. We have,*

$$A_t(G(n,k)) = \frac{1}{t}\Big[\prod_{i=1}^{r}(\delta_i \gcd(\lambda(p_i^{e_i}), k^t - 1) + 1) - \sum_{d|t, d\neq t} d A_d(G(n,k))\Big],$$

and

$$A_t(G_1(n,k)) = \frac{1}{t}\Big[\prod_{i=1}^{r}\delta_i \gcd(\lambda(p_i^{e_i}), k^t - 1) - \sum_{d|t, d\neq t} d A_d(G_1(n,k))\Big],$$

where $\delta_i = 2$ if $2 \mid k^t - 1$ and $8 \mid p_i^{e_i}$, and $\delta_i = 1$ otherwise.

PROPOSITION 2.9. *Let $t_1, t_2, \ldots, t_m$ be distinct positive integers. There exist integers $n > 1$ and $k > 1$ such that $G(n, k)$ has a t_i-cycle for all i.*

PROOF. Take $t = t_1 t_2 \ldots t_m$, and choose $M = k^t - 1$, $N_i = k^{t_i} - 1$, where $k > 1$, so we can have $\mathrm{ord}_M k = t$ and $\mathrm{ord}_{N_i} k = t_i$ for all i. By Dirichlet's Theorem on primes in arithmetic progression, we can choose a prime p such that $p \equiv 1 \pmod{M}$. Then it follows immediately from Lemma 2.7 that $G(p^e, k)$ contains a t_i-cycle for all i and $e \geq 1$. Thus for any positive integer n with $p^e \parallel n$, $G(n, k)$ has a t_i-cycle for all i. □

As an application of the above Lemma, we now prove a generalization of a result in [12]. But first we recall the definition of a Sophie Germain prime and a Fermat prime. A prime number p is a Sophie Germain prime if $2p + 1$ is also a prime, and a Fermat prime is a prime number of the form $2^{2^n} + 1$ for some non-negative integer n.

Proposition 2.10. *There exist positive integers t, m, n, and l such*

$$A_t(G_1(m,k)) > A_t(G_2(m,k)),$$

$$A_t(G_1(n,k)) < A_t(G_2(n,k)),$$

$$A_t(G_1(l,k)) = A_t(G_2(l,k)).$$

Proof. Let t be a prime and take $M = k^t - 1$.

By Proposition 2.9, there exists an integer $m = p_1 p_2$, where p_1 and p_2 are congruent to 1 modulo M, such that both $G_1(m,k)$ and $G_2(m,k)$ has a t-cycle. We then compute the number of t-cycles in $G(m,k)$, $G_1(m,k)$ and $G_2(m,k)$ and obtained

$$A_t(G_2(m,k)) = \frac{2(M-k+1)}{t} < \frac{M^2-(k-1)^2}{t} = A_t(G_1(m,k)).$$

Next, assume k to be odd and take $n = p_1(2q_1+1)(2q_2+1)$, where q_1, q_2 are Sophie Germain primes. The existence of a t-cycle is again assured by Lemma 2.9. Then after some easy computations we get

$$A_t(G_2(n,k)) = \frac{5(k^t-k)}{t} > \frac{4(k^t-k)}{t} = A_t(G_1(n,k)).$$

If k is even, we choose $n = p_1 q_2 q_3$, where q_3, q_4 are Fermat primes, to obtain

$$A_t(G_2(n,k)) = \frac{3(k^t-k)}{t} > \frac{(k^t-k)}{t} = A_t(G_1(n,k)).$$

Finally, let $l = 2p_1$ and we have $A_t(G_1(l,k)) = \frac{(k^t-k)}{t} = A_t(G_2(l,k))$. □

Definition 2.11. Let a be a vertex in $G(n,k)$. We define height of a, denoted by $h(a)$, to be the least non-negative integer j such that a^{k^j} is congruent modulo n to a cycle vertex in $G(n,k)$. We also define $h(C) = \max_{a\in C} h(a)$ for every component C of $G(n,k)$.

For the rest of this section, unless stated otherwise, we let p be a prime, and $e > 1$ an integer.

Lemma 2.12 ([5]). *Let $\lambda(p^e) = uv$, where u is the largest factor of $\lambda(p^e)$ relatively prime to k. Then $h(T(p^e,k,1)) = \min\{i : v \mid k^i\}$.*

Lemma 2.13. *Let $a \neq 0$ be a vertex in $G_2(p^e,k)$ such that $p^{\alpha k^i} \parallel a$, where $\gcd(k,\alpha) = 1$ for some positive integer i. Then a is at height h if and only if h is the least positive integer such that $\alpha k^{h+i-1} < e \leq \alpha k^{h+i}$.*

Proof. The proof is straightforward. □

LEMMA 2.14. *Consider the trees* $T(p^e, k, 0)$ *and* $T(p^e, k^r, 0)$ *for* $r > 1$.

(1) *Suppose* $h(p^{\alpha k^i}) = h'$ *in* $T(p^e, k, 0)$, *where* $\gcd(k, \alpha) = 1$ *and* $i \geq 1$. *Then* $h(p^{\alpha k^i}) = \lceil \frac{h'}{r} \rceil$ *in* $T(p^e, k^r, 0)$. *In particular, if* $h(T(p^e, k, 0)) = h$ *and* $r < h$ *then* $h(T(p^e, k^r, 0)) = \lceil \frac{h}{r} \rceil$.

(2) *Suppose* $h(T(p^e, k^r, 0)) = h$. *Then* $h(T(p^e, k, 0)) = \sum_{i=0}^{r-1} h(p^{k^i})$, *where the* p^{k^i}*'s in the sum are considered as vertices in* $T(p^e, k^r, 0)$. *Moreover, if* $h(p^{k^i}) = h(p^{k^j})$ *in* $T(p^e, k^r, 0)$ *for all* $i, j \leq r - 1$, *then we have* $h(T(p^e, k, 0)) = rh$.

PROOF. Let $h(p^{\alpha k^i}) = h'$ in $T(p^e, k, 0)$ and assume $h(p^{\alpha k^i}) = m$ in $T(p^e, k^r, 0)$. By Lemma 2.13 we have $\alpha k^{h'+i-1} < e \leq \alpha k^{h'+i}$ and $\alpha k^{r(m-1)+i} < e \leq \alpha k^{rm+i}$, which implies that $r(m-1) < h' \leq rm$, and hence $h(p^{\alpha k^i}) = \lceil \frac{h'}{r} \rceil$ in $T(p^e, k^r, 0)$.

For the second part, we note that $\text{indeg}_k^{p^e}(p^{k^i}) = 0$ for all $i < r$, and $\text{indeg}_k^{p^e}(p^{k^j}) > 0$ for all $j \geq r$ in $T(p^e, k^r, 0)$. It is clear that $h(T(p^e, k, 0)) = \sum_{i=0}^{r-1} h(p^{k^i})$, where the p^{k^i}'s in the sum are considered as vertices in $T(p^e, k^r, 0)$. Furthermore, if $h(p^{k^i}) = h(p^{k^j})$ in $T(p^e, k^r, 0)$ for all $i, j \leq r - 1$, then $h(p^{k^i}) = h(p^{k^j}) = h(p) = h$ in $T(p^e, k^r, 0)$. Thus the result follows. □

LEMMA 2.15. *Consider the trees* $T(p^e, k, 1)$ *and* $T(p^e, k^r, 1)$ *for* $r > 1$.

(1) *Suppose* $h = h(T(p^e, k, 1))$ *and* $r < h$. *Then* $h(T(p^e, k^r, 1)) = \lceil \frac{h}{r} \rceil$.

(2) *Suppose* $h(T(p^e, k^r, 1)) = h > 1$. *Then* $h(T(p^e, k, 1)) = rh - i$ *for some* i *such that* $0 \leq i < r$.

PROOF. The proof is an application of Lemma 2.12. □

DEFINITION 2.16. A tree of height h is *complete* if every vertex have positive indegree except the vertices at height h.

LEMMA 2.17. *Let* $h(T(p^e, k, 1)) = h > 1$. $T(p^e, k, 1)$ *is complete if and only if* $\gcd(\lambda(p^e), k^i) = \gcd(\lambda(p^e), k)^i$ *for all* $i \leq h$.

Proof. Let $h(T(p^e,k,1)) = h > 1$ and assume $T(p^e,k,1)$ to be complete. It is enough to show that $\gcd(\lambda(p^e),k^h) = \gcd(\lambda(p^e),k)^h$. Since $\gcd(\lambda(p^e),k^h) = |T(p^e,k,1)|$, then counting the number of vertices in $T(p^e,k,1)$ by working our way up from the cycle vertex 0, we obtain

$$\begin{aligned}
&\gcd(\lambda(p^e),k^h)\\
&\quad= |T(p^e,k,1)|\\
&\quad= 1 + (\mathrm{indeg}_k^{p^e}(1) - 1) + \mathrm{indeg}_k^{p^e}(1) \times (\mathrm{indeg}_k^{p^e}(1) - 1) + \cdots\\
&\qquad + \mathrm{indeg}_k^{p^e}(1) \times \cdots \times (h-1) \times \cdots \times \mathrm{indeg}_k^{p^e}(1) \times (\mathrm{indeg}_k^{p^e}(1) - 1)\\
&\quad= \mathrm{indeg}_k^{p^e}(1) + (\mathrm{indeg}_k^{p^e}(1))^2 - \mathrm{indeg}_k^{p^e}(1)\\
&\qquad + (\mathrm{indeg}_k^{p^e}(1))^3 - (\mathrm{indeg}_k^{p^e}(1))^2 + \cdots\\
&\qquad + (\mathrm{indeg}_k^{p^e}(1))^h - (\mathrm{indeg}_k^{p^e}(1))^{h-1}\\
&\quad= (\mathrm{indeg}_k^{p^e}(1))^h = (\gcd(\lambda(p^e),k))^h.
\end{aligned}$$

Conversely, assume that $T(p^e,k,1)$ is not complete. Then there exists a vertex b at some height $j < h$ such that $\mathrm{indeg}_k^{p^e}(b) = 0$. If j is taken to be the least such positive integer, then we must have $\gcd(\lambda(p^e),k^{j+1}) < \gcd(\lambda(p^e),k)^{j+1}$, and we are done. □

Lemma 2.18. *Let $h(T(p^e,k,1)) = h > 1$ and $r < h$. Then $T(p^e,k^r,1)$ is complete if and only if $T(p^e,k,1)$ is complete and $r \mid h$.*

Proof. First we observe that if $T(p^e,k^r,1)$ is complete then $\gcd(\lambda(p^e),k^r) = \gcd(\lambda(p^e),k)^r$. Now, if $r \nmid h$ then $r\lceil\frac{h}{r}\rceil > h$ and this yields $\gcd(\lambda(p^e),k^{r\lceil\frac{h}{r}\rceil}) = \gcd(\lambda(p^e),k^h) < \gcd(\lambda(p^e),k^r)^{\lceil\frac{h}{r}\rceil}$, which means that $T(p^e,k^r,1)$ is not complete. So by Lemma 2.17, it follows that if $T(p^e,k^r,1)$ is complete, $T(p^e,k,1)$ must also be complete.

The converse is a direct application of Lemma 2.17. □

Lemma 2.19. *Let a and b be two fixed points in $G(n,k_1)$ and $G(n,k_2)$, respectively. Suppose that $T(n,k_1,a) \cong T(n,k_2,b)$, then $T(n,k_1^r,a) \cong T(n,k_2^r,b)$ for all $r \geq 2$.*

Proof. Suppose $\phi\colon T(n,k_1,a) \longrightarrow T(n,k_2,b)$ be an isomorphism. Then we have $|T(n,k_1^r,a)| = |T(n,k_2^r,b)|$ for any $r \geq 1$, and they contain the same vertices. Suppose there is an edge between x and y in $T(n,k_1^r,a)$. Then there exist vertices $x_2,x_3,\ldots,x_{r-1}$ such that $x^{k_1} \equiv x_2 \pmod{n}$, $x_i^{k_1} \equiv x_{i+1} \pmod{n}$ for $i = 2,3,\ldots,r-2$, and $x_{r-1}^{k_1} \equiv y \pmod{n}$ in $T(n,k_1,a)$, which implies that

$\phi(x)^{k_2} \equiv \phi(x_2) \pmod{n}$, $\phi(x_i)^{k_2} \equiv \phi(x_{i+1}) \pmod{n}$ for $i = 2, 3, \ldots, r-2$, $\phi(x_{r-1})^{k_2} \equiv \phi(y) \pmod{n}$, and thus $\phi(x)^{k_2^r} \equiv \phi(y) \pmod{n}$ in $T(n, k_2, b)$. Hence, the result follows. □

Definition 2.20. Let O_t^m denote a component of $G(n,k)$ having a t-cycle and is of height 1, and every vertex of positive indegree has indegree m.

Lemma 2.21 ([5]). *We have $O_1^m \times G \cong O_1^m \times H$ if and only if $G \cong H$ for any digraphs G and H.*

3. Semiregularity

Definition 3.1. A digraph $G(n,k)$ is semiregular if there exists a positive integer d such that every vertex of $G(n,k)$ has indegree d or 0.

The semiregularity property is perhaps the most useful property that the digraphs $G(n,k)$ or its components can have. We know that $G_1(n,k)$ and its components are semiregular. The tree $T(p_i^{e_i}, k, 0)$, which is not always semiregular, has a nice simple structure whenever it is semiregular. This can be seen through Lemma 3.2 below. The situation is similar with the tree $T(n,k,0)$ as we shall see in this section. The semiregular digraphs $G(n,k)$ was characterized in [13] and [11], and it was proved that it has a close relationship with its tree structure. In this section we state some important results on semiregularity that will be needed in this paper, and we prove a new characterization for semiregular digraphs $G(n,k)$. We also provide a relationship between the symmetric property and semiregularity property of $G(n,k)$.

Lemma 3.2 ([13], [5]). *Let p be an odd prime and $e \geq 1$ an integer.*

(1) *Suppose $p^\alpha \parallel k$ and $\gcd(p-1,k) = 1$. Then $G_2(p^e,k)$ is semiregular if and only if $1 \leq e \leq k + \alpha + 1$.*

(2) *Suppose $\gcd(p(p-1),k) = 1$. Then $G_2(p^e,k)$ is semiregular if and only if $1 \leq e \leq k+1$.*

(3) *Suppose $\gcd(p-1,k) > 1$. Then $G_2(p^e,k)$ is semiregular if and only if $1 \leq e \leq k$.*

Furthermore, $G(p^e,k)$ is semiregular if and only if $\gcd(p^{e-1}(p-1),k) = p^{e-1}$.

Let $p = 2$ and $e \geq 2$. $G_2(2^e, k)$ is semiregular if and only if one of the following conditions hold:

(1) $e \in \{1, 2, 3, 4, 6\}$ *whenever* $k = 2$;

(2) $1 \leq e \leq 9$ *whenever* $k = 4$;

(3) $1 \leq e \leq k + \alpha + 2$ *whenever* $k \geq 6$ *and* $2^\alpha \parallel k$.

Moreover, $G(2^e, k)$ is semiregular if and only if one of the following conditions hold:

(1) $e \in \{1, 2, 4\}$ *whenever* $k = 2$;

(2) $1 \leq e \leq 5$ *whenever* $k = 4$;

(3) $1 \leq e \leq \alpha + 2$ *whenever* $k \geq 6$ *and* $2^\alpha \parallel k\ 2^\alpha k$.

REMARK 3.3. (1) Let $2^r \parallel k$, $r > 0$. If $T(2^e, k, 0)$ is not semiregular then $e > k + r + 2$.

(2) Let $k > 2$ be an odd integer. Then, looking at the indegrees of 0 and 2^k in $T(2^e, k, 0)$ one can inspect that $T(2^e, k, 0)$ is semiregular if and only if $1 \leq e \leq k + 1$.

REMARK 3.4. Note that for all $r > 1$ and prime p, $T(p^e, k^r, 0)$ is semiregular whenever $T(p^e, k, 0)$ is semiregular. Thus we can conclude that for any cycle vertex a, $T(n, k^r, a)$ is semiregular whenever $T(n, k, a)$ is semiregular.

THEOREM 3.5 ([11]). *The following statements are equivalent*:

(1) *the digraph $G(n, k)$ is semiregular*;

(2) *the trees attached to all cycle vertices in $G(n, k)$ are isomorphic*;

(3) $\gcd(p_i^{e_i}(p_i - 1), k) = p_i^{e_i - 1}$ *for all i such that $1 \leq i \leq r$*.

REMARK 3.6. If $G(n, k)$ is semiregular and $m = \prod_{i=1}^{r} p_i^{e_i - 1}$, then we can write $G(n, k)$ more explicitly as

$$G(n, k) = a_1 O_{t_1}^m \cup a_2 O_{t_2}^m \ldots \cup a_l O_{t_l}^m,$$

where $a_i = A_{t_i}(G(n, k))$ for all $t_i \in \mathcal{A}(G(n, k))$.

The following result is another characterization for the semiregular digraphs $G(n, k)$.

COROLLARY 3.7. *The digraph $G(n, k)$ is semiregular if and only if $T(n, k, 0) \cong T(n, k, 1)$*.

Proof. In view of Theorem 3.5 we only prove the converse. Since

$$\prod_{i=1}^{r} p_i^{e_i-1} = |T(n,k,0)| = |T(n,k,1)| = \gcd(p_i^{e_i-1}(p_i-1), k^{s_i}),$$

where $s_i = h(T(p_i^{e_i},k))$, we observe that k must be odd and $\gcd(p_i-1,k) = 1$ for all i. We also have $p^{e_i - \lceil \frac{e_i}{k} \rceil} \parallel k$, and the semiregularity of $T(n,k,0)$ implies $\lceil \frac{e_i}{k} \rceil \le 2$ for all i. If $\lceil \frac{e_i}{k} \rceil = 2$ for some i, then $p_i^{e_i-2} \le k < e$ which is a contradiction. Using Theorem 3.5 again, the result will follow. □

Lemma 3.8. *The tree $T(n,k,0)$ is semiregular if and only if $T(p_i^{e_i},k,0)$ is semiregular for all i.*

Proof. Suppose that $T(p_i^{e_i},k,0)$ is not semiregular for some i. Then there exists vertices a and b in $T(p_i^{e_i},k,0)$ such that $\text{indeg}_k^{p_i^{e_i}}(a) \ne \text{indeg}_k^{p_i^{e_i}}(b)$, and it follows that $\mathrm{indeg}_{\mathrm{k}}^{\mathrm{p_i^{e_i}}}(0,0,\dots,\mathrm{a},\dots,0) \ne \mathrm{indeg}_{\mathrm{k}}^{\mathrm{p_i^{e_i}}}(0,0,\dots,\mathrm{b},\dots,0)$ in $T(n,k,0)$. Thus $T(n,k,0)$ is not semiregular.

The converse is straightforward. □

Definition 3.9. Let $M \ge 2$ be an integer. The digraph $G(n,k)$ is symmetric of order M if its set of components can be partitioned into subsets of size M each containing M isomorphic components.

Symmetric digraphs $G(n,k)$ have been characterized completely in [5] and [14]. The following results show the relationship between the symmetric property and the semiregularity property of $G(n,k)$.

Proposition 3.10 ([11]). *Let p be an odd prime. $G(p^e,k)$ is symmetric of order p if and only if $G(p^e,k)$ is semiregular and $k \equiv 1 \pmod{p-1}$.*

There was a result proved in [11], which says that if $G(n_1,k)$ and $G(n_2,k)$ are symmetric of order m_1 and m_2 respectively, where $\gcd(n_1,n_2) = 1$, then $G(n_1 n_2,k)$ is symmetric of order $m_1 m_2$. Using this, we can generalized Proposition 3.10 as follows:

Theorem 3.11. *Let n be an odd integer. Then $G(n,k)$ is symmetric of order $\prod_{i=1}^{r} p_i$ if and only if $G(n,k)$ is semiregular and $k \equiv 1 \ \big(\mathrm{mod}\ \lambda\big(\prod_{i=1}^{r} p_i\big)\big)$.*

Proof. Assume that $G(n,k)$ is symmetric of order $\prod_{i=1}^r p_i$. Then there exists at least $\big(\prod_{i=1}^r p_i\big) - 1$ components, say, $C_1, C_2, \ldots, C_{(\prod_{i=1}^r p_i)-1}$, which are isomorphic to $T(n,k,0)$. Now since $|T(n,k,0)| = |T(p_1^{e_1},k,0)| \times |T(p_2^{e_2},k,0)| \times \cdots \times |T(p_r^{e_r},k,0)| = \prod_{i=1}^r p_i^{e_i-1}$, we obtain $|C_1| + |C_2| + \cdots + |C_{(\prod_{i=1}^r p_i)-1}| + |T(0)| = n$, and by Theorem 3.5, $G(n,k)$ must be semiregular. Also, since $\prod_{i=1}^r p_i = A_1(G(n,k)) = \prod_{i=1}^r [\gcd(\lambda(p_i^{e_i}), k-1) + 1]$ it follows that $k \equiv 1 \pmod{\lambda\big(\prod_{i=1}^r p_i\big)}$.

The converse follows immediately from the result stated above. □

4. The trees $T(n,k,0)$, $T(n,k,1)$, and $T(n,k,(1,0))$

The study of the tree structure of $G(n,k)$ basically comes down to analyzing the structure of the trees attached to cycle vertices of the type 0, 1 and $(1,0)$. In this section, we prove the uniqueness of the factorization of the trees $T(n,k,1)$ and $T(n,k,0)$. The factorization of a tree attached to cycle vertices of the type $(1,0)$ is a little more delicate. Although it is not unique in general, but it does in most cases.

Notation. (1) For every integer a and a prime q, we denote $\upsilon_q(a)$ to be the highest power of q dividing a. Let C be a subdigraph of $G(n,k)$. Then we define

$$\upsilon_q(C) = \{\upsilon_q(\mathrm{indeg}_k^n(c))\colon \text{for every vertex } c \in C\}.$$

(2) Let C be a component of $G(n,k)$. We denote $n(C,h)$ to be the number of vertices of positive indegree at height h in C.

(3) We denote by $\mathcal{F}^t(C)$, the set of all those vertices at height t in a subdigraph C of $G(n,k)$.

For the rest of this paper, we assume $k_1 > 1$ and $k_2 > 1$ to be integers such that $k_1 \neq k_2$.

Proposition 4.1. *$T(n,k_1,1) \cong T(n,k_2,1)$ if and only if* $\gcd(\lambda(n),k_1) = \gcd(\lambda(n),k_2)$.

Proof. Assume $T(n,k_1,1) \cong T(n,k_2,1)$. Since $\mathrm{indeg}_{k_1}^n(1) = \mathrm{indeg}_{k_2}^n(1)$, then it follows easily that $\gcd(\lambda(n),k_1) = \gcd(\lambda(n),k_2)$. Now we prove the converse. Assume that $\gcd(\lambda(n),k_1) = \gcd(\lambda(n),k_2)$, then

$$\mathrm{indeg}_{k_1}^n(1) = \prod_{i=1}^r \gcd(\lambda(p_i^{e_i}),k_1) = \prod_{i=1}^r \gcd(\lambda(p_i^{e_i}),k_2) = \mathrm{indeg}_{k_2}^n(1).$$

Since $\gcd(\lambda(n), k_1^t) = \gcd(\lambda(n), k_2^t)$ we also have $\mathrm{indeg}^n_{k_1^t}(1) = \mathrm{indeg}^n_{k_2^t}(1)$ for all $t > 1$. If both $T(n, k_1, 1)$ and $T(n, k_2, 1)$ are complete trees then we are done. So we consider $T(n, k_1, 1)$ and $T(n, k_2, 1)$ to be non-complete trees. Now for all $t > 1$, we have

$$\begin{aligned}|\mathcal{F}^t(T(n, k_1, 1))| &= \mathrm{indeg}^n_{k_1^t}(1) - \mathrm{indeg}^n_{k_1^{t-1}}(1)\\ &= \mathrm{indeg}^n_{k_2^t}(1) - \mathrm{indeg}^n_{k_2^{t-1}}(1) = |\mathcal{F}^t(T(n, k_2, 1))|,\end{aligned}$$

which follows that

$$\begin{aligned}n(T(n, k_1, 1), h) &= \frac{|\mathcal{F}^{h+1}(T(n, k_1, 1))|}{\mathrm{indeg}^n_{k_1}(1)}\\ &= \frac{|\mathcal{F}^{h+1}(T(n, k_2, 1))|}{\mathrm{indeg}^n_{k_2}(1)} = n(T(n, k_2, 1), h),\end{aligned}$$

at each height h of $T(n, k_j, 1)$ for $j = 1, 2$. If $T(n, k_1, 1) \ncong T(n, k_2, 1)$ then at some height h in $T(n, k_1, 1)$ and $T(n, k_2, 1)$, chosen to be the least, there exist vertices say a and b respectively, such that $\mathrm{indeg}^n_{k_1^2}(a) \neq \mathrm{indeg}^n_{k_2^2}(b)$, which is not possible. Hence, the result follows. □

Remark 4.2. Using Lemma 2.19, the above Theorem is also true in more general terms. That is, if both $T(n, k_1, a)$ and $T(n, k_2, b)$, where a and b are fixed points, are semiregular, then $T(n, k_1, a) \cong T(n, k_2, b)$ if and only if $\mathrm{indeg}^n_{k_1^r}(a) = \mathrm{indeg}^n_{k_2^r}(b)$ for all $r \geq 1$.

Corollary 4.3. *$T(n, k_1, 1) \cong T(n, k_2, 1)$ if and only if $T(p_i^{e_i}, k_1, 1) \cong T(p_i^{e_i}, k_2, 1)$ for all i.*

Before proving the uniqueness of the factorization of $T(n, k, 0)$, we first study the structure of $T(p^e, k, 0)$, particularly when it is not semiregular.

Let p be a prime. If $T(p^e, k, 0)$ is semiregular and is of height 2, then the vertices a of positive indegree at height 1 are of the type where $p^k \parallel a$, and $n(T(p^e, k, 0), 1) = p - 1$. We now look at the structure of $T(p^e, k, 0)$, having height m, when it is not semiregular. Observe that $\lceil \frac{e}{k^i} \rceil > k$ for all i such that $1 \leq i \leq m - 2$, and $\lceil \frac{e}{k^{m-1}} \rceil < k$. Also one can see that $(\lceil \frac{e}{k^i} \rceil - 1)k^i < e \leq (\lceil \frac{e}{k^i} \rceil)k^i$ for all i such that $1 \leq i \leq m - 1$. Denote $a_i = \lceil \frac{e}{k^i} \rceil - 1$.

We claim that

$$\min\{\upsilon_p(T(p^e,k,0))\} = \upsilon_p(\mathrm{indeg}_k^{p^e}(p^k))$$

and

$$\max\{\upsilon_p(T(p^e,k,0))\} = \upsilon_p(\mathrm{indeg}_k^{p^e}(0)),$$

whenever $\lceil \frac{e}{k} \rceil \geq 2$, and

$$\max\{\upsilon_p(T(p^e,k,0)-\{0\})\} = \upsilon_p(\mathrm{indeg}_k^{p^e}(p^{a_1k})),$$

whenever $\lceil \frac{e}{k} \rceil \geq 3$. The claim is obvious if $\lceil \frac{e}{k} \rceil = 2$. So we take $\lceil \frac{e}{k} \rceil \geq 3$. First we consider the case when $p^\alpha \parallel k$, where $\alpha > 0$. Because of Lemmas 2.5 and 2.6, it is enough to consider the indegrees of the vertices of the form

$$\begin{gathered} p^k, p^{2k}, \dots, p^{a_1k}, \\ p^{k^2}, p^{2k^2}, \dots, p^{a_2k^2}, \\ \vdots \\ p^{k^{m-1}}, p^{2k^{m-1}}, \dots, p^{a_{m-1}k^{m-1}}. \end{gathered}$$

Since $e > \left(\lceil \frac{e}{k} \rceil - 1\right)k$ it follows that $e - lk > k > \alpha$ whenever $1 \leq l < \left(\lceil \frac{e}{k} \rceil - 1\right)$, and thus for an odd prime p we obtain

$$\mathrm{indeg}_k^{p^e}(p^{lk}) = p^{(k-1)l+\alpha}\gcd(p-1,k)$$

for all l such that $1 \leq l < \left(\lceil \frac{e}{k} \rceil - 1\right) = a_1$. Also,

$$\mathrm{indeg}_k^{p^e}(p^{a_1k}) = p^{(k-1)a_1+\alpha}\gcd(p-1,k) \text{ or } p^{e-a_1-1}\gcd(p-1,k).$$

Since $(k-1)(a_1-1)+\alpha < e-a_1-1$, then in either case we have $\mathrm{indeg}_k^{p^e}(p^{a_1k}) > \mathrm{indeg}_k^{p^e}(p^{(a_1-1)k})$, and so we can write

$$\mathrm{indeg}_k^{p^e}(p^{lk}) < \mathrm{indeg}_k^{p^e}(p^{(l+1)k}) \quad \text{for all } l \text{ such that } 1 \leq l \leq a_1 - 1. \tag{4.1}$$

Thus we can conclude that

$$\min\{\upsilon_p(T(p^e,k,0))\} = \upsilon_p(\mathrm{indeg}_k^{p^e}(p^k)),$$

and

$$\max\{\upsilon_p(T(p^e,k,0)-\{0\})\} = \upsilon_p(\mathrm{indeg}_k^{p^e}(p^{a_1k}))$$

Finally, it is easy to verify $a_1(k-1)+\min\{e-a_1k-1,\alpha\} \leq e - \lceil \frac{e}{k} \rceil$, and hence the remaining claim follows.

If $p \nmid k$, the claim is obvious.

Now let $p = 2$. The claim is again obvious if k is odd. So we take $2^r \parallel k$, and since $T(2^e, k, 0)$ is not semiregular we must have $\text{indeg}_k^{2^e}(2^{lk}) = 2^{(k-1)l+r+1}$ whenever $l < \left(\lceil \frac{e}{k} \rceil - 1\right) = a_1$. Using Lemma 2.5, we can compute

$$\text{indeg}_k^{2^e}(2^{a_1k}) = \begin{cases} 2^{e-a_1-1} \text{ or } 2^{a_1(k-1)+r+1} & \text{if } e - a_1k \geq 3, \\ 2^{a_1(k-1)+1} & \text{if } e - a_1k = 2, \\ 2^{e-a_1-1} & \text{if } e - a_1k = 1, \end{cases}$$

and it can be easily checked in all the cases that

$$\upsilon_2(\text{indeg}_k^{2^e}(0)) \geq \upsilon_2(\text{indeg}_k^{2^e}(2^{a_1k})),$$

and

$$\upsilon_2(\text{indeg}_k^{2^e}(2^{a_1k})) \geq \upsilon_2(\text{indeg}_k^{2^e}(2^{(a_1-1)k})),$$

as required.

Remark 4.4. Using the fact that $e > \left(\lceil \frac{e}{k} \rceil - 1\right)k$, the following results can be obtained easily.

(1) $\upsilon_p(\text{indeg}_k^{p^e}(0)) \geq \upsilon_p(\text{indeg}_k^{p^e}(1))$ for every odd prime p.

(2) $\upsilon_2(\text{indeg}_k^{2^e}(0)) \geq \upsilon_2(\text{indeg}_k^{2^e}(1))$, whenever $e \neq 3$.

(3) $\upsilon_2(\text{indeg}_k^{2^e}(1)) \geq \upsilon_2(\text{indeg}_k^{2^e}(0))$ whenever $e = 3$. Moreover, the equality holds if $k \geq 3$. This can be proved directly using the definition.

Lemma 4.5. *Let p be a prime, and $e > 1$ be an integer. Let $h(T(p^e, k_i, 0)) > 1$ for $i = 1, 2$. Then $T(p^e, k_1, 0) \cong T(p^e, k_2, 0)$ if and only if $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil = 2$ and $\text{indeg}_{k_1}^{p^e}(p^{k_1}) = \text{indeg}_{k_2}^{p^e}(p^{k_2})$.*

Proof. Assume that $T(p^e, k_1, 0) \cong T(p^e, k_2, 0)$. Since

$$\upsilon_p(\text{indeg}_{k_1}^{p^e}(p^{k_1})) = \upsilon_p(\text{indeg}_{k_2}^{p^e}(p^{k_2})), \tag{4.2}$$

then p must divide at least one of k_1 or k_2.

Without any loss, take $p^\alpha \parallel k_1$, where $\alpha > 0$. The result is clear if both $T(p^e, k_1, 0)$ and $T(p^e, k_2, 0)$ are semiregular. So we assume $T(p^e, k_1, 0)$ and $T(p^e, k_2, 0)$ to be non-semiregular. If $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil \geq 4$, then

$$2(k_1 - 1) + \alpha = \upsilon_p(\text{indeg}_{k_1}^{p^e}(p^{2k_1})) = \upsilon_p(\text{indeg}_{k_2}^{p^e}(p^{2k_2})) = 2(k_2 - 1) + \beta,$$

where $p^\beta \parallel k_2$, $\beta \geq 0$, which along with (4.2) yields $k_1 = k_2$, a contradiction.

Let $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil = 3$, and first take p to be an odd prime such that $p^\beta \parallel k_2$, $\beta > 0$. Without any loss of generality, take $k_1 < k_2$. Then it must follow that $e - 2k_1 - 1 > \alpha$. This is because, using (4.2) and the fact that $p^\beta \parallel \alpha - \beta$, $e - 2k_1 - 1 \leq \alpha$ implies that $e \leq 2k_1 + \alpha + 1 = k_2 + k_1 + \beta + 1 = k_2 + k_2 + 2\beta - \alpha + 1$ which leads to $\lceil \frac{e}{k_2} \rceil = 2$, a contradiction. So $2(k_1 - 1) + \alpha = \upsilon_p(\operatorname{indeg}_{k_1}^{p^e}(p^{2k_1})) = \upsilon_p(\operatorname{indeg}_{k_2}^{p^e}(p^{2k_2})) = 2(k_2 - 1) + \min\{e - 2k_2 - 1, \beta\}$, which in any case leads to a contradiction. Next, if $p \nmid k_2$ then using (4.2) again we automatically have $k_1 < k_2$. Similarly in this case we must have $e - 2k_1 - 1 > \alpha$, and since $\upsilon_p(\operatorname{indeg}_{k_1}^{p^e}(p^{2k_1})) = \upsilon_p(\operatorname{indeg}_{k_2}^{p^e}(p^{2k_2}))$ then solving it with (4.2) we get another contradiction.

Hence, $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil = 2$.

Finally,

$$\begin{aligned}\frac{p^{e-2}(p-1)}{\operatorname{indeg}_{k_1}^{p^e}(p^{k_1})} &= n(T(p^e, k_1, 0), 1) \\ &= n(T(p^e, k_2, 0), 1) = \frac{p^{e-2}(p-1)}{\operatorname{indeg}_{k_2}^{p^e}(p^{k_2})},\end{aligned}$$

implies $\gcd(p-1, k_1) = \gcd(p-1, k_2)$.

Now consider $p = 2$, and let $2^\beta \parallel k_2$ for $\beta > 0$. By inspection it is easy to see that $k_1 \neq 2$ and $k_2 \neq 2$. Again, without any loss of generality, take $k_1 < k_2$. If $e - 2k_2 \geq 3$, or $e - 2k_2 \leq 2$ and $e - 2k_1 \leq 2$, then arguing exactly as in the odd prime case we will get a contradiction. If $e - 2k_2 \leq 2$ and $e - 2k_1 \geq 3$ then

$$\begin{aligned}2(k_1 - 1) + \alpha + 1 &= \upsilon_2(\operatorname{indeg}_{k_1}^{2^e}(2^{2k_1})) \\ &= \upsilon_2(\operatorname{indeg}_{k_2}^{2^e}(2^{2k_2})) = 2(k_2 - 1) + \min\{e - 2k_2 - 1, \beta\},\end{aligned}$$

which implies that $e = 2k_1 + \alpha + 2$, a contradiction, or with (4.2), $\alpha = \beta + 1$, again a contradiction as $2^\beta \parallel \alpha - \beta$. Next, assume k_2 to be odd, then (4.2) implies $k_1 + \alpha = k_2 - 1$ and so $k_1 < k_2$. If $e - 2k_1 \geq 3$ then

$$\begin{aligned}2(k_1 - 1) + \alpha + 1 &= \upsilon_2(\operatorname{indeg}_{k_1}^{2^e}(2^{2k_1})) \\ &= \upsilon_2(\operatorname{indeg}_{k_2}^{2^e}(2^{2k_2})) = 2(k_2 - 1),\end{aligned}$$

which leads to $k_1 = k_2$, a contradiction. If $e - 2k_1 \leq 2$ then it similarly follows that $k_1 = k_2 + 1$, which is also a contradiction. Thus in this case we must also have $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil = 2$.

We now prove the converse. Since $\lceil \frac{e}{k_1} \rceil = \lceil \frac{e}{k_2} \rceil = 2$ and $\gcd(p-1, k_1) = \gcd(p-1, k_2)$, it is enough to show that $n(T(p^e, k_1, 0), 1) = n(T(p^e, k_2, 0), 1)$. But, $n(T(p^e, k_1, 0), 1) = \frac{|\mathcal{F}^2(T(p^e, k_1, 0))|}{\operatorname{indeg}_{k_1}^{p^e}(p^{k_1})} = \frac{|\mathcal{F}^2(T(p^e, k_2, 0))|}{\operatorname{indeg}_{k_2}^{p^e}(p^{k_2})} = n(T(p^e, k_2, 0), 1)$, as desired. □

NOTE. Let p be a prime, and $p \nmid k_1$ and $p \nmid k_2$. Then $T(p^e, k_1, 0) \cong T(p^e, k_2, 0)$ if and only if $k_1 \geq e$ and $k_2 \geq e$.

THEOREM 4.6. *The trees* $T(n, k_1, 0) \cong T(n, k_2, 0)$ *if and only if* $T(p_i^{e_i}, k_1, 0) \cong T(p_i^{e_i}, k_2, 0)$ *for all* i.

PROOF. Assume $T(n, k_1, 0) \cong T(n, k_2, 0)$. Then $\text{indeg}^n_{k_1}(0) = \text{indeg}^n_{k_2}(0)$ which implies that $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil$ for all i. By Lemma 2.21 we can assume $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil > 1$ for all i, and so we obtain $h(T(p_i^{e_i}, k_j, 0)) = 2$ for all i and for $j = 1, 2$. For all primes p_i and $p \neq p_i$, we observe that

$$
\begin{aligned}
\upsilon_{p_i}(\text{indeg}^{p_i^{e_i}}_{k_1}(p_i^{k_1})) &= \min\{\upsilon_{p_i}(T(n, k_1, 0))\} \\
&= \min\{\upsilon_{p_i}(T(n, k_2, 0))\} \\
&= \upsilon_{p_i}(\text{indeg}^{p_i^{e_i}}_{k_2}(p_i^{k_2})),
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
\upsilon_{p_i}\Big(\prod_{j\neq i} \gcd(p_j - 1, k_1)\Big) &= \max\{\upsilon_{p_i}(T(n, k_1, 0))\} \\
&= \max\{\upsilon_{p_i}(T(n, k_2, 0))\} \\
&= \upsilon_{p_i}\Big(\prod_{j\neq i} \gcd(p_j - 1, k_2)\Big),
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
\upsilon_{p}\Big(\prod_{\forall j} \gcd(p_j - 1, k_1)\Big) &= \max\{\upsilon_{p}(T(n, k_1, 0))\} \\
&= \max\{\upsilon_{p}(T(n, k_2, 0))\} \\
&= \upsilon_{p}\Big(\prod_{\forall j} \gcd(p_j - 1, k_2)\Big).
\end{aligned} \tag{4.5}
$$

Then it is easy to see that both $\gcd(p_i - 1, k_1)$ and $\gcd(p_i - 1, k_2)$ have the same prime divisors. Furthermore, if both $\gcd(p_i - 1, k_1)$ and $\gcd(p_i - 1, k_2)$ are greater than 1 for odd primes p_i, and using the fact that

$$
\begin{aligned}
&\min\{\upsilon_p\{a \in T(n, k_1, 0)\colon \upsilon_{p_i}(\text{indeg}^n_{k_1}(a)) \leq \upsilon_{p_i}(\text{indeg}^n_{k_1}(b)), \text{for all } b\}\} \\
&\quad = \min\{\upsilon_p\{a \in T(n, k_2, 0)\colon \upsilon_{p_i}(\text{indeg}^n_{k_2}(a)) \leq \upsilon_{p_i}(\text{indeg}^n_{k_2}(b)), \text{for all } b\}\}
\end{aligned}
$$

for every prime $p \neq p_i$, we obtained $\gcd(p_i - 1, k_1) = \gcd(p_i - 1, k_2)$.

Now we are left to prove that $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil = 2$ for all i. If both $T(p_i^{e_i}, k_1, 0)$ and $T(p_i^{e_i}, k_2, 0)$ are semiregular then by Lemmas 3.2 and 4.5 they must be isomorphic. Suppose for some i, $T(p_i^{e_i}, k_1, 0)$ is semiregular but not $T(p_i^{e_i}, k_2, 0)$. Then using (4.3) we get $\max\{\upsilon_{p_i}(T(p_i^{e_i}, k_2, 0))\} = \min\{\upsilon_{p_i}(T(p_i^{e_i}, k_2, 0))\}$, which is not possible as $\gcd(p_i - 1, k_2) = 1$. Thus we can assume both $T(p_i^{e_i}, k_1, 0)$ and $T(p_i^{e_i}, k_2, 0)$ to be non-semiregular for all i.

Suppose $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil > 2$ for some i. As seen in the proof of Lemma 4.5, it is enough to prove that

$$\upsilon_{p_i}(\mathrm{indeg}_{k_1}^{p_i^{e_i}}(p_i^{2k_1})) = \upsilon_{p_i}(\mathrm{indeg}_{k_2}^{p_i^{e_i}}(p_i^{2k_2})), \tag{4.6}$$

as this will lead to a contradiction. Now, because of (4.3), every prime $p_j \mid n$ must divide at least one of k_1 or k_2. If some prime p_j divides exactly one of them, say, k_1, then every other such primes must also divide only k_1, with $\upsilon_{p_j}(k_1) = \alpha$ for all such primes p_j. Then it follows that $k_1 + \alpha = k_2$, and thus $k_l - 1 > \upsilon_{p_i}(\prod_{j \neq i} \gcd(p_j - 1, k_l))$ for $l = 1, 2$. If $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil \geq 4$, then we look at the ordering of the elements in the sets $\upsilon_{p_i}(T(n, k_1, 0))$ and $\upsilon_{p_i}(T(n, k_2, 0))$ while observing that

$$\upsilon_{p_i}(\mathrm{indeg}_{k_l}^{n}(p_i^{2k_l})) > \upsilon_{p_i}(\mathrm{indeg}_{k_l}^{n}(p_i^{k_l})) + \upsilon_{p_i}\Big(\prod_{j \neq i} \gcd(p_j - 1, k_l)\Big)$$

for $l = 1, 2$, to obtain (4.6) as desired. Finally, let $\lceil \frac{e_i}{k_1} \rceil = \lceil \frac{e_i}{k_2} \rceil = 3$ and notice that we already have

$$\begin{aligned}\upsilon_{p_i}(\mathrm{indeg}_{k_1}^{n}(p_i^{2k_1})) &= 2(k_1 - 1) + \upsilon_{p_i}(k_1) \\ &> \upsilon_{p_i}(\mathrm{indeg}_{k_1}^{n}(p_i^{k_1})) + \upsilon_{p_i}\Big(\prod_{j \neq i} \gcd(p_j - 1, k_1)\Big).\end{aligned}$$

From this, and using the hypothesis, it will also imply that

$$\upsilon_{p_i}(\mathrm{indeg}_{k_2}^{n}(p_i^{2k_2})) > \upsilon_{p_i}(\mathrm{indeg}_{k_2}^{n}(p_i^{k_2})) + \upsilon_{p_i}\Big(\prod_{j \neq i} \gcd(p_j - 1, k_2)\Big),$$

and we are done. The proof is now complete. □

Even though the factorization of a tree attached to the cycle vertices 0 and 1 in $G(n, k)$ is unique, however, this is not always true for trees attached to a cycle vertex of the form $(1, 0)$. To illustrate this we give the following examples.

EXAMPLES. (1) Consider $T(19^2 \times 3^{11}, 9, (1,0))$ and $T(19^2 \times 3^{11}, 15, (1,0))$, where $h(T(19^2, 9, 1)) = 1 < 2 = h(T(19^2, 15, 1))$ and $h(T(3^{11}, 15, 0)) = 1 < 2 = h(T(3^{11}, 9, 0))$. Since $\operatorname{indeg}_{9^i}^{19^2\times 3^{11}}(1,0) = \operatorname{indeg}_{15^i}^{19^2\times 3^{11}}(1,0)$ for $i = 1, 2$, and $T(3^{11}, 9, 0)$ is semiregular, then by Remark 4.2 the isomorphism $T(19^2 \times 3^{11}, 9, (1,0)) \cong T(19^2 \times 3^{11}, 15, (1,0))$ holds.

(2) Similarly, we have $T(5^2 \times 2^{13}, 12, (1,0)) \cong T(5^2 \times 2^{13}, 14, (1,0))$, but $h(T(5^2, 12, 1)) \neq h(T(5^2, 14, 1))$ and $h(T(2^{13}, 12, 0)) \neq h(T(2^{13}, 14, 0))$.

In the above two examples showing the isomorphism of the trees attached to a cycle vertex of the type $(1,0)$, instead of using Remark 4.2, one can show the isomorphism directly by looking into the structure of the trees as they all have height less than or equal to 2.

In the following theorem we show that the only situation preventing the uniqueness of the factorization of trees attached to a cycle vertex of the type $(1,0)$ is the property where $h(T(p_l^{e_l}, k_1, 0)) = 1 < h(T(p_l^{e_l}, k_2, 0))$.

For the rest of this paper, unless stated otherwise, we take $n = n_1 n_2$, where $\gcd(n_1, n_2) = 1$.

NOTATION. Denote a_n to be a vertex in $G(n,k)$.

THEOREM 4.7. *Consider the trees $T(n, k_l, (1_{n_1}, 0_{n_2}))$ such that, for all primes $p_j \mid n_2$ and $l = 1, 2$, either $h(T(p_j^{e_j}, k_l, 0)) > 1$ or $h(T(p_j^{e_j}, k_l, 0)) = 1$. Then $T(n, k_1, (1_{n_1}, 0_{n_2})) \cong T(n, k_2, (1_{n_1}, 0_{n_2}))$ if and only if $T(p_i^{e_i}, k_1, 1) \cong T(p_i^{e_i}, k_2, 1)$ and $T(p_j^{e_j}, k_1, 0) \cong T(p_j^{e_j}, k_2, 0)$ for all primes $p_i^{e_i} \parallel n_1$ and $p_j^{e_j} \parallel n_2$.*

PROOF. Suppose that $T(n, k_1, (1_{n_1}, 0_{n_2})) \cong T(n, k_2, (1_{n_1}, 0_{n_2}))$. By Lemma 2.21 we can assume $h(T(p_j^{e_j}, k_l, 0)) > 1$ for all primes $p_j \mid n_2$ and $l = 1, 2$. Since

$$
\begin{aligned}
&\prod_{p_i \mid n_1} \gcd(p_i^{e_i - 1}(p_i - 1), k_1) \prod_{p_j \mid n_2} p_j^{e_j - \lceil \frac{e_j}{k_1} \rceil} \\
&= \operatorname{indeg}_{k_1}^n (1_{n_1}, 0_{n_2}) \\
&= \operatorname{indeg}_{k_2}^n (1_{n_1}, 0_{n_2}) \\
&= \prod_{p_i \mid n_1} \gcd(p_i^{e_i - 1}(p_i - 1), k_2) \prod_{p_j \mid n_2} p_j^{e_j - \lceil \frac{e_j}{k_2} \rceil},
\end{aligned}
\tag{4.7}
$$

and

$$\begin{aligned}\prod_{p_i|n_1}&\gcd(p_i^{e_i-1}(p_i-1),k_1^{r_i})\\ &=|T(n,k_1,(1_{n_1},0_{n_2}))|\\ &=|T(n,k_2,(1_{n_1},0_{n_2}))|\\ &=\prod_{p_i|n_1}\gcd(p_i^{e_i-1}(p_i-1),k_2^{s_i}),\end{aligned} \tag{4.8}$$

where $r_i = h(T(p_i^{e_i}, k_1, 1))$, $s_i = h(T(p_i^{e_i}, k_2, 1))$, then it follows that $p_i \mid k_1$ if and only if $p_i \mid k_2$ for all primes $p_i \mid n_1$.

Note that for all primes $p_j \mid n_2$, we have

$$\begin{aligned}\upsilon_{p_j}(\mathrm{indeg}^n_{k_1}(1_{n_1},0_{n_2}))&=\upsilon_{p_j}(\mathrm{indeg}^{n_1}_{k_1}(1))+e_j-\left\lceil\frac{e_j}{k_1}\right\rceil\\ &=\upsilon_{p_j}(\mathrm{indeg}^{n_1}_{k_2}(1))+e_j-\left\lceil\frac{e_j}{k_2}\right\rceil\\ &=\upsilon_{p_j}(\mathrm{indeg}^n_{k_2}(1_{n_1},0_{n_2})),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\min\{\upsilon_{p_j}(T(n,k_1,(1_{n_1},0_{n_2})))\}&=\upsilon_{p_j}(\mathrm{indeg}^{n_1}_{k_1}(1))+\upsilon_{p_j}(\mathrm{indeg}^{p_j^{e_j}}_{k_1}(p_j^{k_1}))\\ &=\upsilon_{p_j}(\mathrm{indeg}^{n_1}_{k_2}(1))+\upsilon_{p_j}(\mathrm{indeg}^{p_j^{e_j}}_{k_2}(p_j^{k_2}))\\ &=\min\{\upsilon_{p_j}(T(n,k_2,(1_{n_1},0_{n_2})))\},\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\max\{\upsilon_{p_j}(T(n,k_1,(1_{n_1},0_{n_2})))\}&=\max\left\{\upsilon_{p_j}\left(T\left(\frac{n_2}{p_j^{e_j}},k_1,0\right)\right)\right\}\\ &=\max\left\{\upsilon_{p_j}\left(T\left(\frac{n_2}{p_j^{e_j}},k_2,0\right)\right)\right\}\\ &=\max\{\upsilon_{p_j}(T(n,k_2,(1_{n_1},0_{n_2})))\}.\end{aligned} \tag{4.11}$$

Then it is easy to see that $\gcd(p_j-1,k_1)$ and $\gcd(p_j-1,k_2)$ have the same prime divisors for all primes $p_j \mid n_2$. Next, the trees $T(p_j^{e_j}, k_1, 0)$ and $T(p_j^{e_j}, k_2, 0)$ are either both semiregular or both non-semiregular for all primes $p_j \mid n_2$. Because if for some prime $p_j \mid n_2$, say $T(p_j^{e_j}, k_1, 0)$ is semiregular but not $T(p_j^{e_j}, k_2, 0)$, then from (4.9) and (4.10) we obtain

$$\begin{aligned}\max\{\upsilon_{p_j}(T(p_j^{e_j},k_2,0))\}&=e_j-\left\lceil\frac{e_j}{k_2}\right\rceil\\ &=k_2-1+\upsilon_{p_j}(\mathrm{indeg}^{p_j^{e_j}}_{k_2}(p_j^{k_2}))\\ &=\min\{\upsilon_{p_j}(T(p_j^{e_j},k_2,0))\},\end{aligned}$$

which is a contradiction.

Since

$$\begin{aligned}&\min\{\upsilon_p\{a \in T(n,k_1,(1_{n_1},0_{n_2})): \upsilon_{p_j}(\mathrm{indeg}^n_{k_1}(a)) \le \upsilon_{p_j}(\mathrm{indeg}^n_{k_1}(b)), \text{for all } b\}\}\\ &\quad = \min\{\upsilon_p\{a \in T(n,k_2,(1_{n_1},0_{n_2})):\\ &\qquad\qquad \upsilon_{p_j}(\mathrm{indeg}^n_{k_2}(a)) \le \upsilon_{p_j}(\mathrm{indeg}^n_{k_2}(b)), \text{for all } b\}\}\end{aligned}$$

for every primes $p_j \mid n_2$ and $p \neq p_j$, then we can deduce that $\gcd(p_j - 1, k_1) = \gcd(p_j - 1, k_2)$. Finally, we claim that it suffices to show that $\lceil \frac{e_j}{k_1} \rceil = \lceil \frac{e_j}{k_2} \rceil = 2$ for all e_j such that $p_j^{e_j} \parallel n_2$. Because this would imply, using (4.9) and (4.10), that $T(p_j^{e_j}, k_1, 0) \cong T(p_j^{e_j}, k_2, 0)$ for all primes $p_j \mid n_2$, and from (4.7), Proposition 4.1 and Corollary 4.3 it follows that $T(p_i^{e_i}, k_1, 1) \cong T(p_i^{e_i}, k_2, 1)$ for all primes $p_i \mid n_1$, as desired.

Now we proceed to prove the claim, that is proving $\lceil \frac{e_j}{k_1} \rceil = \lceil \frac{e_j}{k_2} \rceil = 2$.

CASE 1. *For some prime $p_j \mid n_2$, let $p_j \nmid k_1$ or $p_j \nmid k_2$, but not both.*

Using (4.8) we get $\upsilon_{p_j}(T(n_1,k_1,1)) = \upsilon_{p_j}(T(n_1,k_2,1)) = \{0\}$, and it then follows from (4.9) and (4.10) that $\lceil \frac{e_j}{k_1} \rceil = \lceil \frac{e_j}{k_2} \rceil$ and $\upsilon_{p_j}(\mathrm{indeg}^{p_j^{e_j}}_{k_1}(p_j^{k_1})) = \upsilon_{p_j}(\mathrm{indeg}^{p_j^{e_j}}_{k_2}(p_j^{k_2}))$, respectively. If $\lceil \frac{e_j}{k_1} \rceil = \lceil \frac{e_j}{k_2} \rceil = a \geq 3$, then arguing similarly as in the proof of the corresponding part of Theorem 4.6 we will get a contradiction.

CASE 2. *Let $p_j^{\alpha} \parallel k_1$, $p_j^{\beta} \parallel k_2$, where $\alpha > 0$, $\beta > 0$ for some prime $p_j \mid n_2$.*

Denote $a_1 = \lceil \frac{e_j}{k_1} \rceil - 1$, $a_2 = \lceil \frac{e_j}{k_2} \rceil - 1$. Consider the sets

$$S_l = \{\upsilon_{p_j}(\mathrm{indeg}^n_{k_l}(a)) : a \in T(n,k_l,(1_{n_1},0_{n_2}))\}, \quad \text{where } l = 1,2.$$

Since the trees $T(n,k_1,(1_{n_1},0_{n_2}))$ and $T(n,k_2,(1_{n_1},0_{n_2}))$ are isomorphic we must have $S_1 = S_2$. If we assume, without any loss, that $k_1 < k_2$ and $\lceil \frac{e}{k_1} \rceil \geq 3$, then arguing similarly as in the proof of Theorem 4.6, it can be proved that

$$k_l - 1 > \upsilon_{p_j}\Big(\prod_{\substack{i \neq j\\ p_i \mid n_2}} \gcd(p_i - 1, k_l)\Big) \quad \text{for } l = 1,2.$$

Also, since $e_j - a_1k_1 - 1 > \alpha$ then after denoting

$$r = \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{n_1}(1)) \quad \text{and} \quad c = \upsilon_{p_j}\Big(\prod_{\substack{i\neq j\\ p_i|n_2}} \gcd(p_i-1,k_l)\Big),$$

we can order the elements of the set S_1 as follows:

$$\begin{aligned} r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{k_1})) &< \cdots \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{k_1})) + c \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{2k_1})) \\ &< \cdots \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{2k_1})) + c \\ &< \cdots \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{(a_1-1)k_1})) + c \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(p_j^{a_1k_1})) \\ &< \cdots \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(0)) \\ &< \cdots \\ &< r + \upsilon_{p_j}(\mathrm{indeg}_{k_1}^{p_j^{e_j}}(0)) + c. \end{aligned}$$

Again using the hypothesis that $T(n,k_1,(1_{n_1},0_{n_2})) \cong T(n,k_2,(1_{n_1},0_{n_2}))$, and after denoting $s = \upsilon_{p_j}(\mathrm{indeg}_{k_2}^{n_1}(1))$, the elements in the set S_2 can also be ordered as

$$\begin{aligned} s + \upsilon_{p_j}(\mathrm{indeg}_{k_2}^{p_j^{e_j}}(p_j^{k_2})) &< \cdots \\ &< s + \upsilon_{p_j}(\mathrm{indeg}_{k_2}^{p_j^{e_j}}(p_j^{k_2})) + c \\ &< s + \upsilon_{p_j}(\mathrm{indeg}_{k_2}^{p_j^{e_j}}(p_j^{2k_2})) \\ &< \cdots \\ &< s + \upsilon_{p_j}(\mathrm{indeg}_{k_2}^{p_j^{e_j}}(p_j^{2k_2})) + c \\ &< \cdots \end{aligned}$$

$$
\begin{aligned}
&< s + \upsilon_{p_j}(\text{indeg}_{k_2}^{p_j^{e_j}}(p_j^{(a_2-1)k_2})) + c \\
&< s + \upsilon_{p_j}(\text{indeg}_{k_2}^{p_j^{e_j}}(p_j^{a_2k_2})) \\
&< \cdots \\
&< s + \upsilon_{p_j}(\text{indeg}_{k_2}^{p_j^{e_j}}(0)) \\
&< \cdots \\
&< s + \upsilon_{p_j}(\text{indeg}_{k_2}^{p_j^{e_j}}(0)) + c.
\end{aligned}
$$

If $\lceil \frac{e_j}{k_1} \rceil \geq 4$ and $\lceil \frac{e_j}{k_2} \rceil \geq 4$, or if $\lceil \frac{e_j}{k_1} \rceil = 3$ and $\lceil \frac{e_j}{k_2} \rceil = 3$, then from the above two sequences we obtain $r + \upsilon_{p_j}(\text{indeg}_{k_1}^{p_j^{e_j}}(p_j^{2k_1})) = s + \upsilon_{p_j}(\text{indeg}_{k_2}^{p_j^{e_j}}(p_j^{2k_2}))$, which along with (4.10) yields $k_1 = k_2$, a contradiction. Also, if $\lceil \frac{e_j}{k_1} \rceil \geq 4$ and $\lceil \frac{e_j}{k_2} \rceil \leq 3$, or if $\lceil \frac{e_j}{k_1} \rceil = 3$ and $\lceil \frac{e_j}{k_2} \rceil = 2$, then $S_1 \neq S_2$ which is not possible. Thus we can conclude that $\lceil \frac{e_j}{k_1} \rceil = \lceil \frac{e_j}{k_2} \rceil = 2$. The proof is now complete □

Recall a result proved in [15] that the trees attached to all cycle vertices in a fundamental constituent of $G(n, k)$ are isomorphic. We end this section by proving a necessary and sufficient condition on the isomorphism of trees in two distinct fundamental constituents of $G(n, k)$. Let $P = \{p_1, p_2, \ldots, p_r\}$ and $P_1, P_2, P_1 \neq P_2$, be subsets of P. We write

$$n = MNR,$$

where

$$M = \prod_{p_i \in P_1, p_i \notin P_2} p_i^{e_i}, \quad N = \prod_{p_i \in P_2, p_i \notin P_1} p_i^{e_i}, \quad R = \prod_{p_i \in P_1 \cup P_2} p_i^{e_i},$$

and so

$$G_{P_1}^*(n, k) \cong T(M, k, 0) \times G_1(N, k) \times T(R, k, 0),$$

$$G_{P_2}^*(n, k) \cong G_1(M, k) \times T(N, k, 0) \times T(R, k, 0).$$

THEOREM 4.8. *Then the trees attached to all cycle vertices in* $G_{P_1}^*(n, k) \cup G_{P_2}^*(n, k)$ *are isomorphic if and only if* $T(p_i^{e_i}, k, 0)$ *is semiregular for all* $p_i \notin P_1 \cap P_2$ *and* $\text{indeg}_{k^r}^n(1_M, 0_N) = \text{indeg}_{k^r}^n(0_M, 1_N)$ *for all* $r \geq 1$.

Proof. Assume that the trees in $G^*_{P_1}(n,k) \cup G^*_{P_2}(n,k)$ are isomorphic. Since $T(MN,k,(1_M,0_N)) \cong T(MN,k,(0_M,1_N))$ then by Lemma 2.19 it follows that $\text{indeg}^{MN}_{k^r}(1_M,0_N) = \text{indeg}^{MN}_{k^r}(0_M,1_N)$ for all $r \geq 1$. Also note that,

$$\begin{aligned}\upsilon_{p_i}(\text{indeg}^M_k(1)) &= \upsilon_{p_i}(\text{indeg}^{MN}_k(1_M,0_N)) \\ &= \upsilon_{p_i}(\text{indeg}^{MN}_k(0_M,1_N)) \\ &= \upsilon_{p_i}(\text{indeg}^{p_i^{e_i}}_k(0)) + \upsilon_{p_i}(\text{indeg}^N_k(1)),\end{aligned} \tag{4.12}$$

$$\begin{aligned}\upsilon_{p_i}(\text{indeg}^M_k(1)) &= \min\{\upsilon_{p_i}(T(MN,k,(1_M,0_N)))\} \\ &= \min\{\upsilon_{p_i}(T(MN,k,(0_M,1_N)))\} \\ &= \min\{\upsilon_{p_i}(T(p_i^{e_i},k,0))\} + \upsilon_{p_i}(\text{indeg}^N_k(1)),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\upsilon_{p_j}(\text{indeg}^M_k(1)) + \upsilon_{p_j}(\text{indeg}^{p_j^{e_j}}_k(0)) &= \upsilon_{p_j}(\text{indeg}^{MN}_k(1_M,0_N)) \\ &= \upsilon_{p_j}(\text{indeg}^{MN}_k(0_M,1_N)) \\ &= \upsilon_{p_j}(\text{indeg}^N_k(1)),\end{aligned} \tag{4.14}$$

(4.15)

$$\begin{aligned}\upsilon_{p_j}(\text{indeg}^M_k(1)) + \min\{\upsilon_{p_j}(T(p_j^{e_j},k,0))\} &= \min\{\upsilon_{p_j}(T(MN,k,(1_M,0_N)))\} \\ &= \min\{\upsilon_{p_j}(T(MN,k,(0_M,1_N)))\} \\ &= \upsilon_{p_j}(\text{indeg}^N_k(1)),\end{aligned}$$

for all primes $p_i \mid M$ and $p_j \mid N$.

If for some prime p_l, $T(p_l^{e_l},k,0)$ has height greater than 1, then we get

$$\max\{\upsilon_{p_l}(T(p_l^{e_l},k,0))\} = \min\{\upsilon_{p_l}(T(p_l^{e_l},k,0))\}.$$

So we are left to show that $\gcd(p_l - 1, k) = 1$. Since

$$\begin{aligned}&\min\{\upsilon_p\{a \in T(MN,k,(1_M,0_N)): \\ &\qquad \upsilon_{p_l}(\text{indeg}^{MN}_k(a)) \leq \upsilon_{p_l}(\text{indeg}^{MN}_k(b)),\ \text{for all } b\}\} \\ &\quad = \min\{\upsilon_p\{a \in T(MN,k,(0_M,1_N)): \\ &\qquad\qquad \upsilon_{p_l}(\text{indeg}^{MN}_k(a)) \leq \upsilon_{p_l}(\text{indeg}^{MN}_k(b)),\ \text{for all } b\}\},\end{aligned}$$

then using (4.12)–(4.15), one can deduce $\upsilon_p(\gcd(p_l - 1, k)) = 0$ for every prime $p \neq p_l$, as required.

The converse follows immediately from Remark 4.2. □

5. Isomorphism of $G_P^*(n,k_1)$ and $G_P^*(n,k_2)$, and factorization of $G(n,k)$

We start this section by proving a necessary and sufficient condition for the isomorphism of the trees in fundamental constituents $G_P^*(n,k_1)$ and $G_P^*(n,k_2)$ of $G(n,k_1)$ and $G(n,k_2)$, respectively.

THEOREM 5.1. *The trees* $T(n,k_1,(1_{n_1},0_{n_2})) \cong T(n,k_2,(1_{n_1},0_{n_2}))$ *if and only if all the following conditions are satisfied*:

(1) $T(p_j^{e_j},k_1,0) \cong T(p_j^{e_j},k_2,0)$ *whenever*

- $h(T(p_j^{e_j},k_1,0)) > 1$,
- $h(T(p_j^{e_j},k_2,0)) > 1$,

for primes $p_j \mid n_2$.

(2) $\text{indeg}^n_{k_1^r}(1_{n_1},0_{n_2}) = \text{indeg}^n_{k_2^r}(1_{n_1},0_{n_2})$ *for all* $r \geq 1$.

(3) *For primes* $p_i \mid n_1$, $T(p_i^{e_i},k_1,1) \cong T(p_i^{e_i},k_2,1)$ *whenever* $\gcd(\lambda(p_i^{e_i}),k_1) = \gcd(\lambda(p_i^{e_i}),k_2)$.

(4) *For primes* $p_j \mid n_2$, $T(p_j^{e_j},k_{i_1},0)$ *is semiregular whenever* $h(T(p_j^{e_j},k_{i_2},0)) = 1$, *where* $i_1 \nmid i_2$.

PROOF. Suppose that $T(n,k_1,(1_{n_1},0_{n_2})) \cong T(n,k_2,(1_{n_1},0_{n_2}))$. Because of Lemma 2.21 we can assume $\lceil \frac{e_j}{k_1} \rceil > 1$ or $\lceil \frac{e_j}{k_2} \rceil > 1$.

For some prime $p_j \mid n_2$, let $h(T(p_j^{e_j},k_1,0)) > 1$ and $h(T(p_j^{e_j},k_2,0)) > 1$. Running through the arguments of the proof of Theorem 4.7, part 1 follows immediately.

Part 2 follows directly from Lemma 2.19, and part 3 also follows immediately from Proposition 4.1.

Now we prove the last part. For some prime $p_j \mid n_2$, assume, without any loss, that $h(T(p_j^{e_j},k_1,0)) = 1$. So we have to prove that $T(p_j^{e_j},k_2,0)$ is semiregular. To avoid triviality we can take $h(T(p_j^{e_j},k_2,0)) > 1$. First, solving the equalities

$$\min\{\upsilon_{p_j}(T(n,k_1,(1_{n_1},0_{n_2})))\} = \min\{\upsilon_{p_j}(T(n,k_2,(1_{n_1},0_{n_2})))\}$$

and $\upsilon_{p_j}(\text{indeg}^n_{k_1}(1_{n_1},0_{n_2})) = \upsilon_{p_j}(\text{indeg}^n_{k_2}(1_{n_1},0_{n_2}))$ we obtain

$$\min\{\upsilon_{p_j}(T(p_j^{e_j},k_2,0))\} = \max\{\upsilon_{p_j}(T(p_j^{e_j},k_2,0))\}.$$

Also, since

$$(5.1)\quad \begin{aligned} &\min\{\upsilon_p\{a \in T(n,k_1,(1_{n_1},0_{n_2})):\\ &\quad \upsilon_{p_j}(\mathrm{indeg}^n_{k_1}(a)) \le \upsilon_{p_j}(\mathrm{indeg}^n_{k_1}(b)),\ \text{for all } b\}\}\\ &= \min\{\upsilon_p\{a \in T(n,k_2,(1_{n_1},0_{n_2})):\\ &\quad \upsilon_{p_j}(\mathrm{indeg}^n_{k_2}(a)) \le \upsilon_{p_j}(\mathrm{indeg}^n_{k_2}(b)),\ \text{for all } b\}\}\end{aligned}$$

for every prime $p \neq p_j$, we must have $\gcd(p_j - 1, k_2) = 1$, which means that $T(p_j^{e_j}, k_2, 0)$ is semiregular.

For the converse, we take $n_1' = \prod p_i^{e_i}$, where the product runs over all those primes $p_i \mid n_1$ such that $T(p_i^{e_i}, k_1, 1) \ncong T(p_i^{e_i}, k_2, 1)$, and $n_2' = \prod p_j^{e_j}$, where the product runs over all those primes $p_j \mid n_2$ such that $\lceil \frac{e_j}{k_1} \rceil = 1$ or $\lceil \frac{e_j}{k_2} \rceil = 1$, but not both. Then it is enough to show that $T(n_1'n_2', k_1, (1_{n_1'}, 0_{n_2'})) \cong T(n_1'n_2', k_2, (1_{n_1'}, 0_{n_2'}))$. But then, this follows immediately from Remark 4.2. □

Corollary 5.2. *Suppose that* $T(n,k_1,(1_{n_1},0_{n_2})) \cong T(n,k_2,(1_{n_1},0_{n_2}))$. *Then* $T(p_i^{e_i}, k_1^2, 1) \cong T(p_i^{e_i}, k_2^2, 1)$ *and* $T(p_j^{e_j}, k_1^2, 0) \cong T(p_j^{e_j}, k_2^2, 0)$ *for all primes* $p_i \mid n_1$ *and* $p_j \mid n_2$.

Since the trees attached to all cycle vertices in a fundamental constituent of $G(n,k)$ are isomorphic, then looking at the structure of $G_P^*(n,k)$, we have the following result.

Theorem 5.3. *Let* $a_i = (b_1, b_2, \dots, b_r)$, *where* $b_j = 0, 1$, *be the fixed point of* $G_P^*(n,k_i)$ *for* $i = 1, 2$. *Then* $G_P^*(n,k_1) \cong G_P^*(n,k_2)$ *if and only if all the following conditions are satisfied*:

(1) $T(n,k_1,a_1) \cong T(n,k_2,a_2)$,

(2) $\mathcal{A}(G_P^*(n,k_1)) = \mathcal{A}(G_P^*(n,k_2)) = \mathcal{A}$, *and*

(3) $A_t(G_P^*(n,k_1)) = A_t(G_P^*(n,k_2))$ *for all* $t \in \mathcal{A}$.

Now we can prove the following theorem.

Theorem 5.4. *Let* P *be the set of all primes dividing* n_2. *Then* $G_P^*(n,k_1) \cong G_P^*(n,k_2)$ *if and only if the following conditions are satisfied*:

(1) $T(n,k_1,(1_{n_1},0_{n_2})) \cong T(n,k_2,(1_{n_1},0_{n_2}))$;

(2) *there exists a factorization* $\lambda(n_1) = uv$, *where* u *is the largest factor such that* $\gcd(u,k_1) = \gcd(u,k_2) = 1$. *Furthermore,* $\mathrm{ord}_d k_1 = \mathrm{ord}_d k_2$ *for every divisor* d *of* u.

PROOF. We make use of Theorem 5.3. First we assume $G_P^*(n,k_1) \cong G_P^*(n,k_2)$. Since part 1 is obvious, we prove only the second part. Using the factorization of $G_P^*(n,k)$ given in (1.1), we have

$$\mathcal{A}(G_1(n,k_1)) = \mathcal{A}(G_1(n,k_2)) = \mathcal{A},$$

and

$$A_t(G_1(n,k_1)) = A_t(G_1(n,k_2)) \quad \text{for all } t \in \mathcal{A}.$$

Now because of Corollary 5.2, the existence of the factorization $\lambda(n_1) = uv$, where u is the largest factor of $\lambda(n_1)$ such that $\gcd(u,k_1) = \gcd(u,k_2) = 1$ is obvious. Since $A_t(G_1(n_1,k_1)) = A_t(G_1(n_1,k_2))$ for all $t \in \mathcal{A}$, then by induction on t, it is easy to see that $\gcd(\lambda(n_1), k_1^t - 1) = \gcd(\lambda(n_1), k_2^t - 1)$ for all $t \in \mathcal{A}$. Thus it subsequently follows that $\mathrm{ord}_d k_1 = \mathrm{ord}_d k_2$ for every divisor d of u.

For the converse, it suffices to show that

$$\mathcal{A}(G_1(n_1,k_1)) = \mathcal{A}(G_1(n_1,k_2)) = \mathcal{A},$$

and

$$A_t(G_1(n_1,k_1)) = A_t(G_1(n_1,k_2)) \quad \text{for all } t \in \mathcal{A}.$$

Since $\mathrm{ord}_d k_1 = \mathrm{ord}_d k_2$ for every divisor d of u, then by definition the first condition is obvious. For the second part, we note that $\gcd(\lambda(n_1), k_1^t - 1) = \gcd(\lambda(n_1), k_2^t - 1)$ for all $t \in \mathcal{A}$, which then implies that

$$\prod_{p_i \mid n_1} \gcd(\lambda(p_i^{e_i}), k_1^t - 1) = \prod_{p_i \mid n_1} \gcd(\lambda(p_i^{e_i}), k_2^t - 1),$$

and hence the desired result follows by induction on t. □

LEMMA 5.5. *Let p be an odd prime, and $e > 1$ be an integer. If* $\mathrm{indeg}_k^{p^e}(1) = \mathrm{indeg}_k^{p^e}(0)$ *then* $T(p^e,k,1) \cong T(p^e,k,0)$. *Moreover, if k is even the same holds for $p = 2$, except when $k = 2$ and $e = 5$.*

PROOF. Let p be an odd prime, and assume $\mathrm{indeg}_k^{p^e}(1) = \mathrm{indeg}_k^{p^e}(0)$. Then we must have $\lceil \frac{e}{k} \rceil = 1$. Because, if $\lceil \frac{e}{k} \rceil > 1$ then it follows from Lemma 3.2 that $T(p^e,k,0)$ is not semiregular, that is $e \geq k + e - \lceil \frac{e}{k} \rceil + 2$, which is a contradiction as $e - \lceil \frac{e}{k} \rceil + 2 \leq p^{e - \lceil \frac{e}{k} \rceil} \leq k$.

Let $p = 2$ and k be an even integer. Since the case when $e = 1, 2$ is trivial, we take $e \geq 3$. By inspection we see that $T(2^5, 2, 0)$ is not semiregular even though $\mathrm{indeg}_2^{2^5}(1) = \mathrm{indeg}_2^{2^5}(0)$. Now assume $\mathrm{indeg}_k^{2^e}(1) = \mathrm{indeg}_k^{2^e}(0)$, then we claim that this would imply the semiregularity of $G(2^e, k)$. First it is easy to verify that the claim holds when $k = 2, 4$. So we take $k \geq 6$ and assume $T(2^e, k, 0)$ is not semiregular. Then

$$e \geq k + e - \left\lceil \frac{e}{k} \right\rceil + 2 > 2e - \left(2\left\lceil \frac{e}{k} \right\rceil - 2\right),$$

which is a contradiction as $2\lceil \frac{e}{k} \rceil - 2 < e$.

Next, using Lemma 2.12 it can be proved, by treating the cases when $\lceil \frac{e}{k} \rceil = 1$ and $\lceil \frac{e}{k} \rceil = 2$ separately, that $h(T(2^e, k, 1)) = h(T(2^e, k, 0)) = 1$ or 2. Since $G(2^e, k)$ has exactly two components, $T(2^e, k, 0)$ and $T(2^e, k, 1)$ respectively, then we can see that $n(T(2^e, k, 1), 1) = n(T(2^e, k, 0), 1)$, and hence the isomorphism $T(2^e, k, 1) \cong T(2^e, k, 0)$ follows. □

Remark 5.6. It is obvious that any fixed point of $G(n, k)$ will also be a fixed point of $G(n, k^r)$ for all $r \geq 2$. If $C(n, k)$ is a component of $G(n, k)$ containing a t-cycle, then $C(n, k^r)$ is the union of $\gcd(t, r)$ number of components each containing a $\frac{t}{\gcd(t,r)}$-cycle for all $r \geq 2$. Conversely, if a vertex c is a fixed point in $G(n, k^r)$ for some $r \geq 2$, then c as a vertex in $G(n, k)$ is either a fixed point or in a t-cycle, where $\gcd(t, r) = t$.

Lemmas 2.14 and 2.15 together with Remark 5.6 gives us an idea about the cycle and tree structure of $G(n, k^r)$ in terms of the cycle and tree structure of $G(n, k)$.

Proposition 5.7. *Suppose that $G(n, k_1) \cong G(n, k_2)$. Then we have $G(n, k_1^r) \cong G(n, k_2^r)$ for all $r \geq 2$.*

Proof. Assume $G(n, k_1) \cong G(n, k_2)$. We write

$$G(n, k_l) = J_1(n, k_l) \cup J_2(n, k_l) \cup \cdots \cup J_s(n, k_l),$$

where each $J_i(n, k_l)$ is a union of components of $G(n, k_l)$ having isomorphic trees, and any two trees from $J_i(n, k_l)$ and $J_j(n, k_l)$, where $i \neq j$, respectively, are not isomorphic. Then $J_i(n, k_1) \cong J_i(n, k_2)$ for all i, and so it is enough to prove that $J_i(n, k_1^r)$ and $J_i(n, k_2^r)$ for all $r \geq 2$. From Lemma 2.19 it immediately follows that the trees attached to all cycle vertices in $J_i(n, k_1^r)$ and $J_i(n, k_2^r)$ for all i and $r \geq 2$, are isomorphic. Using Remark 5.6 we obtained $\mathcal{A}(J_i(n,k_1^r)) = \mathcal{A}(J_i(n,k_2^r)) = \mathcal{A}_r^i$ for all i and $r \geq 1$, and

$$\begin{aligned} A_m(J_i(n,k_1^r)) &= \sum_{\substack{t\in\mathcal{A}_1^i \\ \frac{t}{\gcd(r,t)}=m}} \gcd(r,t) A_t(J_i(n,k_1)) \\ &= \sum_{\substack{t\in\mathcal{A}_1^i \\ \frac{t}{\gcd(r,t)}=m}} \gcd(r,t) A_t(J_i(n,k_2)) \\ &= A_m(J_i(n,k_2^r)) \end{aligned}$$

for all $m \in \mathcal{A}_r^i$, i and $r \geq 2$. Hence, the result follows. □

Finally we prove the uniqueness of the factorization of $G(n,k)$.

THEOREM 5.8. *Suppose that $G(n,k_1) \cong G(n,k_2)$. Then*

$$G(p_i^{e_i},k_1) \cong G(p_i^{e_i},k_2),$$

whenever they are not semiregular, and

$$G\Big(\prod p_j^{e_j},k_1\Big) \cong G\Big(\prod p_j^{e_j},k_2\Big),$$

where the products run over all those primes p_j such that both $G(p_j^{e_j},k_1)$ and $G(p_j^{e_j},k_2)$ are semiregular.

We note that in this theorem we are not able to prove completely the uniqueness of the factorization of $G(n,k)$, in particular, it does not say anything about the uniqueness of the semiregular factors of $G(n,k)$. But still we are more or less satisfied because the structure of semiregular digraphs is well known and have been characterized (see Theorem 3.5), and as seen in Remark 3.6 we can actually write them explicitly. One important observation about semiregular digraphs is its tree structure, that is, trees attached to all its cycle vertices are isomorphic. Ironically, as far as the techniques employed in this proof are concerned, this very property of semiregular digraphs prevents us from having a complete uniqueness on the factorization of $G(n,k)$.

Proof. Assume $G(n,k_1) \cong G(n,k_2)$. We divide the proof into two parts.

Claim 1. *In view of Corollary 4.3 and Theorem 4.6, we claim that* $T(n,k_1,1) \cong T(n,k_2,1)$ *and* $T(n,k_1,0) \cong T(n,k_2,0)$.

First we note that k_1 and k_2 are of the same parity. This is because if say k_1 is even and k_2 is odd; then for an odd n, the indegree of every cycle vertex in $G(n,k_2)$ is odd but the indegree of the cycle vertex 1 in $G(n,k_1)$ is even, and similarly for an even n, the indegree of every cycle vertex in $G(n,k_1)$ is even but the indegree of the cycle vertex 1 in $G(n,k_2)$ is odd.

First take n, k_1, k_2 to be odd integers. Since

$$\begin{aligned}
&\min\{\upsilon_{p_i}\{a=(a_1,\ldots,a_r)\colon a_j=0,1\}\}\\
&\quad=\min\{\upsilon_{p_i}\{a\in G(n,k_1)\colon a \text{ is a cycle vertex}\}\}\\
&\quad=\min\{\upsilon_{p_i}\{b\in G(n,k_2)\colon b \text{ is a cycle vertex}\}\}\\
&\quad=\min\{\upsilon_{p_i}\{b=(b_1,\ldots,b_r)\colon b_j=0,1\}\}
\end{aligned}$$

for all i, and using the fact that $\upsilon_p(\mathrm{indeg}_k^{p^e}(0)) \geq \upsilon_p(\mathrm{indeg}_k^{p^e}(1))$ for any odd prime p, we obtain $\upsilon_{p_i}(\mathrm{indeg}_{k_1}^{p_i^{e_i}}(1)) = \upsilon_{p_i}(\mathrm{indeg}_{k_2}^{p_i^{e_i}}(1))$ for all i. Now let $n = n_1 n_2$, where $p_i^{e_i} \parallel n_1$ if and only if $p_i^{e_i-1} \mid k_l$ for $l = 1,2$. By an application of Theorem 3.5 and Corollary 3.7, it can be seen that $T(n_2,k_l,0)$ cannot be isomorphic to any other trees in $G(n_2,k_l)$ for $l = 1,2$. Next, by comparing the indegrees of the cycle vertices in $G(n,k_1)$ and $G(n,k_2)$, we observe that $\gcd(p_i-1,k_1)=1$ if and only if $\gcd(p_i-1,k_2)=1$. Then from Lemma 5.5 we see that the tree $T(n_2,k_1,0)$ is not isomorphic to $T(n_2,k_2,c)$ for any cycle vertex $c \neq 0$ in $G(n_2,k_2)$, and vice versa. Thus one can deduce that the only option we have is

$$\begin{aligned}
T(n,k_1,0) &= O_1^{\prod_{p_i\mid n_1} p_i^{e_i-1}} \times T(n_2,k_1,0)\\
&\cong O_1^{\prod_{p_i\mid n_1} p_i^{e_i-1}} \times T(n_2,k_2,0)\\
&= T(n,k_2,0).
\end{aligned}$$

Next we show that $T(n,k_1,1) \cong T(n,k_2,1)$. Since

$$\begin{aligned}
\mathrm{indeg}_{k_1}^n(1_{n_1},0_{n_2}) &= \max\{\mathrm{indeg}_{k_1}^n(c_1,c_2,\ldots,c_r)\colon c_i=0,1\}\\
&= \max\{\mathrm{indeg}_{k_2}^n(d_1,d_2,\ldots,d_r)\colon d_i=0,1\}\\
&= \mathrm{indeg}_{k_2}^n(1_{n_1},0_{n_2}),
\end{aligned}$$

and

$$\begin{aligned}\operatorname{indeg}_{k_1}^n(0_{n_1}, 1_{n_2}) &= \min\{\operatorname{indeg}_{k_1}^n(c_1, c_2, \ldots, c_r) \colon c_i = 0, 1\}\\ &= \min\{\operatorname{indeg}_{k_2}^n(d_1, d_2, \ldots, d_r) \colon d_i = 0, 1\}\\ &= \operatorname{indeg}_{k_2}^n(0_{n_1}, 1_{n_2}),\end{aligned}$$

then it follows that $T(n_1, k_1, 1) \cong T(n_1, k_2, 1)$ and $T(n_2, k_1, 1) \cong T(n_2, k_2, 1)$, respectively.

Now let n be odd and k_1, k_2 be even integers. Since

$$\upsilon_2(\operatorname{indeg}_{k_l}^n(0)) < \upsilon_2(\operatorname{indeg}_{k_l}^n(a_1, a_2, \ldots, a_r)) < \upsilon_2(\operatorname{indeg}_{k_l}^n(1)),$$

where $a_i = 0, 1$, not all equal to 0 or 1, and $l = 1, 2$, then we must have $T(n, k_1, 0) \cong T(n, k_2, 0)$ and $T(n, k_1, 1) \cong T(n, k_2, 1)$.

Let $n = 2^e \prod_{i=1}^r p_i^{e_i}$ be even and k_1, k_2 be odd integers. Then $T(2^e, k_l, 1)$ is trivial, and we have

$$G_1(2^e, k_1) \times G\Big(\frac{n}{2^e}, k_1\Big) \cong G_1(2^e, k_2) \times G\Big(\frac{n}{2^e}, k_2\Big)$$

and

$$G_2(2^e, k_1) \times G\Big(\frac{n}{2^e}, k_1\Big) \cong G_2(2^e, k_2) \times G\Big(\frac{n}{2^e}, k_2\Big).$$

Similar to the first case, it follows that

$$T(n, k_1, 1) \cong T(n, k_2, 1) \quad \text{and} \quad T\Big(\frac{n}{2^e}, k_1, 0\Big) \cong T\Big(\frac{n}{2^e}, k_2, 0\Big),$$

and since $\upsilon_2(G(n, k_l)) = \upsilon_2(T(2^e, k_l, 0))$ for $l = 1, 2$, we derive

$$T(2^e, k_1, 0) \cong T(2^e, k_2, 0).$$

Finally, let $n = 2^e \prod_{i=1}^r p_i^{e_i}$, k_1, k_2 be even integers. In this case, $G(2^e, k_1)$ and $G(2^e, k_2)$ consist of only two components. Also note that

$$\begin{aligned}|T(n, k_l, (1_{2^e}, 0, \ldots, 0))| &= |T(n, k_l, (0_{2^e}, 0, \ldots, 0))|\\ &= 2^e \prod_{i=1}^r p_i^{e_i - 1}\\ &\neq |T(n, k_l, (c_{2^e}, c_1, c_2, \ldots, c_r))|,\end{aligned}$$

where at least one $c_i = 1$, and $l = 1, 2$, and

$$\begin{aligned}&\max\{\upsilon_2(|T(n, k_l, (c_{2^e}, c_1, c_2, \ldots, c_r))|) \colon c_i = 0, 1\}\\ &\quad = \{\upsilon_2(|T(n, k_l, (0_{2^e}, 1, 1, \ldots, 1))|), \upsilon_2(|T(n, k_l, (1_{2^e}, 1, \ldots, 1))|)\},\end{aligned}$$

where $l = 1, 2$. If $e \neq 3$ then by comparing the indegrees of the cycle vertices in $G(n, k_l)$ and using Lemma 5.5 we can conclude that $T(n, k_1, 1) \cong T(n, k_2, 1)$ and $T(n, k_1, 0) \cong T(n, k_2, 0)$. In view of Remark 4.4(3) and Lemma 5.5, let $e = 3$ and say $k_1 = 2$. Comparing the indegrees again, we have

$$T(n, k_1, (1_{2^e}, 0, \dots, 0)) \cong T(n, k_2, (0_{2^e}, 0, \dots, 0)),$$

and

$$T(n, k_1, (0_{2^e}, 0, \dots, 0)) \cong T(n, k_2, (1_{2^e}, 0, \dots, 0)),$$

which impies that $\operatorname{indeg}^n_{k_1}(1) = \operatorname{indeg}^n_{k_2}(0) = \operatorname{indeg}^n_{k_2}(1) = \operatorname{indeg}_{k_1}(0)$, a contradiction. Thus $T(n, k_1, (0_{2^e}, 0, \dots, 0))$ and $T(n, k_2, (0_{2^e}, 0, \dots, 0))$ must be isomorphic. Next, we have $T(n, k_1, (1_{2^e}, 1, \dots, 1)) \cong T(n, k_2(0_{2^e}, 1, 1, \dots, 1))$ and $T(n, k_1, (0_{2^e}, 1, \dots, 1)) \cong T(n, k_2(1_{2^e}, 1, 1, \dots, 1))$, and since $\operatorname{indeg}^{2^e}_{k_2}(0) = \operatorname{indeg}^{2^e}_{k_2}(1)$, it follows from Lemma 5.5 that $\operatorname{indeg}^{2^e}_{k_1}(0) = \operatorname{indeg}^{2^e}_{k_1}(1)$, which is again a contradiction. Hence, the claim.

Claim 2. *If $G(p_i^{e_i}, k_1)$ and $G(p_i^{e_i}, k_2)$ are not semiregular, then*

$$\mathcal{A}(G_1(p_i^{e_i}, k_1)) = \mathcal{A}(G_1(p_i^{e_i}, k_2)) = \mathcal{A}_i,$$

and

$$A_t(G_1(p_i^{e_i}, k_1)) = A_t(G_1(p_i^{e_i}, k_2)) \quad \textit{for all } t \in \mathcal{A}_i.$$

Also, $G\big(\prod p_j^{e_j}, k_1\big) \cong G\big(\prod p_j^{e_j}, k_2\big)$, where the products run over all those primes p_j such that both $G(p_j^{e_j}, k_1)$ and $G(p_j^{e_j}, k_2)$ are semiregular.

For the first part, it suffices to prove $G_1\big(\prod p_i^{e_i}, k_1\big) \cong G_1\big(\prod p_i^{e_i}, k_2\big)$, where the products run over all those primes p_i such that both $G(p_i^{e_i}, k_1)$ and $G(p_i^{e_i}, k_2)$ are not semiregular. Because this would imply, under the same products, that

$$\mathcal{A}\Big(G_1\Big(\prod p_i^{e_i}, k_1\Big)\Big) = \mathcal{A}\Big(G_1\Big(\prod p_i^{e_i}, k_2\Big)\Big),$$

and

$$A_t\Big(G_1\Big(\prod p_i^{e_i}, k_1\Big)\Big) = A_t\Big(G_1\Big(\prod p_i^{e_i}, k_2\Big)\Big) \quad \text{for all } t,$$

and as seen in the proof of Theorem 5.4, $\operatorname{ord}_d k_1 = \operatorname{ord}_d k_2$ must hold for every positive divisor d of u, where $\lambda(\prod p_i^{e_i}) = uv$ and u is the largest factor such that $\gcd(u, k_1) = \gcd(u, k_2) = 1$. Then by a property of the Carmichael function given in Lemma 2.2, it is easy to see that $\operatorname{ord}_{d'} k_1 = \operatorname{ord}_{d'} k_2$ for every positive divisor d' of u_i, where $\lambda(p_i^{e_i}) = u_i v_i$ and u_i is the largest factor such that $\gcd(u_i, k_1) = \gcd(u_i, k_2) = 1$. Since $\gcd(\lambda(p_i^{e_i}), k_1) = \gcd(\lambda(p_i^{e_i}), k_2)$ holds for all i, then using the same arguments as in the proof of Theorem 5.4 we get the desired result.

Now we prove $G_1\big(\prod p_i^{e_i}, k_1\big) \cong G_1\big(\prod p_i^{e_i}, k_2\big)$, where the products run over all those primes p_i such that both $G(p_i^{e_i}, k_1)$ and $G(p_i^{e_i}, k_2)$ are not semiregular. First, using equations (1.1) and (1.2), we obtain $\mathcal{A}(G_1\big(\prod p_i^{e_i}, k_1\big)) = \mathcal{A}(G_1\big(\prod p_i^{e_i}, k_2\big))$, over the same products. Let k_1, k_2 be even integers. Then all factors of $G(n, k_1)$ and $G(n, k_2)$, corresponding at least to odd primes, are non-semiregular. If n is odd then for $l = 1, 2$, we have

$$\max\{\upsilon_2(|T(n, k_l, (a_1, a_2, \ldots, a_r)|): a_i = 0, 1\} = \upsilon_2(|T(n, k_l, (1, 1, \ldots, 1)|),$$

which is unique. Since the trees attached to all cycle vertices in a fundamental constituent of $G(n, k)$ are isomorphic, the condition $A_t(G(n, k_1)) = A_t(G(n, k_2))$ must hold for all t as desired. Let $n = 2^e \prod_{i=1}^{r} p_i^{e_i}$ be even. If $\operatorname{indeg}_{k_l}^{2^e}(1) \neq \operatorname{indeg}_{k_l}^{2^e}(0)$ then $T(n, k_l, 1)$ is not isomorphic to any $T(n, k_l, (a_1, a_2, \ldots, a_r))$, where at least one $a_j = 0$, for $l = 1, 2$, and so we must have $G_1(n, k_1) \cong G_1(n, k_2)$. However, if $\operatorname{indeg}_{k_l}^{2^e}(1) = \operatorname{indeg}_{k_l}^{2^e}(0)$ then after applying Lemma 5.5 it follows that the trees attached to all cycle vertices in $G_{\{2\}}^*(n, k_l) \cup G_1(n, k_l)$ are isomorphic, and thus

$$G_{\{2\}}^*(n, k_1) \cup G_1(n, k_1) \cong G_{\{2\}}^*(n, k_2) \cup G_1(n, k_2).$$

This implies that

$$\begin{aligned} 2 \times A_t(G_1(n, k_1)) &= A_t(G_{\{2\}}^*(n, k_1) \cup G_1(n, k_1)) \\ &= A_t(G_{\{2\}}^*(n, k_2) \cup G_1(n, k_2)) \\ &= 2 \times A_t(G_1(n, k_2)) \end{aligned}$$

for all t, as required.

Now consider k_1, k_2 to be odd, and first assume $n = \prod_{i=1}^{r} p_i^{e_i}$ also to be odd. Since $\upsilon_{p_i}(\operatorname{indeg}_{k_1}^{p_i^{e_i}}(1)) = \upsilon_{p_i}(\operatorname{indeg}_{k_2}^{p_i^{e_i}}(1))$ for all i, and the fact that $\gcd(p_i - 1, k_1) = 1$ if and only if $\gcd(p_i - 1, k_2) = 1$, we have $T(p_i^{e_i}, k_1, 1) \cong T(p_i^{e_i}, k_1, 0)$ if and only if $T(p_i^{e_i}, k_2, 1) \cong T(p_i^{e_i}, k_2, 0)$ for all i. Let us take $n = n_1 n_2$, where $p_i^{e_i} \parallel n_1$ if and only if $p_i^{e_i - 1} \mid k_l$ for $l = 1, 2$. Then for all primes p_j such that $p_j^{e_j} \mid n_2$, $T(p_j^{e_j}, k_1, 0)$ and $T(p_j^{e_j}, k_2, 0)$ are semiregular and $\lceil \frac{e_j}{k_1} \rceil = 1$, $\lceil \frac{e_j}{k_2} \rceil = 1$. This is because if $\lceil \frac{e_j}{k_l} \rceil = 2$ we obtained $k_l < e_j \leq k_l + \upsilon_{p_j}(k_l)$+1 for $l = 1, 2$, which follows that $p_j^{\upsilon_{p_j}(k_l)} < k_1 - k_2 < \upsilon_{p_j}(k_l) + 1$, a contradiction. Since $G(n_1, k_l) \times T(n_2, k_l, 0)$ is the union of all those components in $G(n, k_l)$ for $l = 1, 2$ such that $p_j^{e_j - 1}$ divides every cycle vertex for all $p_j \mid n_2$, then

$$G(n_1, k_1) \times T(n_2, k_1, 0) \cong G(n_1, k_2) \times T(n_2, k_2, 0),$$

which after using Lemma 2.21 it becomes $G(n_1,k_1) \cong G(n_1,k_2)$. Similarly, since $G(n_1,k_l)\times G_1(n_2,k_l)$ is the union of all those components in $G(n,k_l)$ for $l = 1,2$ such that $p_j^{\upsilon_{p_j}(k_l)}$ divides every cycle vertex for all $p_j \mid n_2$, we obtained

$$G(n_1,k_1)\times G_1(n_2,k_1) \cong G(n_1,k_2)\times G_1(n_2,k_2).$$

Since all the trees in $G(n_1,k_l)\times G_1(n_2,k_l)$, for $l = 1,2$, are isomorphic, then from Proposition 5.7 we get $A_1(G_1(n_2,k_1^r)) = A_1(G_1(n_2,k_2^r))$ for all $r \geq 2$. Hence, $G_1(n_2,k_1) \cong G_1(n_2,k_2)$, as it can be proved inductively on t that $A_t(G_1(n_2,k_1)) = A_t(G_1(n_2,k_2))$ for all t, by observing that

$$A_t(G_1(n_2,k_l)) = \frac{1}{t}\Big[A_1(G_1(n_2,k_l^t)) - \sum_{d\mid t, d\neq t} dA_d(G_1(n_2,k_l))\Big] \quad \text{for } l = 1,2.$$

Finally, for an even integer $n = 2^e\prod_{i=1}^r p_i^{e_i}$, the trees $T(2^e,k_l,1)$ for $l = 1,2$ are trivial, and as seen in the corresponding part of Claim 1, $G(2^e,k_1) \cong G(2^e,k_2)$. The rest then follows exactly as in the preceding case. This completes the proof. □

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