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Groups, Geometry, and Dynamics

On Beauville surfaces

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Abstract. We prove that if a finite group G acts freely on a product of two curves $C_1 \times C_2$ so that the quotient $S = C_1 \times C_2/G$ is a Beauville surface then C_1 and C_2 are both non hyperelliptic curves of genus ≥ 6; the lowest bound being achieved when $C_1 = C_2$ is the Fermat curve of genus 6 and $G = (\mathbb{Z}/5\mathbb{Z})^2$. We also determine the possible values of the genera of C_1 and C_2 when G equals S_5, $\mathrm{PSL}_2(\mathbb{F}_7)$ or any abelian group. Finally, we produce examples of Beauville surfaces in which G is a p-group with $p = 2, 3$.

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1. Introduction and statement of results

A complex algebraic curve C will be termed *triangle curve* if it admits a finite group of automorphisms $G < \mathrm{Aut}(C)$ so that $C/G \cong \mathbb{P}^1$ and the natural projection $C \to C/G$ ramifies over three values, say $0, 1, \infty$. If the branching orders at these points are p, q and r we will say that C/G is an orbifold of type (or signature) (p, q, r) or, more simply, that C/G is of type (p, q, r). Triangle curves are also known as arithmetic curves or regular Belyi (Riemann) surfaces. This is because of the following theorem

Belyi's Theorem. *A complex algebraic curve C can be defined over a number field if and only if there is a (Belyi) function $f : C \to \mathbb{P}^1$ ramified over three values.*

For complex surfaces S an analogous criterion in which Belyi functions are replaced by Lefschetz functions is given in [6]. Among the complex surfaces defined over a number field an important class is that of Beauville surfaces defined as follows

Definition ([3]). A *Beauville surface* is a compact complex surface S satisfying the following properties:

1) It is isogenous to a higher product, that is $S \cong C_1 \times C_2/G$, where C_i $(i = 1, 2)$ are curves of genera $g_i \geq 2$ and G is a finite group acting freely on $C_1 \times C_2$ by holomorphic transformations.

2) If $G^o < G$ denotes the subgroup consisting of the elements which preserve each of the factors, then G^o acts effectively on each curve C_i so that $C_i/G^o \cong \mathbb{P}^1$ and $C_i \to C_i/G^o$ ramifies over three points.

It is easy to see ([5]) that an automorphism of the product of two curves as above $f : C_1 \times C_2 \to C_1 \times C_2$ either preserves each factor or interchanges them. Clearly, if two such automorphisms f_1, f_2 interchange the factors, which can only occur when $C_1 \cong C_2$, then $f_1 \circ f_2$ will preserve them, hence G^o has at most index 2 in G . A Beauville surface $C_1 \times C_2/G$ is said to be of *mixed* or *unmixed type* according to whether $[G : G^o] = 2$ or $G = G^o$. Accordingly the group G is said to *admit a mixed or unmixed Beauville structure*.

It is clear that any Beauville surface of mixed type $S = C_1 \times C_2/G$ automatically gives rise to a Beauville surface of unmixed type $S^o = C_1 \times C_2/G^o$. This is the reason why for the purpose of this article Beauville surfaces of mixed type will play no role.

Beauville surfaces were introduced by F. Catanese in [5] generalizing the following construction of A. Beauville which appears as exercise number 4 on page 159 of [4].

Beauville's example. Let $C_1 = C_2$ be the Fermat curve $x_0^5 + x_1^5 + x_2^5 = 0$ and G the group $(\mathbb{Z}/5\mathbb{Z})^2$ acting on $C_1 \times C_2$ by the rule

$$\begin{aligned}(a,b)\cdot([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \\ = \left([\xi^a x_0 : \xi^b x_1 : x_2], [\xi^{a+3b} y_0 : \xi^{2a+4b} y_1 : y_2]\right)\end{aligned}$$

where $\xi = e^{\frac{2\pi i}{5}}$. Then $S = C_1 \times C_2/G$ is a Beauville surface with $g_1 = g_2 = 6$, $G^o = G$ and C_i/G of type $(5, 5, 5)$.

In fact, Bauer and Catanese [1] have shown that when one considers all possible actions of G on $C_1 \times C_2$ one gets exactly two isomorphic classes of Beauville surfaces.

The relevance of Beauville surfaces lies mainly on the fact that they are the rigid ones among surfaces isogenous to a product. We recall that an algebraic variety is called rigid when it does not admit any non trivial deformation.

Most, if not all, of what is known about them is due to work done by Catanese on his own ([5]) or jointly with Bauer and Grunewald ([3], [2]).

As noted in [3] there are many interesting open problems regarding Beauville surfaces. The most obvious ones are questions such as which finite groups can occur, which curves C_i, which genera g_i, which curves or genera for a given group G, etc.

The following (necessarily) partial answers are known.

Theorem ([3]). 1) *A finite abelian group G admits an unmixed Beauville structure iff $G = (\mathbb{Z}/n\mathbb{Z})^2$, $(n, 6) = 1$.*

2) *The following groups admit unmixed Beauville structures:*

a) A_n *for large* n.
b) S_n *for* $n \in \mathbb{N}$ *with* $n \geq 7$.
c) $\mathrm{SL}(2, \mathbb{F}_p)$, $\mathrm{PSL}(2, \mathbb{F}_p)$ *for* $p \neq 2, 3, 5$.

In this paper we prove the following results

1) If $S = C_1 \times C_2/G$ is a Beauville surface, then neither C_1 nor C_2 can be a hyperelliptic curve (Theorem 3).

2) The minimum genera (in the lexicographic order) that can occur in the construction of a Beauville surface is $(g_1, g_2) = (6, 6)$; in other words, although in the definition of Beauville surface it is only required $g_i \geq 2$, one, in fact, has $g_i \geq 6$. The minimum being achieved by Beauville's seminal example described above (Theorem 9).

3) S_5 and S_6, hence S_n for all $n \geq 5$, admit a Beauville structure (Corollary 15).

4) We determine which pair of genera g_1, g_2 occur when G is S_5, $\mathrm{PSL}_2(\mathbb{F}_7)$ or an abelian group (Theorem 12, Theorem 13, Theorem 16).

5) Finally, we produce examples of Beauville surfaces in which G is a p-group with $p = 2, 3$. These appear to be the first known examples of 2 and 3-groups admitting Beauville structure (see [2], p. 38).

2. A criterion for G to admit Beauville structures

As shown in [2] and [3] there is a purely algebraic criterion to detect when a finite group G admits a Beauville structure.

Definition 1. Let G be a finite group and a, b, c three generators of order p, q, r respectively. We shall say that (a, b, c) is a *hyperbolic triple of generators* if the following conditions hold:

(1) $abc = 1$,
(2) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

Now, set

$$\Sigma(a, b) := \bigcup_{g \in G} \bigcup_{i=1}^{\infty} \{g a^i g^{-1}, g b^i g^{-1}, g c^i g^{-1}\}$$

Criterion ([2], [3]). G admits an unmixed Beauville structure if and only if it has two hyperbolic triples of generators (a_i, b_i, c_i) of order (p_i, q_i, r_i), $i = 1, 2$, satisfying the following compatibility condition:

$$\Sigma(a_1, b_1) \cap \Sigma(a_2, b_2) = 1.$$

The curves C_i on which G acts to produce the required Beauville surface $S = C_1 \times C_2/G$ arise as follows:

The triangle group

$$\Gamma_{(p_i,q_i,r_i)} = \langle x, y, z : x^{p_i} = y^{q_i} = z^{r_i} = xyz = 1\rangle$$

acts on the upper half-plane $\mathbb{H}$ as a discrete group of isometries (i.e. as a *Fuchsian group*) and $C_i = \mathbb{H}/K_i$ where K_i is the kernel of the epimorphism $\theta_i : \Gamma_{(p_i,q_i,r_i)} \to G$ which sends $x \to a_i$, $y \to b_i$ and $z \to c_i$. It is well known that K_i is a torsion free group that acts freely on $\mathbb{H}$ and that, according to the Riemann–Hurwitz formula, the genus g_i of the curve C_i is given by

$$2g_i - 2 = |G|\left(1 - \left(\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i}\right)\right).$$

We see that $g_i \geq 2$ precisely because (p_i, q_i, r_i) is of hyperbolic type.

This is a useful criterion. It permits to check through a computer program whether or not a group (of not very large order) admits Beauville structures. For instance, one has

Proposition 2. *The only non abelian group of order* ≤ 120 *which admits a Beauville structure is* S_5.

This result has been obtained in [2] with the help of MAGMA. Incidentally, we note that S_5 is missing in the printed version of the article.

3. Some restrictions on the product $C_1 \times C_2$

Theorem 3. *Let* $S = C_1 \times C_2/G$ *be a Beauville surface, then neither* C_1 *nor* C_2 *is a hyperelliptic curve.*

Proof. Let us suppose that one of them, let us say C_1, is a hyperelliptic curve. Then, the group generated by any automorphism τ of order bigger than 2 must fix points on C_1 because otherwise the obvious quotient map $C_1 \to C_1/\langle\tau\rangle$ would provide an unramified cyclic covering of degree bigger than 2 between two hyperelliptic curves. According to Maclachlan [12] this is impossible. Therefore, as G acts on $C_1 \times C_2$ fixed point freely, only the elements of order 2 can fix points on C_2. Hence C_2/G would have to have type $(2, 2, 2)$ which is not a hyperbolic type. □

Corollary 4. *If* $S = C_1 \times C_2/G$ *is a Beauville surface neither* C_1 *nor* C_2 *has genus* 2.

Proposition 5. *If* $S = C_1 \times C_2/G$ *is a Beauville surface neither* C_1 *nor* C_2 *has genus* 3.

Proof. Assume that the genus of C_1 is $g_1 = 3$. The list of triangle curves of genus 3 and their corresponding groups is well known (see e.g. [10] or [15]). In view of Proposition 2, the only group of this list which could admit a Beauville structure is $\mathrm{PSL}_2(\mathbb{F}_7)$, the simple group with 168 elements.

Now, the Beauville structures of this group are analyzed in Theorem 13 below. It is there found that the genera g_i of C_i have to satisfy $g_i \geq 8$. □

Proposition 6. *If $S = C_1 \times C_2/G$ is a Beauville surface neither C_1 nor C_2 has genus 4.*

Proof. Assume C_1 has genus 4. The list of triangle curves of genus 4 and their corresponding groups is also well known ([11], [15], see also [14]). Matching this list with Proposition 2, as we did before, we are left again with only one possibility, namely $G = S_5$. But S_5 can only produce a Beauville surface if the genera of C_1 and C_2 are 19 and 21 (see Theorem 12 below). □

The proof of the fact that none of the curves C_1 and C_2 can have genus 5 is based on the following group theoretical result.

Proposition 7. *Let p be an odd prime and G a group of order $2^k p$. Assume that G maps onto a cyclic group of order p or onto a dihedral group of order $2p$. Then G does not admit a Beauville structure.*

Proof. Note that if x, y are generators of a dihedral group of order $2p$, then x, y or xy are of order p. Hence if G is generated by a and b, then the order of a, b or ab is divisible by p and so a Sylow p-subgroup of G is contained in $\langle a\rangle \cup \langle b\rangle \cup \langle ab\rangle$. Since two Sylow p-subgroups of G are conjugated, G can not admit a Beauville structure. □

Corollary 8. *If $S = C_1 \times C_2/G$ is a Beauville surface neither C_1 nor C_2 has genus 5.*

Proof. Kuribayashi and Kimura [9] have also obtained the list of all possible groups than can occur as groups of automorphisms of a genus 5 Riemann surface. Comparing again this list with Proposition 2 we are left with only two groups, a group G_1 of order $160 = 2^5 \cdot 5$ and a group G_2 of order $192 = 2^6 \cdot 3$. From the description of these two groups given in [9] it is obvious that the first group maps onto a dihedral group of order 10 and the second one onto a dihedral group of order 6. Now the result follows from Proposition 7. □

Assembling together the results in this section, we obtain the following

Theorem 9. *Let C_1, C_2 be curves of genera g_1, g_2 and G a finite group acting on $C_1 \times C_2$ so that $S = C_1 \times C_2/G$ is a Beauville surface. Then $g_1, g_2 \geq 6$. Furthermore, the minimum $(g_1, g_2) = (6, 6)$ is achieved by Beauville's own example.*

We record in passing a collateral implication of Proposition 7

Corollary 10. *Let p be an odd prime and G a group of order $2^k p$. If $(p-1)/2$ is odd, then G does not admit a Beauville structure.*

Proof. Let $N = O_2(G)$ be the maximal normal 2-subgroup of G. Since G/N is soluble, its order is equal to $2^l p$ and $O_2(G/N) = 1$, G/N has only one minimal normal subgroup and this subgroup is of order p. Now, using that $(p-1)/2$ is odd, we obtain that G/N is a cyclic group of order p or a dihedral group of order $2p$. Thus, we may apply Proposition 7. □

4. The Beauville genus spectrum of a group

Definition 11. Let G be a finite group. By the Beauville genus spectrum of G we mean the set $\mathrm{Spec}(G)$ of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves C_1, C_2 of genera g_1 and g_2 with an action of G on $C_1 \times C_2$ such that $S = C_1 \times C_2/G$ is a Beauville surface.

We observe that $\mathrm{Spec}(G)$ is always a finite set, for by the Riemann–Hurwitz formula g_i is bounded by $1 + \frac{|G|}{2}$. In fact for many groups $\mathrm{Spec}(G) = \emptyset$. For instance, the only abelian groups G for which $\mathrm{Spec}(G) \neq \emptyset$ are $G = (\mathbb{Z}/n\mathbb{Z})^2$ with $(n, 6) = 1$ ([3]).

We also observe that knowledge of $\mathrm{Spec}(G)$ provides knowledge of the possible numerical invariants of the Beauville surfaces $S = C_1 \times C_2/G$. Indeed, as the irregularity of a Beauville surface is always $q = 0$ the geometric genus p_g is obtainable from the identity $1 + p_g(X) - q(X) = \frac{1}{4}\chi_{\mathrm{top}}(S) = \frac{(g_1-1)(g_2-1)}{|G|}$ (see [5]).

In this section we compute the spectrum of a few groups.

Theorem 12. $\mathrm{Spec}(S_5) = \{(19, 21)\}$.

Proof. A Beauville surface $S = C_1 \times C_2/S_5$ with curves C_1 of genus $g_1 = 19$ and C_2 of genus $g_2 = 21$ is provided by the following two triples of generators of S_5:

$$a_1 = (125), \quad b_1 = (13)(254), \quad c_1 = (123)(45)$$

and

$$a_2 = (2345), \quad b_2 = (1234), \quad c_2 = (14253).$$

Let now (a_i, b_i, c_i) $(i = 1, 2)$ be two compatible triples of generators of hyperbolic type (p_i, q_i, r_i). We have to show that $(p_1, q_1, r_1) = (3, 6, 6)$ and $(p_2, q_2, r_2) = (4, 4, 5)$.

We first note that the identity $a_i b_i c_i = 1$ implies that each of the triples (a_i, b_i, c_i) consists of one even permutation and two odd permutations. With this observation at our disposal the rest of the proof consists of a case by case analysis of all possible types (p_i, q_i, r_i). To simplify the task we assume that $p_i \leq q_i \leq r_i$. This can be done because if instead of a_i, b_i, c_i we choose e.g. the generators $a_i b_i a_i^{-1}$, a_i, c_i then the type becomes (q_i, p_i, r_i).

We shall only consider the types with $p_1 = 2$, the remaining cases being similar.

Since the types of the form $(2, 2, r)$, $(2, 3, r)$ and $(2, 4, 4)$ are not hyperbolic types we are left with only four possible types:

• $(2, 4, 5)$. Clearly a_1 has to be a transposition and b_1^2 a product of two disjoint transpositions. On the other hand, if c_2 has order 6, then c_2^3 is conjugate to a_1. Thus, a compatible triple (a_2, b_2, c_2) could only have type $(3, 3, 3)$ which is not a hyperbolic type.

• $(2, 4, 6)$. In this case b_1^2 (resp. c_1^3) would be an even (resp. odd) order two permutation. Thus, the only candidate left for the type of (a_2, b_2, c_2) is $(5, 5, 5)$ which is ruled out by our observation above.

• $(2, 5, 6)$. In this case a_1 would have to be odd, hence the only possible type left for (a_2, b_2, c_2) would be $(4, 4, 4)$ which also contradicts the main observation above.

• $(2, 6, 6)$. In this case a_1 would have to be even. Now, since b_1^2 is a permutation of order 3 and b_1^3 is conjugate to a transposition, the only candidate for (p_2, q_2, r_2) is $(5, 5, 5)$ which, again by the observation above, is forbidden. □

Theorem 13. $\mathrm{Spec}(\mathrm{PSL}_2(\mathbb{F}_7)) = \{(8, 49), (15, 49), (17, 22), (22, 33), (22, 49)\}$.

Proof. The pairs of genera above correspond to the following pairs of hyperbolic type:

(i) $(3, 3, 4)$, $(7, 7, 7)$,
(ii) $(3, 4, 4)$, $(7, 7, 7)$,
(iii) $(3, 3, 7)$, $(4, 4, 4)$,
(iv) $(4, 4, 4)$, $(3, 7, 7)$,
(v) $(4, 4, 4)$, $(7, 7, 7)$.

Obviously, any pair of triples of generators (a_i, b_i, c_i) realizing any of these five pairs of types will be necessarily compatible. Thus, we have to show that such generating triples exist and that no other pair of hyperbolic types can be realized by a pair of compatible generating triples.

Consider the following elements of $\mathrm{PSL}_2(\mathbb{F}_7)$:

$$\alpha = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & -3 \\ -3 & 3 \end{pmatrix}, \quad \delta = \begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}.$$

Next we exhibit a generating triple $(a, b, c = (ab)^{-1})$ for each of the six types involved.

- $(3,3,7)$: $a=\alpha,\ \ b=\beta\alpha\beta^{-1}$,
- $(7,7,7)$: $a=\beta^4,\ \ b=\alpha\beta\alpha^{-1}$,
- $(3,3,4)$: $a=\beta\alpha^2\beta^{-1},\ \ b=\alpha$,
- $(4,4,4)$: $a=\gamma,\ \ b=\beta^{-3}\alpha$,
- $(3,7,7)$: $a=\alpha,\ \ b=\alpha\beta^{-1}\alpha^{-1}$,
- $(3,4,4)$: $a=\alpha^2\delta\alpha^{-2}, b=\alpha^2\delta$.

Now, if one bears in mind that in $\mathrm{PSL}_2(\mathbb{F}_7)$ there is exactly one conjugacy class of elements of order 1, 2, 3, 4 and two of order 7 ([8], p. 289), it is easy to show that no other pair of types (p_i, q_i, r_i) is possible. □

While the spectrum of S_6 can be found by computer means (in fact, $\mathrm{Spec}(S_6) = \{(49,91),(91,121),(91,169),(121,169),(151,169)\}$), a theoretical explanation of this fact similar to the one given for the groups S_5 and $\mathrm{PSL}_2(\mathbb{F}_7)$ will take too long to be included here and so we content ourselves with showing that $\mathrm{Spec}(S_6) \neq \emptyset$ since this is all we need to state Corollary 13.

Proposition 14. $\mathrm{Spec}(S_6) \neq \emptyset$

Proof. The following triples of generators produce a Beauville structure with group S_6:

$$\begin{aligned} a_1 &= (12), & b_1 &= (123456), & c_1 &= (26543)\\ a_2 &= (1526), & b_2 &= (1234), & c_2 &= (16)(2543). \end{aligned}$$ □

Corollary 15. $\mathrm{Spec}(S_n)$ *admits a Beauville structure if and only if* $n \geq 5$

By part 1 of the theorem in [3] quoted in the introduction we know that $\mathrm{Spec}(G) = \emptyset$ for all abelian groups except for the groups $(\mathbb{Z}/n\mathbb{Z})^2$, with $(n,6)=1$. For these exceptional groups we have

Theorem 16. *For* $(n,6)=1$,

$$\mathrm{Spec}((\mathbb{Z}/n\mathbb{Z})^2) = \left\{\left(\frac{(n-1)(n-2)}{2}, \frac{(n-1)(n-2)}{2}\right)\right\}.$$

Moreover, the corresponding curves C_1, C_2 *are both isomorphic to the Fermat curve* $x_0^n + x_1^n + x_2^n = 0$.

Proof. Let us denote by F_n the Fermat curve of equation $x_0^n + x_1^n + x_2^n = 0$ and by H_n the subgroup of $\mathrm{Aut}(F_n)$ generated by the automorphisms

$$\tau_1([x_0 : x_1 : x_2]) = [\xi_n x_0 : x_1 : x_2], \quad \tau_2([x_0 : x_1 : x_2] = [x_0 : \xi_n x_1 : x_2]$$

where $\xi_n = e^{\frac{2\pi i}{n}}$.

It is clear that $H_n \cong (\mathbb{Z}/n\mathbb{Z})^2$, and it is easy to see that F_n/H_n is an orbifold of type (n, n, n) [7].

Obviously, the proof will be a consequence of the following two statements.

1) If $(\mathbb{Z}/n\mathbb{Z})^2$ is generated by two elements $a, b \in (\mathbb{Z}/n\mathbb{Z})^2$, then $\mathrm{order}(a) = \mathrm{order}(b) = \mathrm{order}(ab) = n$.

2) For any curve C which, like F_n, admits a group of automorphisms $H \cong (\mathbb{Z}/n\mathbb{Z})^2$ such that C/H is an orbifold of hyperbolic type (n, n, n) there is an isomorphism $\alpha\colon C \to F_n$ such that $\alpha H \alpha^{-1} = H_n$.

1) To prove the first statement set $\mathrm{order}(a) = k$, $\mathrm{order}(b) = m$ and consider the homomorphism

$$\begin{aligned} \varphi\colon \mathbb{Z}/(k) \times \mathbb{Z}/(m) &\to (\mathbb{Z}/n\mathbb{Z})^2, \\ (1,0) &\mapsto a, \\ (0,1) &\mapsto b. \end{aligned}$$

As φ is an epimorphism we have $km \geq n^2$, and since, by definition, k and m are divisors of n we infer that $k = m = n$ and that φ is in fact an isomorphism.

2) For the proof of the second statement we translate matters into the language of section 2 and so we write $C = \mathbb{H}/K$, $C/H = \mathbb{H}/\Gamma_n$ and $H = \Gamma_n/K$, where $\Gamma_n = \Gamma_{(n,n,n)}$ and K is a torsion free subgroup $K \lhd \Gamma_n$. Moreover, since H is abelian, we must also have $[\Gamma_n, \Gamma_n] \lhd K$. We then have obvious epimorphisms

$$\Gamma_n/[\Gamma_n, \Gamma_n] \to \Gamma_n/K = H$$

and

$$\begin{aligned} H \cong (\mathbb{Z}/n\mathbb{Z})^2 &\to \Gamma_n/[\Gamma_n, \Gamma_n], \\ (1,0) &\mapsto \bar{\alpha}, \\ (0,1) &\mapsto \bar{\beta} \end{aligned}$$

where $\alpha, \beta \in \Gamma_n$ is any pair of elements of order n generating Γ_n.

It follows that

$$|\Gamma_n/[\Gamma_n, \Gamma_n]| \geq |\Gamma_n/K| \geq |\Gamma_n/[\Gamma_n, \Gamma_n]|\,.$$

Therefore $C \cong \mathbb{H}/[\Gamma_n, \Gamma_n]$ and $H = \Gamma_n/[\Gamma_n, \Gamma_n]$. The conclusion is that a pair such as (C, H) is unique, hence isomorphic to the Fermat pair (F_n, H_n). □

5. Beauville structures on finite p-groups

While for $p \neq 2, 3$ the groups $(\mathbb{Z}/p\mathbb{Z})^2$ provide examples of p-groups admitting a Beauville structure, it seems that no such examples exist in the literature when $p = 2, 3$. The goal of this section is to construct such groups. We want to stress that it is very plausible that most 2-generated finite p-groups of sufficiently large order have this property. Our examples have orders 2^{12} and 3^{12}. It is very likely that they are not of the smallest possible order.

We will use the following notation. If G is a group and g, h are two elements of G we denote by g^h the element $h^{-1}gh$ and by $[g, h]$ the commutator $g^{-1}h^{-1}gh$. If H_1 and H_2 are two subgroups of G, then $[H_1, H_2]$ will denote the subgroup of G, generated by $\{[g, h] \mid g \in H_1, h \in H_2\}$. We will write $[a, b, c]$ for $[[a, b], c]$. If n is a natural number, we denote by G^n the subgroup of G generated by all the nth powers of the elements of G. We also set $\Phi_n(G) = G^n[G, G]$.

Our examples will be presented as quotients of a free group. We will use the following well-known proposition.

Proposition 17. *Let F be a free group of rank d, p a prime and H a subgroup of index n. Then $|H : H^p[H, H]| = p^{(d-1)n+1}$.*

Proof. This follows from the Nielsen–Schreier formula for the minimal number of generators of H and the fact that H is a free group. □

Most of our calculations will be based on the following well-known proposition.

Proposition 18. *Let F be a free group with free generators $x, y_1, \ldots, y_s$. Let N be a normal subgroup of F such that all the y_i lie in N and F/N is a cyclic group of order n generated by xN. Then*

$$x^n, \ y_1, \ y_1^x, \ \ldots, \ y_1^{x^{n-1}}, \ \ldots, \ y_s^x, \ \ldots, \ y_s^{x^{n-1}}$$

are free generators of N.

Proof. This set of generators is easily obtained by applying the Reidemeister–Schreier rewriting process (see [13] for full details) □

In the following lemma we will show how this proposition may be applied.

Lemma 19. *Let F be a free group on free generators x and y. Then*

(i) $[x, y^2] \notin (F^2)^2$,

(ii) $[y, x, y, y][y, x^2, y, y] \notin \Phi_3(\Phi_3(F))$.

Proof. (i) By Proposition 18, y^2, x, x^y are free generators of $\langle x, F^2 \rangle$. Applying again Proposition 18, we see that

$$x^2, y^2, (y^2)^x, \ldots$$

are free generators of F^2. Hence, $[x, y^2] = ((y^2)^x)^{-1} y^2 \notin (F^2)^2$.

(ii) Using Proposition 18 we obtain that x^3, y, y^x, y^{x^2} are free generators of $\langle y, \Phi_3(F)\rangle$. Hence, $x^3, y, [y,x], [y,x^2]$ are also free generators of $\langle y, \Phi_3(F)\rangle$. Applying again Proposition 18 we see that

$$y^3,\ [y,x],\ [y,x]^y,\ [y,x]^{y^2},\ [y,x^2],\ [y,x^2]^y,\ [y,x^2]^{y^2}, x^3,\ \ldots$$

are free generators of $\Phi_3(F)$. Note that $[y,x,y,y] \in H_1 = \langle [y,x], [y,x]^y, [y,x]^{y^2}\rangle$ and $[y,x,y,y] \notin \Phi_3(\Phi_3(F))$. Also $[y,x^2,y,y] \in H_2 = \langle [y,x], [y,x]^y, [y,x]^{y^2}\rangle$ and $[y,x^2,y,y] \notin \Phi_3(\Phi_3(F))$. Thus, $[y,x,y,y][y,x^2,y,y] \notin \Phi_3(\Phi_3(F))$. □

Theorem 20. *There is a 2-group G of order 2^{12} which admits a Beauville structure.*

Proof. Let F be a free group of rank 2 generated by x and y. Denote by H the normal subgroup generated (as a normal subgroup of F) by $\{x^2, y^2, (xy)^4\}$. Then H has index 8 in F (note that F/H is isomorphic to the dihedral group of order 8). Put $N = H^2 = H^2[H,H]$ and $G = F/N$. Then, by Proposition 17, G has order 2^{12}.

Put $a = xN$, $b = yN$, $a_1 = ab^2$ and $b_1 = b(ab)^4$. We want to show that $\Sigma(a,b)$ and $\Sigma(a_1,b_1)$ have a trivial intersection. Since the order of all elements from $\Sigma(a,b)$ and $\Sigma(a_1,b_1)$ is a power of 2 and these two sets are closed by taking powers, it is enough to compare only elements of order 2. The elements of order 2 in $\Sigma(a,b)$ are the conjugacy classes of $a^2, b^2, (ab)^4$ and the elements of order 2 in $\Sigma(a_1,b_1)$ are the conjugacy classes of $a_1^2, b_1^2, (a_1b_1)^4$.

Let $K = H/N$ and $L = [K,G]$. Note that a^2L is not conjugate to $b_1^2L = b^2L$ and $(a_1b_1)^4L = (ab)^4L$ in G/L, because these three elements are central in G/L and they are different. Using the same argument, we conclude that b^2 is not conjugate to a_1^2 and $(a_1b_1)^4$ and $(ab)^4$ is not conjugate to a_1^2 and b_1^2.

Thus, we only have to check that a^2 is not conjugate to a_1^2, b^2 is not conjugate to b_1^2 and $(ab)^4$ is not conjugate to $(a_1b_1)^4$.

Since the centralizer $C_G(a^2)$ of a^2 in G is $\langle a, K\rangle$, the conjugacy class of a^2 consists of four elements:

$$(a^2)^G = \{a^2, (a^2)^b, (a^2)^{[b,a]}, (a^2)^{ba}\}.$$

Note that $C_G(a_1^2) = \langle a_1, K\rangle = C_G(a^2) = C_G(a^2)^{[a,b]}$, $C_G(a_1^2) \neq C_G(a^2)^b$ and $C_G(a_1^2) \neq C_G(a^2)^{ba}$. Hence, $a_1^2 \neq (a^2)^b$ and $a_1^2 \neq (a^2)^{ba}$.

Observe, now, that $a^2 \equiv (a^2)^{[b,a]} \pmod{(G^2)^2}$ and from part (i) of Lemma 19 it follows that $a_1^2 = (ab^2)^2 = a^2[a,b^2] \not\equiv a^2 \pmod{(G^2)^2}$. Thus, $a_1^2 \notin (a^2)^G$.

We have that $(b_1)^2 = (b(ab)^4)^2 = b^2[b,(ab)^4]$. Note that $[b,(ab)^4]$ commutes with ab and is different from 1. On the other hand, the conjugacy class of b^2 consists of four elements:

$$(b^2)^G = \{b^2[b^2,(ab)^i] \mid i = 0,1,2,3\}$$

and $[b^2,(ab)^i]$ commutes with ab if and only if $i \equiv 0 \pmod 4$. Thus, $b_1^2 \notin (b^2)^G$.

Finally, we have that

$$((ab)^4)^G = \{(ab)^4, ((ab)^4)^a = (ba)^4\}$$

and

$$\begin{aligned}(a_1 b_1)^4 &= (ab^2 b(ab)^4)^4 = ((ab)^5 b^2)^4 \\ &= (ab)^4 [b^2, (ab)^5, (ab)^5, (ab)^5] = (ab^{-1})^4.\end{aligned}$$

Note that xy, yx, xy^{-1} form a free set of generators of $\langle xy, F^2\rangle$ and so

$$(xy)^4, (yx)^4, (xy^{-1})^4$$

are a part of a free set of generators of H. Thus, $(ab)^4$, $(ba)^4$, $(ab^{-1})^4$ are different, whence $(a_1 b_1)^4 \notin ((ab)^4)^G$. □

Theorem 21. *There is a 3-group G of order 3^{12} which admits a Beauville structure.*

Proof. Let F be a free group of rank 2 generated by x and y and $N = \Phi_3(\Phi_3(F))$. We put $G = F/N$. Then, by Proposition 17, G has order 3^{12}.

Put $a = xN$, $b = yN$, $a_1 = ab^3$ and $b_1 = b[a,b]$. We want to show that $\Sigma(a,b)$ and $\Sigma(a_1, b_1)$ have a trivial intersection. Arguing as in the proof of the previous theorem, we see that it is enough to show that a^3 is not conjugate to a_1^3, b^3 is not conjugate to b_1^3 and $(ab)^3$ is not conjugate to $(a_1 b_1)^3$.

For example, let us see that b^3 is not conjugate to b_1^3. First note that

$$b_1^3 = (b[a,b])^3 = b^3[a,b,b,b].$$

On the other hand

$$(b^3)^G = \{b^3, (b^3)^a = b^3[b,a,b,b], (b^3)^{a^2} = [b,a^2,b,b]\}.$$

Note that $[a,b]^{-1} = [b,a]$. Since $[a,b,b,b] \neq 1$,

$$[a,b,b,b] = [b,a,b,b]^{-1} \neq [b,a,b,b],$$

and so $b_1^3 \neq (b^3)^a$.

We can now use part (ii) of Lemma 19 to obtain that $b_1^3 \neq (b^3)^{a^2}$. Thus, we conclude that $b_1^3 \notin (b^3)^G$. The other cases are proved in a similar way. □

Corollary 22. *For every prime number p there is a p-group G which admits a Beauville structure.*

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