

The cohomology of Bestvina–Brady groups

Ian J. Leary and Müge Saadetoğlu*

Dedicated to Warren Dicks, for the occasion of his 60th birthday

Abstract. For each subcomplex of the standard CW-structure on any torus, we compute the homology of a certain infinite cyclic regular covering space. In all cases when the homology is finitely generated, we also compute the cohomology ring. For aspherical subcomplexes of the torus, our computation gives the homology of the groups introduced by M. Bestvina and N. Brady in [3]. We compute the cohomological dimension of each of these groups.

Mathematics Subject Classification (2010). 20J06, 13F55, 20F36, 57M07.

Keywords. Cohomological dimension, cohomology ring, Bestvina–Brady group, Artin group.

1. Introduction

In 1995, Bestvina and Brady introduced a construction which takes as input a finite flag complex L , and outputs a group H_L , the Bestvina–Brady group, whose homological finiteness properties are controlled by the homotopy type of L [3]. Bestvina and Brady’s most remarkable application of this construction was in the case when L is acyclic but not 1-connected, in which case they showed that H_L is type FP but not finitely presentable [3]. Many other groups having surprising combinations of finiteness conditions can be constructed as Bestvina–Brady groups. For example, by choosing L to be acyclic over some rings but not acyclic over others, one obtains a group H_L which is FP over some rings but not FP over others. With the benefit of hindsight, Stallings’ group that is finitely presented but not FP_3 is isomorphic to H_L in the case when L is the octahedron [17], and Bieri’s higher dimensional generalizations of Stallings’ group have a similar interpretation [4].

The original proof of the Bestvina–Brady theorem, and many later generalizations [3], [14], [6] have used techniques from $\text{CAT}(0)$ geometry. The theme of this paper is that many of the homological properties of the groups H_L can be derived using more standard methods from algebraic topology. This is similar in spirit to [8], in

*The first author is partially supported by NSF grant DMS-0505471. The second author is supported by the British Council and by the Ohio State Mathematical Research Institute.

which Dicks and Leary used algebraic techniques to find a presentation for H_L . In particular, we compute the ordinary homology and cohomology of each group H_L , with arbitrary coefficients (in Sections 5 and 6), and we compute the cohomological dimension of H_L over many rings R (in Section 8). One point of interest is that the cohomological dimension of H_L can vary with the choice of ring. Another surprise is that the cohomological dimension of H_L and many of the finiteness properties of H_L are detected by ordinary cohomology. In the case when L is acyclic over a ring R , we also compute the ring structure on the R -cohomology of H_L (Section 6).

One advantage of the algebraic approach is that it applies also in the case when L is infinite. Bestvina–Brady groups H_L for infinite L have found application in [13], so this is not an empty generalization. Moreover, something can be said when L is not a flag complex. The Bestvina–Brady construction really yields an Eilenberg–Mac Lane space for the group H_L rather than the group itself. In the case when L is not flag, this construction still yields a space, although this space is no longer aspherical.

In the next Section, we give a detailed description of the Bestvina–Brady construction and of our results.

2. The spaces T_L and \tilde{T}_L

Let \mathbb{T} be the circle, or 1-dimensional unitary group, given a CW-structure with one 0-cell and one 1-cell. Suppose also that the identity element of the group is chosen to be the 0-cell. For a set V , let $T(V)$ denote the direct sum $T(V) = \bigoplus_{v \in V} \mathbb{T}$. There is a natural CW-structure on $T(V)$ in which the i -cells are in bijective correspondence with i -element subsets of V .

For the purposes of this paper, a simplicial complex will be defined abstractly as a non-empty set of finite sets which is closed under inclusion. The one element members of the set of sets are the vertices of the simplicial complex. Every simplicial complex (including the empty simplicial complex) contains a unique -1 -simplex corresponding to the empty set.

If σ is a finite subset of V , the closure in $T(V)$ of the cell corresponding to σ is equal to $T(\sigma)$, and consists of all the cells corresponding to subsets of σ . It follows that there is a bijective correspondence between simplicial complexes whose vertex set is contained in V and non-empty subcomplexes of $T(V)$ (see [10], 3.23, for this statement in the case when V is finite). The empty simplicial complex corresponds to the subcomplex T_\emptyset consisting of just the single 0-cell of $T(V)$, and a non-empty simplicial complex L corresponds to the complex T_L defined by

$$T_L = \bigcup_{\sigma \in L} T(\sigma).$$

The fundamental group of T_L and the cohomology ring of T_L are easily described in terms of L , and there is a characterisation of those L for which T_L is aspherical. (We shall describe all of these results below.)

A point in $T(V)$ is a vector (t_v) of elements of \mathbb{T} indexed by V , such that only finitely many t_v are not the identity element. The group multiplication induces a map $\mu: T(V) \rightarrow \mathbb{T}$ which takes the point (t_v) to the product of all of the non-identity t_v 's. For each L this induces a cellular map $\mu_L: T_L \rightarrow \mathbb{T}$, and μ_L is surjective when L is non-empty. Our aim is to study the homology and cohomology of the space \tilde{T}_L , the infinite cyclic cover of T_L obtained by pulling back the universal cover of \mathbb{T} via μ_L .

$$\begin{array}{ccc} \tilde{T}_L & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ T_L & \xrightarrow{\mu_L} & \mathbb{T} \end{array}$$

For each L , we describe the homology of \tilde{T}_L , together with information about the \mathbb{Z} -action induced by the action of \mathbb{Z} by deck transformations on \tilde{T}_L . We deduce that for any non-trivial ring R , the R -homology of \tilde{T}_L is finitely generated as an R -module if and only if L is finite and R -acyclic. In all cases when L is R -acyclic, we give a complete description of the cohomology ring $H^*(\tilde{T}_L; R)$.

Let G_L denote the fundamental group of T_L . Since the fundamental group of a CW-complex depends only on its 2-skeleton, the group G_L depends only on the 1-skeleton of the simplicial complex L . The presentation for G_L coming from the cell structure on T_L has one generator for each vertex of L , subject only to the relation that the generators v and w commute whenever $\{v, w\}$ is an edge in L . These groups are known as right-angled Artin groups. It can be shown that T_L is aspherical if and only if L is a flag complex. Every simplicial complex may be completed to a flag complex with the same 1-skeleton (just add in a simplex for each finite complete subgraph of the 1-skeleton) and so one sees that the spaces of the form T_L include models for the classifying spaces of all right-angled Artin groups.

When L is non-empty, \tilde{T}_L is connected and the fundamental group of \tilde{T}_L is the kernel of the induced map $\mu_*: G_L \rightarrow \mathbb{Z}$, which sends each of the generators for G_L to $1 \in \mathbb{Z}$. Call this group H_L . The groups H_L are known as Bestvina–Brady groups. In the case when L is a finite flag complex, M. Bestvina and N. Brady showed that the homological finiteness properties of H_L are determined by the homotopy type of L . For example, they show that H_L is finitely presented if and only if L is 1-connected [3]. For an explicit presentation for H_L for any L , see [8].

Let L be an n -dimensional flag complex. It is easy to show that in this case, the cohomological dimension of the group G_L is equal to $n + 1$. The cohomological dimension of G_L over any non-trivial ring R is also equal to $n + 1$. It also follows easily that the cohomological dimension of H_L is equal to either n or $n + 1$. Our computations together with some of the results from [3] allow us to determine the cohomological dimension of H_L , at least in the case when R is either a field or a subring of the rationals. If $n = 0$ and L is a single point, then H_L is the trivial group. Otherwise, if there exists an R -module A such that $H^n(L; A) \neq 0$, then H_L has cohomological dimension $n + 1$ over R . If there exists no such A , then H_L has cohomological dimension n over R . Note that in contrast to the case of G_L , the

cohomological dimension of H_L may vary with the choice of ring R . As a corollary we deduce that the trivial cohomological dimension and cohomological dimension of H_L are equal.

Some of these results appeared, with slightly different proofs, in the Southampton PhD thesis of the second named author. For some results, we give a brief sketch of a second proof. Some computations of low-dimensional ordinary cohomology (and many other algebraic invariants) for a special class of the Bestvina–Brady groups also appear in a recent paper of S. Papadima and A. Suciu [15].

3. Homology and cohomology of T_L

The differential in the cellular chain complex for $T(V)$ is trivial, and hence so is the differential in the cellular chain complex for T_L , for any L . It follows that for any ring R , $H_i(T_L; R)$ is a free R -module with basis the i -cells of T_L , or equivalently the $(i - 1)$ -simplices of L . The differential in the cellular cochain complex is also trivial. The group $H^i(T_L; R)$ is isomorphic to a direct product of copies of R indexed by the $(i - 1)$ -simplices of L . To describe the ring structure on the cohomology, we first consider the case of the torus $T(V)$.

The cohomology ring $H^*(T(V); R)$ can be described as the exterior algebra $\Lambda_{R,V}^*$. A homogeneous element $f \in \Lambda_{R,V}^n$ is an alternating function $f: V^n \rightarrow R$, where we say that a function is alternating if the following two conditions are satisfied:

- (1) $f(v_1, \dots, v_n) = 0$ whenever there exists $1 \leq i < j \leq n$ with $v_i = v_j$;
- (2) $f(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = -f(v_1, \dots, v_{i+1}, v_i, \dots, v_n)$ for any i with $1 \leq i < n$.

If $f \in \Lambda^i$ and $g \in \Lambda^{n-i}$, the product $f.g$ is the so-called ‘shuffle product’. This is defined in terms of the pointwise product by the equation

$$f.g(v_1, \dots, v_n) = \sum_{\pi} \epsilon(\pi) f(v_{\pi(1)}, \dots, v_{\pi(i)}) g(v_{\pi(i+1)}, \dots, v_{\pi(n)}),$$

where $\epsilon(\pi) \in \{\pm 1\}$ denotes the sign of the permutation π , and the summation ranges over all permutations π such that

$$\pi(1) < \pi(2) < \dots < \pi(i) \quad \text{and} \quad \pi(i+1) < \pi(i+2) < \dots < \pi(n).$$

(The ‘shuffles’ or permutations of the above type are chosen because they are a set of coset representatives in S_n for the subgroup $S_i \times S_{n-i}$, so that each i -element subset of $\{1, \dots, n\}$ is equal to $\{\pi(1), \dots, \pi(i)\}$ for exactly one such π . Any other set of coset representatives could be used instead.)

There is a similar description of the ring structure on $H^*(T_L; R)$ for any simplicial complex L , as the exterior face ring $\Lambda_R^*(L)$ of L . If V is the vertex set of L , $\Lambda_R^*(L)$ is the quotient of $\Lambda_{R,V}^*$ by the homogeneous ideal I_L , with generators the functions

that vanish on every n -tuple (v_1, \dots, v_n) which does not span a simplex of L . The inclusion of T_L in $T(V)$ induces a homomorphism of cohomology rings

$$\Lambda_{R,V}^* \cong H^*(T(V); R) \rightarrow H^*(T_L; R),$$

and it is easy to check (from the additive description of $H^*(T_L; R)$ given above) that this homomorphism is surjective and that its kernel is I_L . Hence one obtains a theorem which was first stated in [12] in the case when L is finite:

Theorem 1. *For any simplicial complex L and any ring R , the cohomology ring $H^*(T_L; R)$ is isomorphic to the exterior face ring $\Lambda_R^*(L)$.*

For any path-connected space X , there is a natural isomorphism between $H^1(X; \mathbb{Z})$ and $\text{Hom}(\pi_1(X), \mathbb{Z})$. The element of $H^1(T_L; \mathbb{Z}) = \Lambda_{\mathbb{Z}}^1(L)$ that corresponds to the homomorphism $\mu_*: G_L \rightarrow \mathbb{Z}$ is the element β_L , the constant function which takes each vertex of L to $1 \in \mathbb{Z}$. By a slight abuse of notation, write β_L also for the element of $\Lambda_R^1(L)$ that takes each vertex of L to $1 \in R$.

In any anticommutative ring, multiplication by an element of odd degree gives rise to a differential. The cochain complex structure on $\Lambda_R^*(L)$ given by multiplication by β_L is easily described.

Theorem 2. *For any ring R , there is a natural isomorphism of cochain complexes*

$$(\Lambda_R^*(L), \beta \times) \cong C_+^{*-1}(L; R)$$

between the exterior face ring of L with differential given by left multiplication by β_L , and the augmented simplicial cochain complex of L shifted in degree by one.

Proof. In degree i , each of the two graded R -modules is isomorphic to a direct product of copies of R indexed by the $(i - 1)$ -simplices of L , or equivalently the R -valued functions on the oriented $(i - 1)$ -simplices of L , where $f(-\sigma) = -f(\sigma)$ if $-\sigma$ is the same simplex as σ with the opposite orientation. It remains to show that this isomorphism is compatible with the differentials on the two cochain complexes.

Let f be an R -valued function on the $(i - 1)$ -simplices of L , and compare the functions $\beta \cdot f$ and δf , the image of f under the differential on $C_+^{*-1}(L; R)$. If (v_0, \dots, v_i) is the vertex set of an oriented i -simplex of L , then

$$\begin{aligned} \beta \cdot f(v_0, \dots, v_i) &= \sum_{j=0}^i (-1)^j \beta(v_j) f(v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_i) \\ &= \sum_{j=0}^i (-1)^j f(v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_i) \\ &= \delta f(v_0, \dots, v_i). \end{aligned}$$

This completes the proof. □

4. Higher homotopy of T_L

Recall that a full subcomplex M of a simplicial complex L is a subcomplex such that if σ is any simplex of L and each vertex of σ is in M , then σ is in M .

Proposition 3. *If M is a full subcomplex of L , then T_M is a retract of T_L .*

Proof. Let W be a subset of V . There is an isomorphism of topological groups $T(V) \cong T(W) \oplus T(V - W)$. The inclusion

$$i : T(W) \cong T(W) \oplus \{1\} \rightarrow T(V)$$

and projection

$$\pi : T(V) \rightarrow T(V)/\{1\} \oplus T(V - W) \cong T(W)$$

satisfy $\pi \circ i = 1_{T(W)}$, and show that $T(W)$ is a retract of $T(V)$.

Now suppose that L is a simplicial complex with vertex set V and that M is the full subcomplex with vertex set $W \subseteq V$. Then T_L is a subcomplex of $T(V)$, and T_M is a subcomplex of $T(W)$. The maps i and π , when restricted to T_M and T_L , show that T_M is a retract of T_L as claimed. \square

Recall that a simplicial complex L is said to be flag if every finite complete subgraph of the 1-skeleton of L is the 1-skeleton of a simplex of L . Any full subcomplex of a flag complex is flag.

Proposition 4. *T_L is aspherical if and only if L is a flag complex.*

Proof. A subset of a CW-complex that meets the interior of infinitely many cells contains an infinite discrete set, so cannot be compact. Hence any map from a sphere to a CW-complex has image inside a finite subcomplex and any homotopy between maps of a sphere into a CW-complex has image contained in a finite subcomplex. Thus it suffices to consider the case when L is finite.

Suppose that L is a finite flag complex with vertex set V . If L is an n -simplex, then T_L is an $(n + 1)$ -torus, and so T_L is aspherical. If L is not a simplex, then there exist $v_1, v_2 \in V$ so that there is no edge in L from v_1 to v_2 . For $i = 1, 2$, let L_i be the full subcomplex of L with vertex set $V - \{v_i\}$, and define L_3 by $L_3 = L_1 \cap L_2$. Then $L = L_1 \cup L_2$, and each of L_1, L_2 and L_3 is flag. By induction, T_{L_i} is aspherical for $i = 1, 2, 3$. Also $T_{L_3} = T_{L_1} \cap T_{L_2}$ is a subcomplex of both T_{L_1} and T_{L_2} . The fundamental group of T_{L_3} maps injectively to the fundamental group of each of T_{L_1} and T_{L_2} , since T_{L_3} is a retract of each of T_{L_1} and T_{L_2} . A theorem of Whitehead [10], 1.B.11, implies that $T_L = T_{L_1} \cup T_{L_2}$ is aspherical.

Conversely, suppose that L is not flag. Then L contains a full subcomplex M which is equal to the boundary of an n -simplex for some $n > 1$. Then T_M is the n -skeleton of an $(n + 1)$ -torus, and so $\pi_n(T_M)$ is non-zero. Since T_M is a retract of T_L it follows that $\pi_n(T_L)$ is non-zero too. \square

Remark 5. There is also a metric proof that T_L is aspherical whenever L is a finite flag complex [3]. (When L is flag, the geodesic metric on $T(L)$ induced by the standard product metric on $T(V)$ is locally CAT(0). A version of the Cartan–Hadamard theorem shows that any locally CAT(0) metric space is aspherical.) We give the above proof instead to emphasize that the metric technology is not needed.

5. Homology of \tilde{T}_L

Let Z denote the fundamental group of \mathbb{T} , an infinite cyclic group, and let z denote a generator for Z . Since \tilde{T}_L is defined in terms of $\mu_L: T_L \rightarrow \mathbb{T}$ by pulling back the universal covering space of \mathbb{T} , Z acts via deck transformations on \tilde{T}_L . When L is non-empty, the map μ_L induces an isomorphism $G_L/H_L \cong Z$. In this section we describe the cellular chain complex and homology of \tilde{T}_L as a Z -module, for every L . Let $C_*^+(L)$ denote the augmented cellular chain complex of L , and let $d = d_L$ be its differential.

Proposition 6. *The cellular chain complex $C_*(\tilde{T}_L)$ is isomorphic to the chain complex $\mathbb{Z}[Z] \otimes C_{*-1}^+(L)$ with differential $(1 - z) \otimes d_L$.*

Proof. Each $(n - 1)$ -simplex σ of L corresponds to a cubical n -cell in T_L , whose opposite faces are identified. In \tilde{T}_L this lifts to a free Z -orbit of n -cells. The i th opposite pair of faces are no longer identified, but differ by the translation action of z . By picking an orbit representative in each orbit of cells, we establish a Z -equivariant bijection between the set of n -cells of \tilde{T}_L and the direct product of Z with the set of $(n - 1)$ -simplices of L . The free abelian group with this basis is naturally isomorphic to $\mathbb{Z}[Z] \otimes C_{n-1}^+(L)$. Let v be any fixed 0-cell of \tilde{T}_L . In each orbit of higher-dimensional cells, pick the orbit representative that has v as a vertex but does not have $z^{-1}v$ as a vertex. With respect to this choice of orbit representatives, the boundary map is as claimed. \square

Corollary 7. *For any L , for any abelian group A , and for any $n \geq 0$, there are short exact sequences of $\mathbb{Z}[Z]$ -modules:*

$$0 \rightarrow B_{n-1}^+(L; A) \rightarrow H_n(\tilde{T}_L; A) \rightarrow \mathbb{Z}[Z] \otimes \bar{H}_{n-1}(L; A) \rightarrow 0,$$

$$0 \rightarrow \mathbb{Z}[Z] \otimes \bar{H}_{n-1}(L; A) \rightarrow H_n(\tilde{T}_L; A) \rightarrow Z_{n-1}^+(L; A) \rightarrow 0,$$

where Z acts trivially on $Z_*^+(L; A)$ and on $B_*^+(L; A)$, the cycles and boundaries in $C_*^+(L; A)$. The inclusion of the Z -fixed points in $H_n(\tilde{T}_L; A)$ gives rise to the first sequence, and the map of $H_n(\tilde{T}_L; A)$ onto its largest Z -invariant quotient gives rise to the second sequence.

In the case when $A = R$, a ring, each sequence admits an $R[Z]$ -module structure. In this case, the first sequence is split if $\bar{H}_{n-1}(L; R)$ is R -projective, and the second sequence always admits an R -module splitting.

Proof. Take elements $p(z) \in \mathbb{Z}[Z]$, and $c \in C_{n-1}^+(L; A)$. The chain $p(z) \otimes c$ is a cycle for $(1-z) \otimes d$ if and only if $d(c) = 0$. The boundary of $p(z) \otimes c$ is $(1-z)p(z) \otimes d(c)$. Thus the cycles Z_n in $C_*(\tilde{T}_L; A)$ may be identified with $\mathbb{Z}[Z] \otimes Z_{n-1}^+(L; A)$, and the boundaries B_n may be identified with $(1-z)\mathbb{Z}[Z] \otimes B_{n-1}^+(L; A)$. Between these lies $B'_n = \mathbb{Z}[Z] \otimes B_{n-1}^+(L; A)$, and B'_n is a $\mathbb{Z}[Z]$ -submodule of Z_n . This gives a short exact sequence of $\mathbb{Z}[Z]$ -modules

$$0 \rightarrow B'_n/B_n \rightarrow H_n(\tilde{T}_L; A) \rightarrow Z_n/B'_n \rightarrow 0,$$

and one sees that $B'_n/B_n \cong B_{n-1}^+(L; A)$ with trivial Z -action and that $Z_n/B'_n \cong \mathbb{Z}[Z] \otimes \bar{H}_{n-1}(L; A)$.

Now define Z'_n to be $(1-z)\mathbb{Z}[Z] \otimes Z_{n-1}^+(L; A)$, a $\mathbb{Z}[Z]$ -submodule of Z_n . As abelian groups, $Z_n = Z'_n \oplus (1 \otimes Z_{n-1}^+(L; A))$. It follows that the short exact sequence

$$0 \rightarrow Z'_n/B_n \rightarrow H_n(\tilde{T}_L; A) \rightarrow Z_n/Z'_n \rightarrow 0$$

is always \mathbb{Z} -split. One sees that $Z'_n/B_n \cong \mathbb{Z}[Z] \otimes \bar{H}_{n-1}(L; A)$ and that $Z_n/Z'_n \cong Z_{n-1}^+(L; A)$.

To compute the Z -fixed points in $H_n(\tilde{T}; A)$, apply the Z -fixed point functor to the first sequence. Since this functor is left-exact, one obtains an exact sequence:

$$0 \rightarrow B_{n-1}^+(L; A) \rightarrow H_n(\tilde{T}_L; A)^Z \rightarrow 0.$$

To compute the maximal Z -fixed quotient of $H_n(\tilde{T}; A)$, start with the short exact sequence

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0,$$

and apply the invariant quotient functor $H_0(Z; -)$. This functor is right-exact, and it is easy to see that $H_0(Z; Z_n) = Z_n/Z'_n$. Hence one obtains an exact sequence:

$$Z_n/Z'_n \rightarrow H_n(\tilde{T}_L; A)_Z \rightarrow 0.$$

Since we have already shown that Z_n/Z'_n is a Z -invariant quotient of the homology group $H_n(\tilde{T}_L; A)$, we see that the maximal Z -invariant quotient is isomorphic to Z_n/Z'_n as claimed.

In the case when $A = R$, a ring, the $\mathbb{Z}[Z]$ -modules and maps that appear in the short exact sequences also admit an R -module structure which commutes with the Z -action. If $\bar{H}_{n-1}(\tilde{T}; R)$ is R -projective, then $\mathbb{Z}[Z] \otimes \bar{H}_{n-1}(\tilde{T}; R)$ is $R[Z]$ -projective, and so the first short exact sequence of $R[Z]$ -modules splits. In any case, $Z_{n-1}^+(L; R)$ is free as an R -module, and so the second short exact sequence admits an R -splitting. \square

Corollary 8. *Let L be a finite complex and R a ring such that $\bar{H}_i(L; R) = 0$ for $i < n$. Suppose also that L has f_i i -dimensional simplices for $i \geq 0$, and define*

$f_{-1} = 1$. For each i with $0 \leq i \leq n$, $H_i(\tilde{T}_L; R)$ is a free R -module of rank

$$\sum_{j=0}^i (-1)^{i+j} f_{j-1}.$$

Proof. For each j , $C_j^+(L; R)$ is a free R -module of rank f_j . For each $i \leq n$, we know that

$$H_i(\tilde{T}_L; R) \cong Z_{i-1}^+(L; R) = B_{i-1}^+(L; R),$$

$$C_i^+(L; R) \cong Z_{i-1}^+(L; R) \oplus Z_i^+(L; R),$$

and that $Z_i^+(L; R)$ is a free R -module. Solving for the rank of $Z_i^+(L; R)$ gives the claimed result. \square

Corollary 9. *If L is finite and \mathbb{Q} -acyclic, then the Euler characteristic of \tilde{T}_L is defined and is given by the formula:*

$$\chi(\tilde{T}_L) = \sum_{i \geq 0} (-1)^i (i + 1) f_i.$$

Proof. Since L is \mathbb{Q} -acyclic, the reduced Euler characteristic $\sum_{i \geq -1} (-1)^i f_i$ of L is equal to zero. Let L be n -dimensional. Then the expression for the rank of $H_i(\tilde{T}_L; \mathbb{Q})$ given by the previous corollary gives

$$\chi(\tilde{T}_L) = \sum_{i=0}^n (-1)^i \sum_{j=0}^i (-1)^{i+j} f_{j-1} = \sum_{i=0}^n (-1)^i (n + 1 - i) f_{i-1}.$$

Hence we see that

$$\chi(\tilde{T}_L) = \sum_{i=0}^n (-1)^i (n + 1 - i) f_{i-1} - (n + 1) \sum_{i=0}^{n+1} (-1)^i f_{i-1} = \sum_{i=0}^{n+1} (-1)^{i-1} i f_{i-1},$$

as claimed. \square

Remark 10. There is a way to deduce Corollary 9 from results of Bestvina and Brady from [3]. This alternative proof was given in the second named author’s PhD thesis [16]. Let $\tilde{\mu}_L$ be the map defined by the pullback square

$$\begin{array}{ccc} \tilde{T}_L & \xrightarrow{\tilde{\mu}_L} & \mathbb{R} \\ \downarrow & & \downarrow \\ T_L & \xrightarrow{\mu_L} & \mathbb{T}. \end{array}$$

In the case when L is finite and \mathbb{Q} -acyclic, Bestvina and Brady show that the inclusion $\tilde{\mu}_L^{-1}(x)$ in \tilde{T}_L is a rational homology isomorphism for any real number x [3]. In the

case when x is not an integer, there is a cellular structure on $\tilde{\mu}_L^{-1}(x)$ with $(i + 1)f_i$ i -cells for each $i \geq 0$.

On the other hand, we know of no other proof of Corollary 7 or Corollary 8 than the proofs given above.

6. Cohomology of \tilde{T}_L

Proposition 11. *Let $C^*(\tilde{T}; A)$ denote the cellular cochain complex for \tilde{T}_L with coefficients in the abelian group A . Each $C^n(\tilde{T}_L; A)$ is isomorphic to a coinduced Z -module:*

$$C^n(\tilde{T}_L; A) \cong \text{Hom}(\mathbb{Z}[Z], C_+^{n-1}(L; A)) \cong \prod_{i \in \mathbb{Z}} C_+^{n-1}(L; A).$$

If A is an R -module for some ring R , then this is an isomorphism of $R[Z]$ -modules. The coboundary map is given by $\delta((f_i)_{i \in \mathbb{Z}}) = (\delta f_i - \delta f_{i+1})_{i \in \mathbb{Z}}$. The action of Z is the ‘shift action’: $z(f_i)_{i \in \mathbb{Z}} = (f_{i+1})$. The image of $C^*(T_L)$ in $C^*(\tilde{T}_L)$ is identified with the ‘constant sequences’, i.e., those with $f_i = f_j$ for all i, j .

Proof. Most of the assertions follow immediately from the description of $C_*(\tilde{T}_L)$ given in Proposition 6, since

$$\begin{aligned} C^*(\tilde{T}_L; A) &\cong \text{Hom}(C_*(\tilde{T}_L), A) \cong \text{Hom}(\mathbb{Z}[Z] \otimes C_{*-1}^+(L), A) \\ &\cong \text{Hom}(\mathbb{Z}[Z], C_+^{*-1}(L; A)). \end{aligned}$$

The claim concerning the image of $C^*(T_L)$ is clear, since cochains that factor through the projection $\tilde{T}_L \rightarrow \tilde{T}_L/Z = T_L$ may be identified with cochains that are fixed by the Z -action. \square

Corollary 12. *For any L , any abelian group A , and any $n \geq 0$ there are short exact sequences of $\mathbb{Z}[Z]$ -modules:*

$$\begin{aligned} 0 &\rightarrow C_+^{n-1}(L; A)/B_+^{n-1}(L; A) \rightarrow H^n(\tilde{T}_L; A) \rightarrow M \rightarrow 0, \\ 0 &\rightarrow \prod_Z \bar{H}^{n-1}(L; A) \rightarrow H^n(\tilde{T}_L; A) \rightarrow C_+^{n-1}(L; A)/Z_+^{n-1}(L; A) \rightarrow 0, \end{aligned}$$

where Z acts trivially on $C_+^*(L; A)$, and $Z_+^*(L; A)$ (resp. $B_+^*(L; A)$) denotes the cocycles (resp. coboundaries) in $C_+^*(L; A)$. The module M fits in to a short exact sequence:

$$0 \rightarrow \bar{H}^{n-1}(L; A) \xrightarrow{\Delta} \prod_Z \bar{H}^{n-1}(L; A) \rightarrow M \rightarrow 0,$$

where Z acts by the ‘shift action’ on the product and where Δ is the inclusion of the constant sequences. The first short exact sequence is the inclusion of the Z -fixed points in $H^n(\tilde{T}_L; A)$. If A is an R -module, then both short exact sequences admit an $R[Z]$ -action.

Proof. Using the description of $C^* = C^*(\tilde{T}_L; A)$ given in Proposition 11, we obtain descriptions of the coboundaries B^n and cocycles Z^n in C^n :

$$B^n \cong \prod_{i \in \mathbb{Z}} B_+^{n-1}(L; A),$$

$$Z^n \cong \{(f_i) \in \prod_{i \in \mathbb{Z}} C_+^{n-1}(L; A) : \delta f_i = \delta f_{i+1} \text{ for all } i \in \mathbb{Z}\}.$$

Let B^m be the submodule of Z^n generated by B^n and the constant sequences $\{(f_i)_{i \in \mathbb{Z}} : f_i = f_j\}$, and let Z^m be the submodule of Z^n consisting of sequences of cocycles, i.e., $Z^m = \prod_{i \in \mathbb{Z}} Z_+^{n-1}(L; A)$. The first short exact sequence in the statement is equal to

$$0 \rightarrow B^m/B^n \rightarrow H^n(\tilde{T}_L; A) \rightarrow Z^n/B^m \rightarrow 0,$$

and the second one to

$$0 \rightarrow Z^m/B^n \rightarrow H^n(\tilde{T}_L; A) \rightarrow Z^n/Z^m \rightarrow 0.$$

The computation of the Z -fixed points in $H^n(\tilde{T}_L; A)$ follows by applying the Z -fixed point functor to the first exact sequence. □

Theorem 13. *Let R be a ring. The image of the map*

$$H^*(T_L; R) \rightarrow H^*(\tilde{T}_L; R)$$

is equal to the Z -fixed point subring of $H^(\tilde{T}_L; R)$ and is isomorphic to the quotient $H^*(T_L; R)/(\beta_L)$. In degree n , the cokernel of this map is isomorphic to an infinite product of copies of $\bar{H}^{n-1}(L; R)$. In particular, the map is a ring isomorphism if and only if L is R -acyclic.*

Proof. This follows from Corollary 12 and Theorem 2. □

Corollary 14. *Suppose that L is R -acyclic. There is an R -algebra isomorphism*

$$H^*(\tilde{T}_L; R) \cong \Lambda_R^*(L)/(\beta_L).$$

For each n , $H^n(\tilde{T}_L; R)$ is isomorphic to a direct product of copies of R .

Proof. This follows from Theorem 13 and Theorem 1 □

Remark 15. The second named author’s PhD thesis contained a different proof of Corollary 14 in the case when L is finite, flag and R -acyclic [16]. Recall that we denote by G_L the fundamental group of T_L , and by H_L the fundamental group of \tilde{T}_L . Also recall from Proposition 4 that when L is flag, each of T_L and \tilde{T}_L is an Eilenberg–Mac Lane space for its fundamental group. In [16] an explicit chain homotopy was used to show that the action of Z on $H^*(\tilde{T}_L; R)$ is trivial when L is finite, flag and R -acyclic. The long exact sequence in group cohomology coming from the isomorphism $G_L = H_L : Z$ was then used to establish the isomorphism of Corollary 14.

7. Cohomological dimension

We define the trivial cohomological dimension of a space X , $\text{tcd}(X)$, to be the supremum of those integers n for which there exists an abelian group A for which the singular cohomology group $H^n(X; A)$ is non-zero. For any non-trivial ring R , we define $\text{tcd}_R(X)$ similarly except that only abelian groups A admitting an R -module structure are considered. We have been unable to find a reference for the notion of trivial cohomological dimension (although we do not believe that it can be a new idea), and so we include proofs for many of its properties below.

Now suppose that X is path-connected, and that G is the fundamental group of X . For M a G -module, we write $H^*(X; M)$ for the singular cohomology of X with twisted coefficients in M . If X admits a universal covering space \tilde{X} , then this is just the cohomology of the cochain complex $\text{Hom}_G(C_*(\tilde{X}), M)$ of G -equivariant singular cochains on \tilde{X} . The cohomological dimension $\text{cd}(X)$ of X is the supremum of those integers n such that there is a G -module M for which $H^n(X; M) \neq \{0\}$. For a non-trivial ring R , $\text{cd}_R(X)$ is defined similarly except that only RG -modules M are considered.

Each of the invariants depends only on the homotopy type of X . In the case when X is a classifying space or Eilenberg–Mac Lane space for G , we write $\text{cd}(G)$, $\text{cd}_R(G)$, $\text{tcd}(G)$ and $\text{tcd}_R(G)$ for the corresponding invariants of X .

We summarize the properties of these invariants below.

Proposition 16. *In the following statements, X is any path-connected space, and R is any non-trivial ring. Rings are assumed to have units, and ring homomorphisms are assumed to be unital.*

- (1) $\text{tcd}(X) = \text{tcd}_{\mathbb{Z}}(X)$, $\text{cd}(X) = \text{cd}_{\mathbb{Z}}(X)$.
- (2) $\text{tcd}_R(X) \leq \text{cd}_R(X)$.
- (3) *If there is a ring homomorphism $\phi: R \rightarrow S$, then $\text{tcd}_S(X) \leq \text{tcd}_R(X)$ and $\text{cd}_S(X) \leq \text{cd}_R(X)$.*
- (4) *If R is a subring of S in such a way that R is a direct summand of S as an R - R -bimodule, then $\text{tcd}_R(X) = \text{tcd}_S(X)$ and $\text{cd}_R(X) = \text{cd}_S(X)$.*
- (5) *If Y is a covering space of X , then $\text{cd}_R(Y) \leq \text{cd}_R(X)$.*
- (6) *If $H \leq G$, then $\text{cd}_R(H) \leq \text{cd}_R(G)$.*
- (7) *For any group G , $\text{cd}_R(G) = \text{proj.dim.}_{RG}(R)$.*

Proof. A $\mathbb{Z}G$ -module is the same thing as a G -module, establishing (1). Similarly, any R -module A may be viewed as an RG -module by letting each element of G act via the identity, which establishes (2). A ring homomorphism ϕ as above allows one to define an R -module structure on any S -module and an RG -module structure on any SG -module, which proves (3). Under the hypotheses of (4), any R -module A is isomorphic to a direct summand of the S -module $S \otimes_R A$ and any RG -module M is isomorphic to a direct summand of the SG -module $SG \otimes_{RG} M$. Since cohomology

commutes with finite direct sums, this shows that $\text{tcd}_R(X) \leq \text{tcd}_S(X)$ and $\text{cd}_R(X) \leq \text{cd}_S(X)$. The opposite inequalities follow from (3).

If the fundamental group of Y is the subgroup $H \leq G$, and M is any RH -module, define the coinduced RG -module by

$$\text{Coind}(M) = \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

Then there is an isomorphism $H^*(X; \text{Coind}(M)) \cong H^*(Y; M)$, which establishes (5). Now (6) is just the special case of (5) in which X is a classifying space for G , since then the covering space of X with fundamental group H is a classifying space for H .

If X is a classifying space for G , then the universal covering space \tilde{X} is contractible, and so $C_*(\tilde{X})$ is a free resolution of \mathbb{Z} over $\mathbb{Z}G$. Similarly, $C_*(\tilde{X}; R)$ is a free resolution of R over RG . Hence $H^*(X; M)$ is isomorphic to $\text{Ext}_{\mathbb{Z}G}(\mathbb{Z}, M)$, and if M is an RG -module $H^*(X; M)$ is also isomorphic to $\text{Ext}_{RG}(R, M)$, which establishes (7). (See for example [5] for more details.) \square

Remark 17. It is easy to find groups G for which $\text{tcd}(G) < \text{cd}(G)$. There are many acyclic groups, or groups for which the group homology $H_i(G; \mathbb{Z}) = 0$ for all $i > 0$ (see for example [1]). For any such G , $\text{tcd}(G) = 0$, while $\text{cd}(G) \neq 0$ unless G is the trivial group.

Before stating the next proposition, we remind the reader that the dimension of a simplicial complex is the supremum of the dimensions of its simplices, so that the empty simplicial complex has dimension -1 .

Proposition 18. *For any simplicial complex L and any non-trivial ring R ,*

$$\text{tcd}_R(T_L) = \text{cd}_R(T_L) = \dim(T_L) = \dim(L) + 1.$$

Proof. Immediate from Theorem 1. \square

Proposition 19. *For any simplicial complex L with at least two vertices and for any R ,*

$$\text{tcd}_R(\tilde{T}_L) = \max\{\dim(L), 1 + \text{tcd}_R(L)\}.$$

Proof. Immediate from Corollary 12. \square

The following theorem is the first result in this paper for which we rely on techniques from Bestvina–Brady [3].

Theorem 20. *Let L be a simplicial complex with at least two vertices and let R be a non-trivial ring.*

$$(1) \dim(L) \leq \text{tcd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_L) \leq \dim(L) + 1.$$

(2) If L is R -acyclic, then $\mathrm{tcd}_R(\tilde{T}_L) = \mathrm{cd}_R(\tilde{T}_L) = \dim(L)$.

Proof. Claim 1 follows easily from earlier results. Proposition 19 implies that $\dim(L) \leq \mathrm{tcd}_R(\tilde{T}_L)$. By Proposition 16, one has that $\mathrm{tcd}_R(\tilde{T}_L) \leq \mathrm{cd}_R(\tilde{T}_L) \leq \mathrm{cd}_R(T_L)$, and of course $\mathrm{cd}_R(T_L) \leq \dim(T_L) = \dim(L) + 1$.

It remains to show that when L is R -acyclic, $\mathrm{cd}_R(\tilde{T}_L) \leq \dim(L)$. Following Bestvina and Brady [3], let X_L be the universal covering space of T_L , or equivalently of \tilde{T}_L . Now let $f_L: X_L \rightarrow \mathbb{R}$ be the composite

$$X_L \rightarrow X_L/H_L = \tilde{T}_L \xrightarrow{\tilde{\mu}} \mathbb{R},$$

where $\tilde{\mu}$ is the lift of the map $\mu_L: T_L \rightarrow \mathbb{T}$. Now define $Y_L = f^{-1}(0) \subseteq X_L$. There is a natural cubical CW-structure on X_L , whose cells are the lifts to X_L of the cells of T_L . One can also put a CW-structure on Y_L , such that each cell of Y_L is the intersection of Y_L with a cell of X_L . For this CW-structure, the dimension of Y_L is equal to $\dim(L)$. It can be shown that X_L is homotopy equivalent to Y_L with infinitely many subspaces homotopy equivalent to L coned off (this is from [3], but see also [13] which explicitly checks this in the case when L is infinite).

Since L is R -acyclic, it follows that the inclusion of Y_L in X_L induces an isomorphism of R -homology. The cellular chain complexes $C_*(Y_L; R)$ and $C_*(X_L; R)$ consist of free RH_L -modules, and so it follows that for any RH_L -module M ,

$$H^*(\tilde{T}_L; M) \cong H^*(Y_L/H_L; M).$$

This shows that $\mathrm{cd}_R(\tilde{T}_L) \leq \dim(L)$ as required. \square

Remark 21. In the case when L is empty, \tilde{T}_L is 0-dimensional, and consists of a single free Z -orbit of points. In the case when L is a single point, \tilde{T}_L is homeomorphic to \mathbb{R} , with Z acting via the translation action of \mathbb{Z} . It is clear that the case when L is empty is exceptional for \tilde{T}_L . The reason why the case when L is a single point needs to be excluded from Proposition 19 and from Theorem 20 is that the formulae for $H^*(\tilde{T}_L; R)$ given in Section 6 involve the *reduced* cohomology of L , whereas the definition of $\mathrm{tcd}(L)$ involves the *unreduced* cohomology of L . This only makes a difference when L is both 0-dimensional and R -acyclic, i.e., the case when L is a single point.

8. Bestvina–Brady groups

In this section, we give a complete calculation of the cohomological dimension of Bestvina–Brady groups, or equivalently the cohomological dimension of the spaces \tilde{T}_L for L a flag complex. We impose some conditions on the coefficient ring R that were not previously required. The reason why we work only with flag complexes in this section is that we need to know that when $L \leq K$ is a full subcomplex of K , then \tilde{T}_L is homotopy equivalent to a covering space of \tilde{T}_K .

Theorem 22. *Let R be either a field or a subring of \mathbb{Q} . Let L be a flag complex with at least two vertices. (This implies that H_L is infinite, and that \tilde{T}_L is a classifying space for H_L .) The following equations hold:*

$$\text{cd}_R(H_L) = \text{tcd}_R(H_L) = \max\{\dim(L), 1 + \text{tcd}_R(L)\}.$$

The proof of the theorem will require two lemmas, the second of which is a strengthening of a lemma from [2].

Lemma 23. *Suppose that L is a flag complex and is a subcomplex of a simplicial complex K . The relative barycentric subdivision $(K, L)'$ is a flag complex containing L as a full subcomplex.*

Proof. Before defining the relative barycentric subdivision $(K, L)'$, we remind the reader that each simplicial complex contains a unique -1 -simplex corresponding to the empty subset of its vertices. The vertex set of $(K, L)'$ is the disjoint union of the vertex set of L and the set of simplices of K not contained in L . An n -simplex of $(K, L)'$ has the form $(\sigma_0 < \sigma_1 < \dots < \sigma_r)$, for some r satisfying $0 \leq r \leq n + 1$. Here σ_0 denotes an $(n - r)$ -simplex of L , each σ_i is a simplex of M , and σ_i is not contained in L if $i > 0$.

Suppose that a finite subset S of the vertices of $(K, L)'$ has the property that any two of its members are joined by an edge. Since L is flag, this implies that the set $S \cap L$ is the vertex set of a simplex σ_0 of L . Each element of $S - L$ is a simplex σ_i of M not contained in L . The existence of an edge between each element of $S \cap L$ and each element of $S - L$ implies that each σ_i contains σ_0 , and the existence of an edge between each pair of elements of $S - L$ implies that the σ_i are totally ordered by inclusion. This shows that $(K, L)'$ is a flag complex.

From the description of the simplices of $(K, L)'$, it is easy to see that any simplex of $(K, L)'$ whose vertex set lies in L is in fact a simplex in L , which verifies that L is a full subcomplex of $(K, L)'$. \square

Lemma 24. *Let L be a flag complex, let R be either a subring of \mathbb{Q} or a field of prime order, and suppose that $\text{tcd}_R(L) < \dim(L)$. Then there is an R -acyclic flag complex K containing L as a full subcomplex such that $\dim(L) = \dim(K)$.*

Proof. We may assume that $\dim(L) = n$ is finite, and we may assume that $n \geq 2$. Let $C'L$ denote the cone on the $(n - 2)$ -skeleton of L , and let L_1 be the union $L_1 = L \cup C'L$. Now L_1 is $(n - 2)$ -connected, and the inclusion map $L \rightarrow L_1$ induces an isomorphism $H^n(L_1; A) \rightarrow H^n(L; A)$ for any abelian group A .

In each case, R is a principal ideal domain, and so by the universal coefficient theorem for cohomology [11], V.3.3, for any R -module A we have that

$$H^n(L_1; A) \cong \text{Hom}_R(H_n(L_1, R), A) \oplus \text{Ext}_R^1(H_{n-1}(L_1, R), A).$$

The hypotheses therefore imply that $H_n(L_1; R) = 0$, and that $H_{n-1}(L_1; R)$ is a projective R -module. (Since L_1 is $(n-2)$ -connected, one also has that $\bar{H}_i(L_1; R) = 0$ for each $i < n-1$.) Every projective module for a principal ideal domain is free [11], I.5.1, and so $H_{n-1}(L_1; R)$ is a free R -module. Since $\bar{H}_{n-2}(L_1; \mathbb{Z}) = 0$ (because L_1 is $(n-2)$ -connected), the universal coefficient theorem for homology [11], V.2.5, tells us that $H_{n-1}(L_1; R) \cong H_{n-1}(L_1; \mathbb{Z}) \otimes R$.

If R is a field of prime order, the map $H_{n-1}(L_1; \mathbb{Z}) \rightarrow H_{n-1}(L_1; R)$ is surjective. If R is a subring of \mathbb{Q} , every element $\xi \in H_{n-1}(L_1; R)$ has the property that there exists λ a unit in R such that $\lambda\xi$ is in the image of the map $H_{n-1}(L_1; \mathbb{Z}) \rightarrow H_{n-1}(L_1; R)$. In each case, it follows that there exist integral cycles $z_i \in C_{n-1}(L_1; \mathbb{Z})$ whose images in the group $C_{n-1}(L_1; R)$ map to an R -basis for $H_{n-1}(L_1; R)$. By the Hurewicz theorem, each z_i is realized by some map $f_i: S^{n-1} \rightarrow L_1$. Replace f_i by a simplicial approximation $f'_i: S_i \rightarrow L_1$, where S_i is some triangulation of the $(n-1)$ -sphere. Using a simplicial mapping cylinder construction as in [10], 2C.5, use each f'_i to attach a triangulated n -cell to L_1 to produce L_2 , an R -acyclic simplicial complex with $\dim(L_2) = n$ and such that $L \leq L_1 \leq L_2$. By Lemma 23, we may take K to be the relative barycentric subdivision $K = (L_2, L)'$. \square

Proof of Theorem 22. If S is any field, and R is the smallest subfield of S , then R and S satisfy the conditions of statement (4) of Proposition 16. Hence there are equalities of functions $\text{cd}_R = \text{cd}_S$ and $\text{tcd}_R = \text{tcd}_S$. Thus it suffices to prove Theorem 22 in the case when R is either the field of p elements or a subring of \mathbb{Q} .

We may assume that $\dim(L) = n < \infty$. In the case when $\text{tcd}_R(L) = n$, Proposition 19 and part (1) of Theorem 20 imply that

$$n + 1 = \text{tcd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_L) \leq \dim(\tilde{T}_L) = n + 1,$$

as claimed. If $\text{tcd}_R(L) < n$, then by Lemma 24, there is an n -dimensional R -acyclic flag complex K containing L as a full subcomplex. Part (2) of Theorem 20 tells us that $\text{tcd}_R(K) = \text{cd}_R(K) = n$. Now H_L is a retract of H_K , and so by part (6) of Proposition 16

$$\text{cd}_R(H_L) = \text{cd}_R(\tilde{T}_L) \leq \text{cd}_R(H_K) = \text{cd}_R(\tilde{T}_K) = n.$$

This gives

$$n = \text{tcd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_K) = n,$$

as claimed. \square

9. Examples

Example 25. Let $L = L(m)$ be a flag triangulation of the space constructed by attaching a disc to a circle via a map of degree m . This L has the property that $\text{tcd}_R(L) = 0$ if m is a unit in R and $\text{tcd}_R(L) = 2$ if m is not a unit in R . Let $H = H_L$

be the corresponding Bestvina–Brady group. From Theorem 20 and Proposition 19, it follows that for any ring R , $\text{cd}_R(H) = 2$ if m is a unit in R and $\text{cd}_R(H) = 3$ if m is not a unit in R .

Example 26. Let L be a flag triangulation of a 2-dimensional Eilenberg–Mac Lane space for the additive group of \mathbb{Q} . Then $\text{tcd}_F(L) = 1$ for any field F , while $\text{tcd}_{\mathbb{Z}}(L) = 2$. From Theorem 22 it follows that the Bestvina–Brady group $H = H_L$ has the property that $\text{cd}_F(H) = 2$ for any field F , while $\text{cd}_{\mathbb{Z}}(H) = 3$.

Groups having similar properties to those given in Example 25 have been constructed previously by many authors, using finite-index subgroups of right-angled Coxeter groups [2], [7], [9]. The examples coming from Coxeter groups have the advantage that they are of type FP, whereas they have the disadvantage that the trivial cohomological dimension of these groups appears to be unknown.

Finite-index subgroups of (non-finitely generated) Coxeter groups were used in [7] to construct groups having similar properties to those given in Example 26. Again, the trivial cohomological dimension of those examples appears to be unknown. It can be shown that for any group G of type FP, there is a field F such that $\text{cd}_F(G) = \text{cd}_{\mathbb{Z}}(G)$. (See Proposition 9 of [7].)

References

- [1] G. Baumslag, E. Dyer, and A. Heller, The topology of discrete groups. *J. Pure Appl. Algebra* **16** (1980), 1–47. [Zbl 0419.20026](#) [MR 549702](#)
- [2] M. Bestvina, The virtual cohomological dimension of Coxeter groups. In *Geometric group theory* (Sussex, 1991), Vol. 1, London Math. Soc. Lecture Note Ser. 181, Cambridge University Press, Cambridge 1993, 19–23. [Zbl 0826.20034](#) [MR 1238512](#)
- [3] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups. *Invent. Math.* **129** (1997), 445–470. [Zbl 0888.20021](#) [MR 1465330](#)
- [4] R. Bieri, *Homological dimension of discrete groups*. Queen Mary College Mathematics Notes, Queen Mary College, London 1976. [Zbl 0357.20027](#) [MR 0466344](#)
- [5] K. S. Brown, *Cohomology of groups*. Graduate Texts in Math. 87, Springer-Verlag, New York 1982. [Zbl 0584.20036](#) [MR 0672956](#)
- [6] K.-U. Bux and C. Gonzalez, The Bestvina–Brady construction revisited: geometric computation of Σ -invariants for right-angled Artin groups. *J. London Math. Soc.* (2) **60** (1999), 793–801. [Zbl 1025.20026](#) [MR 1753814](#)
- [7] W. Dicks and I. J. Leary, On subgroups of Coxeter groups. In *Geometry and cohomology in group theory* (Durham, 1994), London Math. Soc. Lecture Note Ser. 252, Cambridge University Press, Cambridge 1998, 124–160. [Zbl 0913.20031](#) [MR 1709956](#)
- [8] W. Dicks and I. J. Leary, Presentations for subgroups of Artin groups. *Proc. Amer. Math. Soc.* **127** (1999), 343–348. [Zbl 0923.20032](#) [MR 1605948](#)

- [9] A. N. Dranishnikov, On the virtual cohomological dimensions of Coxeter groups. *Proc. Amer. Math. Soc.* **125** (1997), 1885–1891. [Zbl 0943.20037](#) [MR 1422863](#)
- [10] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge 2002. [Zbl 1044.55001](#) [MR 1867354](#)
- [11] P. J. Hilton and U. Stambach, *A course in homological algebra*. Graduate Texts in Math. 4, Springer-Verlag, New York 1971. [Zbl 0238.18006](#) [MR 0346025](#)
- [12] K. H. Kim and F. W. Roush, Homology of certain algebras defined by graphs. *J. Pure Appl. Algebra* **17** (1980), 179–186. [Zbl 0444.05066](#) [MR 567067](#)
- [13] I. J. Leary, On finite subgroups of groups of type VF. *Geom. Topol.* **9** (2005), 1953–1976. [Zbl 1166.20033](#) [MR 2175161](#)
- [14] J. Meier, H. Meinert, and L. VanWyk, Higher generation subgroup sets and the Σ -invariants of graph groups. *Comment. Math. Helv.* **73** (1998), 22–44. [Zbl 0899.57001](#) [MR 1610579](#)
- [15] S. Papadima and A. Suciuc, Algebraic invariants for Bestvina–Brady groups. *J. Lond. Math. Soc. (2)* **76** (2007), 273–292. [Zbl 1176.20038](#) [MR 2363416](#)
- [16] M. Saadetoğlu, Finiteness conditions and Bestvina–Brady groups. Ph.D. thesis, University of Southampton, Southampton 2005.
- [17] J. Stallings, A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.* **85** (1963), 541–543. [Zbl 0122.27301](#) [MR 0158917](#)

Received April 2, 2008; revised September 10, 2009

I. J. Leary, Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210-1174, U.S.A.

and

School of Mathematics, University of Southampton, Southampton, SO17 1BJ, UK

E-mail: leary@math.ohio-state.edu

M. Saadetoğlu, School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom

and

AS277, Department of Mathematics, Eastern Mediterranean University, Gazimağusa, Mersin 10, Turkey

E-mail: muge.saadetoglu@emu.edu.tr