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Analytic general solutions of nonlinear second-order q-difference equations with a double characteristic value

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Abstract – As far as the author knows, it seems that an existence theorem of a solution of a general nonlinear q-difference equation is not known. In this paper we will investigate a nonlinear second order q-difference equation whose characteristic equation has only one solution and will show analytic general solutions of such an equation. Further, we will show an example.

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1. Introduction

In this paper we will study *a nonlinear* second-order q-difference equation for a complex number q. There are many researches on *linear* q-difference equations ([1, 7, 10]). For example, growth properties of transcendental entire or meromorphic solutions of linear homogeneous q-difference equations have been studied by means of analytic methods ([5, 6, 12]). Also, nonlinear q-difference equations in which q is real have been studied ([2, 4, 9, 11, 16]). But, for a complex number q, there are not so many researches on nonlinear q-difference equations. In particular, a result concerning the existence of analytic general solutions of nonlinear second-order q-difference equations is not known. In this paper we will show analytic general solutions of the nonlinear second order q-difference equation

$$f(q^2 z) = u(f(z), f(qz)), \quad z \in \mathbb{C}, \tag{1}$$

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where $u(x, y)$ is an entire function of x, y and q is a complex number satisfying $|q| < 1$ under an assumption that there is an equilibrium point f^*, i.e., $f^* = u(f^*, f^*)$. Note that, without loss of generality, we may assume that $f^* = 0$, that is $u(0, 0) = 0$. Further, we assume that the characteristic equation of (1) has only one solution. The case that the characteristic equation has two different solutions will be treated in another paper.

One of our aims is to obtain the existence of an analytic solution; another aim is to obtain general solutions $f(z)$ of (1) such that $f(q^n z) \to 0$ as $n \to +\infty$.

We define $u(x, y)$ in (1) such that

$$u(x, y) = -\beta x - \alpha y + v(x, y), \quad \beta \neq 0, \tag{2}$$

where v consists of higher order terms for x, y such that

$$v(x, y) = \sum_{\substack{i,j \geq 0 \\ i+j \geq 2}} b_{i,j} x^i y^j \not\equiv 0,$$

and α, β, $b_{i,j}$ are constants. We assume that the modulus of the characteristic value is neither 0 nor 1.

We proceed as follows:

(1) determination of formal solutions;

(2) deriving a particular solution by Schauder's Fixed Point Theorem in a locally convex topological space;

(3) obtaining general solutions.

2. Analytic solutions

2.1 – *A formal solution*

The characteristic equation of (1) with (2) is

$$D(\lambda) = \lambda^2 + \alpha\lambda + \beta = 0. \tag{3}$$

Let λ_1, λ_2 be roots of the characteristic equation and suppose that $\lambda_1 = \lambda_2$. We will study the case $\lambda_1 \neq \lambda_2$ in another paper.

Let $\lambda = \lambda_1 = \lambda_2$. We consider an integer p such that

$$q^p = \lambda. \tag{4}$$

In the case of $|\lambda| > 1$, p is a negative integer. From the definition of q, we have $q^{p'} \neq \lambda$ for all integers $p' > p$. That is,

$$D(q^p) = 0, \quad D(q^{p'}) \neq 0 \quad \text{for all integers } p' \neq p. \tag{5}$$

Here we consider solutions of (1) such that

$$f(q^n z) \to 0 \quad \text{as } n \to +\infty.$$

We observe that a formal solution of (1) is given by a power series at the origin

$$f(z) = \sum_{k=p}^{\infty} a_k z^k \quad \text{with } a_p \neq 0. \tag{6}$$

We substitute

$$\begin{aligned} f(qz) &= \sum_{k=p}^{\infty} a_k (qz)^k = \sum_{k=p}^{\infty} a_k q^k z^k, \\ f(q^2 z) &= \sum_{k=p}^{\infty} a_k (q^2 z)^k = \sum_{k=p}^{\infty} a_k q^{2k} z^k \end{aligned}$$

into (1):

$$\begin{aligned} f(q^2 z) &= -\beta f(z) - \alpha f(qz) + v(f(z), f(qz)) \\ &= -\beta f(z) - \alpha f(qz) + \sum_{\substack{i,j \geq 0 \\ i+j \geq 2}} b_{i,j} f(z)^i f(qz)^j, \end{aligned}$$

i.e.,

$$\begin{aligned} \sum_{k=p}^{\infty} a_k q^{2k} z^k = & -\beta \sum_{k=p}^{\infty} a_k z^k - \alpha \sum_{k=p}^{\infty} a_k q^k z^k \\ & + \sum_{\substack{i,j \geq 0 \\ i+j \geq 2}} b_{i,j} \Big(\sum_{k=p}^{\infty} a_k z^k\Big)^i \Big(\sum_{k=p}^{\infty} a_k q^k z^k\Big)^j \\ = & -\beta \sum_{k=p}^{\infty} a_k z^k - \alpha \sum_{k=p}^{\infty} a_k q^k z^k \\ & + \Big(b_{2,0} \Big(\sum_{k=p}^{\infty} a_k z^k\Big)^2 \Big(\sum_{k=p}^{\infty} a_k q^k z^k\Big)^0 \\ & \quad + b_{1,1} \Big(\sum_{k=p}^{\infty} a_k z^k\Big)^1 \Big(\sum_{k=p}^{\infty} a_k q^k z^k\Big)^1 \\ & \quad + b_{0,2} \Big(\sum_{k=p}^{\infty} a_k z^k\Big)^0 \Big(\sum_{k=p}^{\infty} a_k q^k z^k\Big)^2\Big) \end{aligned} \tag{7}$$

$$
\begin{aligned}
&+\Big(b_{3,0}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^3\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^0 \\
&\quad + b_{2,1}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^2\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^1 \\
&\quad + b_{1,2}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^1\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^2 \\
&\quad + b_{0,3}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^0\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^3\Big) \\
&+\Big(b_{4,0}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^4\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^0 \\
&\quad + b_{3,1}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^3\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^1 \\
&\quad + b_{2,2}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^2\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^2 \\
&\quad + b_{1,3}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^1\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^3 \\
&\quad + b_{0,4}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^0\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^4\Big) \\
&+\sum_{\substack{i,j\geq 0\\ i+j\geq 5}} b_{i,j}\Big(\sum_{k=p}^{\infty}a_k z^k\Big)^i\Big(\sum_{k=p}^{\infty}a_k q^k z^k\Big)^j.
\end{aligned}
$$

Comparing the coefficients of z^l, $l \geq p \geq 1$, we have

$$a_p q^{2p} = -\beta a_p - \alpha a_p q^p,$$

i.e.,

$$a_p(q^{2p} + \alpha q^p + \beta) = a_p D(q^p) = 0. \tag{8}$$

From the definition of p, we can take any value as a_p. From the coefficients of z^k for $k \geq p+1$, we have

$$a_k q^{2k} + \alpha a_k q^k + \beta a_k = C_k(a_p, a_{p+1}, \ldots, a_{k-1}),$$

i.e.,

$$a_k D(q^k) = C_k(a_p, a_{p+1}, \dots, a_{k-1}), \tag{9}$$

where $C_k(a_p, a_{p+1}, \dots, a_{k-1})$ are polynomials of $a_p, a_{p+1}, \dots, a_{k-1}$ with coefficients $q, b_{i,j}, 0 \le i \le k, 0 \le j \le k, 2 \le i + j \le k$. From (5), we have

$$\begin{cases} a_p = \text{any value}, \\ a_k = \frac{C_k(a_p, a_{p+1}, \dots, a_{k-1})}{D(q^k)}, \quad k \ge p+1. \end{cases} \tag{10}$$

Thus, we have non trivial formal solutions of (1) such that

$$f(z) = \sum_{k=p}^{\infty} a_k z^k. \tag{11}$$

On the other hand, if any integer p satisfies $q^p \ne \lambda_1 = \lambda_2$, then we cannot determine an integer p and we cannot have a non-trivial formal solution of (1). Therefore, in this paper we assume that there exists an integer p such that $q^p = \lambda_1 = \lambda_2$.

2.2 – *Existence of an analytic solution*

Here we put $f(z) = s$, $f(qz) = t$, $f(q^2 z) = w$ and $H(s, t, w) = -w + u(s, t)$. Then, equation (1) can be written as

$$H(f(z), f(qz), f(q^2 z)) = 0. \tag{12}$$

$H(s, t, w)$ is holomorphic in a neighborhood of $(0, 0, 0)$ and we have $H(0, 0, 0) = 0$ easily. Furthermore, we have $\frac{\partial H}{\partial s}(0, 0, 0) = \frac{\partial u}{\partial s}\Big|_{s=t=0} = -\beta \ne 0$, as remarked in (2). From the implicit function theorem, for the equation $H(s, t, w) = 0$, we have a holomorphic function ϕ such that

$$s = \phi(t, w), \quad \text{for } |t|, |w| \le \rho, \tag{13}$$

for some $\rho > 0$. Furthermore, we have a constant K such that

$$|s| = |\phi(t, w)| \le K, \quad \text{for } |t|, |w| \le \rho. \tag{14}$$

Let N be a positive integer. Put the partial sum of the formal solution as

$$R_N(z) = \sum_{k=p}^{N} a_n z^k,$$

and put $r_N(z) = f(z) - R_N(z)$. Here we rewrite $r(z) = r_N(z)$.

Moreover, we define the following sets:

$$S(\eta) = \{z \in \mathbb{C} \colon |z| \le \eta\},$$
$$J(A,\eta) = \{r \colon r(z) \text{ is holomorphic and } |r(z)| \le A|z|^{N+1}, \text{ for } z \in S(\eta)\}.$$

in which $A > 0$ and η, $0 < \eta < 1$ are constants. The precise values of A and η will be fixed afterwards.

Suppose there exists a solution $f(z)$ of (1) in $S(\eta)$. Then

$$r_N(z) = f(z) - R_N(z)$$

would belong to $J(A,\eta)$ for some suitably chosen constants A, η, and would satisfy the equation

$$r(q^2 z) = u(r(z) + R_N(z), r(qz) + R_N(qz)) - R_N(q^2 z), \tag{15}$$

with $r(z) = r_N(z)$.

Conversely, suppose there exists a solution $r(z)$ of (15), then

$$f(z) = r(z) + R_N(z)$$

would be a solution of (1). So, hereafter, we concentrate on proving the existence of $r(z) \in J(A,\eta)$ such that $f(z) = r(z) + R_N(z)$ satisfies (15).

From the definition of ϕ, the existence of a solution $f(t)$ of (15) is equivalent to the existence of an $r(z)$ which satisfies

$$r(z) = \phi(r(qz) + R_N(qz), r(q^2 z) + R_N(q^2 z)) - R_N(z). \tag{16}$$

For $r(z) \in J(A,\eta)$, we put

$$T[r](z) = \phi(r(qz) + R_N(qz), r(q^2 z) + R_N(q^2 z)) - R_N(z). \tag{17}$$

Then we show the following theorem.

Theorem 2.1. *Let λ_1, λ_2 be roots of $D(\lambda) = 0$ in (2), with $|q| < 1$. Assume that $\lambda_1 = \lambda_2 = \lambda$ and that there exists an integer p such that $q^p = \lambda$.*

Then, there is an $\eta > 0$ such that we have a holomorphic solution $f(z) = \sum_{k=p}^{\infty} a_k z^k$ of (1) in $S(\eta) = \{z; |z| < \eta\}$.

Proof. From the assumption, we can determine a formal solution as in (11). Thus we will prove the existence of an analytic solution.

At first, we prove that T maps $J(A,\eta)$ into itself. We put

$$\begin{aligned} T[r](z) &= \phi(r(qz)+R_N(qz), r(q^2z)+R_N(q^2z)) - R_N(z) \\ &= v_1(z, r(qz), r(q^2z)) + v_2(z) = v_3(z, r(qz), r(q^2z)), \end{aligned}$$

in which

(18a) $$\begin{aligned} v_1(z, r(qz), r(q^2z)) = \phi(r(qz)+R_N(qz), r(q^2z)+R_N(q^2z)) \\ - \phi(R_N(qz), R_N(q^2z)), \end{aligned}$$

(18b) $$v_2(z) = \phi(R_N(qz), R_N(q^2z)) - R_N(z).$$

Since ϕ is holomorphic on $|t| \le \rho$, $|w| \le \rho$, using Cauchy's integral formula [3], we have

$$\frac{\partial\phi}{\partial t}(t,w) = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{\phi(\xi,w)}{(\xi-t)^2}d\xi.$$

Therefore, when $|t| \le \frac{\rho}{2}$, we have $|\xi - t| \ge |\xi| - |t| \ge \rho - \frac{\rho}{2} = \frac{\rho}{2}$ and

$$\Big|\frac{\partial\phi}{\partial t}(t,w)\Big| \le \frac{1}{\pi}\int_{|\xi|=\rho} \frac{|\phi(\xi,w)|}{(\frac{\rho}{2})^2}|d\xi| \le \frac{1}{\pi}\int_{|\xi|=\rho} \frac{K}{(\frac{\rho}{2})^2}|d\xi| = \frac{8K}{\rho}.$$

When $|w| \le \frac{\rho}{2}$, similarly for w we obtain

$$\Big|\frac{\partial\phi}{\partial w}(t,w)\Big| \le \frac{8K}{\rho}.$$

Hence, we have

(19) $$\Big|\frac{\partial\phi}{\partial t}\Big|, \Big|\frac{\partial\phi}{\partial w}\Big| \le \frac{8K}{\rho}, \quad \text{for } |t|, |w| \le \frac{\rho}{2}.$$

Next, we take A, and take η sufficiently small such that $A\eta^{N+1} < \frac{\rho}{4}$. Then, for sufficiently small $|z|$, we have

$$|r(z)| \le A|z|^{N+1} \le A\eta^{N+1} < \frac{\rho}{4}.$$

Thus, we have

$$\begin{aligned} |r(qz)| &\le A|qz|^{N+1} = A|q|^{N+1}|z|^{N+1} < \frac{\rho}{4}, \\ |r(q^2z)| &\le A|q^2z|^{N+1} = A|q|^{2(N+1)}|z|^{N+1} < \frac{\rho}{4}. \end{aligned}$$

Furthermore, we can take z so small that $|R_N(qz)|, |R_N(q^2z)| < \frac{\rho}{4}$. Then, we obtain

$$|t| = |r(qz) + R_N(qz)| \le \frac{\rho}{2},$$
$$|w| = |r(q^2z) + R_N(q^2z)| \le \frac{\rho}{2}.$$

By (18a), since

$$v_1(z, r(qz), r(q^2z)) = \int_0^1 \frac{d}{dx}\phi(xr(qz) + R_N(qz), xr(q^2z) + R_N(q^2z))dx$$
$$= \int_0^1 \Big(r(qz)\frac{\partial \phi}{\partial t}(*) + r(q^2z)\frac{\partial \phi}{\partial w}(*)\Big)dx,$$

where

$$(*) = (xr(qz) + R_N(qz), xr(q^2z) + R_N(q^2z)),$$

from (19) we have

$$\begin{aligned} &|v_1(z, r(qz), r(q^2z))| \\ &\le \int_0^1 \Big(|r(qz)|\Big|\frac{\partial \phi}{\partial t}(*)\Big| + |r(q^2z)|\Big|\frac{\partial \phi}{\partial w}(*)\Big|\Big)dx \\ &\le \int_0^1 \Big(A|q|^{N+1}|z|^{N+1} \cdot \frac{8K}{\rho} + A|q|^{2(N+1)}|z|^{N+1} \cdot \frac{8K}{\rho}\Big)dx \\ &\le \frac{16K}{\rho}A|q|^{N+1}|z|^{N+1}. \end{aligned} \tag{20}$$

From definition of R_N, ϕ, and (18b), we have

$$|v_2(z)| \le K_1(N)|z|^{N+1}, \tag{21}$$

with a constant $K_1(N)$ which depends on N. From (20) and (21), we have

$$\begin{aligned} |T[r](z)| &\le |v_1(z, r(qz), r(q^2z))| + |v_2(z)| \\ &\le \Big(\frac{16K}{\rho}A|q|^{N+1} + K_1(N)\Big)|z|^{N+1}. \end{aligned}$$

Since $|q| < 1$, if we suppose N is so large that

$$\frac{16K}{\rho}|q|^{N+1} < \frac{1}{4},$$

then we have

$$|T[r](z)| \le \Big(\frac{1}{4}A + K_1(N)\Big)|z|^{N+1}.$$

Furthermore, we take A so large that

$$A > \frac{4}{3}K_1(N),$$

then

$$|T[r](z)| < A|z|^{N+1}.$$

So we obtain that T in (17) maps $J(A,\eta)$ into itself.

The map T is obviously continuous if $J(A,\eta)$ is endowed with topology of uniform convergence on compact sets in $S(\eta)$. Furthermore, $J(A,\eta)$ is clearly convex, and is relatively compact by Montel's theorem [3].

Therefore, by Schauder's fixed point theorem ([8, p. 74] and [13, p. 32]), we obtain the existence of a fixed point $r(z) = r_N(z) \in J(A,\eta)$ of T in $S(\eta)$.

Next, we prove uniqueness of the fixed point.

Suppose that there is another fixed point $r^*(z) = r_N^*(z) \in J(A^*,\eta^*)$. Put

$$\begin{aligned} A_0 &= \max(A, A^*),\\ \eta_0 &= \min(\eta,\eta^*), \end{aligned}$$

$$\begin{aligned} f(z) &= r_N(z) + R_N(z) = f_N(z),\\ f^*(z) &= r_N^*(z) + R_N(z) = f_N(z), \end{aligned}$$

$$h(z) = r_N^*(z) - r_N(z) = h_N(z).$$

Then we have $|h(z)| \le 2A_0|z|^{N+1}$. From (16), we have

$$\begin{aligned} h(z) &= \big(\phi(r_N^*(qz) + R_N(qz), r_N^*(q^2z) + R_N(q^2z)) - R_N(z)\big)\\ &\quad - \big(\phi(r_N(qz) + R_N(qz), r_N(q^2z) + R_N(q^2z)) - R_N(z)\big)\\ &= \phi\big(r_N^*(qz) - r_N(qz) + r_N(qz) + R_N(qz),\\ &\qquad r_N^*(q^2z) - r_N(q^2z) + r_N(q^2z) + R_N(q^2z)\big)\\ &\quad - \phi\big(r_N(qz) + R_N(qz), r_N(q^2z) + R_N(q^2z)\big)\\ &= \phi\big(h_N(qz) + f_N(qz), h_N(q^2z) + f_N(q^2z)\big)\\ &\quad - \phi(f_N(qz), f_N(q^2z))\\ &= \int_0^1 \Big(h_N(qz)\frac{\partial\phi}{\partial t}(**) + h_N(q^2z)\frac{\partial\phi}{\partial w}(**)\Big)dx, \end{aligned}$$

where

$$(**) = (xh_N(qz) + f_N(qz), xh(q^2z) + f_N(q^2z)).$$

If η_0 is sufficiently small, for a constant K_1, from (19) we have

$$\left|\frac{\partial\phi}{\partial t}(**)\right|, \quad \left|\frac{\partial\phi}{\partial w}(**)\right| < \frac{8K_1}{\rho},$$

and we suppose that N is sufficiently large such that $|q|^{N+1} < \frac{\rho}{64K_1}$. Thus we have

$$\begin{aligned}
|h(z)| &\leq \int_0^1 \frac{8K_1}{\rho}(|h_N(qz)| + |h_N(q^2z)|)dx \\
&\leq \int_0^1 \frac{8K_1}{\rho}(2A_0|qz|^{N+1} + 2A_0|q^2z|^{N+1})\, dx \\
&< \int_0^1 \frac{8K_1}{\rho}|q|^{N+1}(2A_0|z|^{N+1} + 2A_0|z|^{N+1})dx \\
&< \frac{1}{2}A_0|z|^{N+1}.
\end{aligned}$$

Then,

$$|h_N(z)| = |r_N^*(z) - r_N(z)| \leq \frac{1}{2}A_0|z|^{N+1} = \left(\frac{1}{4}\right)\cdot 2A_0|z|^{N+1}, \quad \text{for } z \in S(\eta_0).$$

Next, we consider $h_N(z)$ is which

$$|h_N(z)| \leq \frac{1}{4}\cdot 2A_0|z|^{N+1}$$

and repeat this procedure. Then we have

$$|h_N(z)| \leq \left(\frac{1}{4}\right)^2\cdot 2A_0|z|^{N+1}.$$

Repeating the procedure i times we obtain

$$|r_N^*(z) - r_N(z)| < \left(\frac{1}{4}\right)^i(2A_0)|z|^{N+1}, \quad i = 1, 2, \ldots$$

Letting $i \to \infty$, we have

$$r_N^*(z) = r_N(z), \quad z \in S(\eta_0).$$

Thus, $r_N^*(z) = r^*(z)$ and $r_N(z) = r(z)$ are holomorphic in $|z| \leq \min(\eta, \eta^*)$ and $r^*(z) \equiv r(z)$ in $z \in S(\eta_0)$. Hence $r_N^*(z) = r_N(z)$ can be continued analytically to $S(\eta_1)$, $\eta_1 = \max(\eta, \eta^*)$.

Finally, we show the independence from N. Here we will show that the solution $f_N(z)$ given by $f_N(z) = r_N(z) + R_N(z)$ does not depend on N. Let $r_N(z) \in J(A_N, \eta_N)$ and $r_{N+1}(z) \in J(A_{N+1}, \eta_{N+1})$ be fixed points of T, and

$$\begin{aligned} f_{N+1}(z) &= r_{N+1}(z) + R_{N+1}(z) \\ &= r_{N+1}(z) + a_{N+1}z^{N+1} + R_N(z) \\ &= \tilde{r}_N(z) + R_N(z). \end{aligned}$$

Then,

$$\begin{aligned} |\tilde{r}_N(z)| &= |r_{N+1}(z) + a_{N+1}z^{N+1}| \\ &\le A_{N+1}|z|^{N+2} + |a_{N+1}| \cdot |z|^{N+1} \\ &= (A_{N+1}|z| + |a_{N+1}|)|z|^{N+1} \\ &= A_N^*|z|^{N+1}, \end{aligned}$$

where

$$A_N^* = A_{N+1}|z| + |a_{N+1}|.$$

We put $A = \max(A_N, A_N^*)$. By the uniqueness of the fixed point, we have $\tilde{r}_N(z) = r_N(z)$ for $z \in S(\eta_N) \cap S(\eta_{N+1})$. Thus,

$$f_{N+1}(z) = f_N(z) \quad \text{in } S(\eta_N) \cap S(\eta_{N+1}).$$

By analytic prolongation [3], both of $f_N(z)$ and $f_{N+1}(z)$ are holomorphic in $S(\eta_N) \cap S(\eta_{N+1})$ and coincide there. Hence, both of them are continued analytically to $S(\eta_N) \cup S(\eta_{N+1})$ and

$$f_{N+1}(z) = f_N(z) \quad \text{in } S(\eta_N) \cup S(\eta_{N+1}).$$

Hence we have an analytic solution $f(z)$ in $S(\eta)$. ∎

The function $\phi(t, w)$ in (13), $s = \phi(t, w)$ for $|t|, |w| \le \rho$, is defined only locally, though we can also analytically continue $f(z)$, keeping out of branch points. The solution obtained is multi-valued.

3. Analytic general solutions

3.1 – *A relation with a some functional equation*

Let $f(z)$ be a solution of (1), and $g(z) = f(qz)$. Then (1) can be written as a system of simultaneous equations

$$\begin{pmatrix} f(qz) \\ g(qz) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} + \begin{pmatrix} 0 \\ v(f(z), g(z)) \end{pmatrix} \tag{22}$$

Let λ_1, λ_2 be roots of equation (3) in which $\lambda_1 = \lambda_2$. Hence, we have $\alpha^2 - 4\beta = 0$. That is,

$$\beta = \frac{1}{4}\alpha^2. \tag{23}$$

Further, from

$$D(\lambda) = \lambda^2 + \alpha\lambda + \beta = \lambda^2 + \alpha\lambda + \frac{1}{4}\alpha^2 = \left(\lambda + \frac{1}{2}\alpha\right)^2 = 0,$$

we have

$$\lambda = -\frac{1}{2}\alpha.$$

Put

$$A = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}\alpha^2 & -\alpha \end{pmatrix}.$$

Since $\operatorname{rank}(A - \lambda E) = 1$, set

$$P = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda + 1 \end{pmatrix}, \tag{24}$$

and we have

$$P^{-1} = \begin{pmatrix} \lambda + 1 & -1 \\ -\lambda & 1 \end{pmatrix}. \tag{25}$$

Put

$$\begin{pmatrix} f \\ g \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}. \tag{26}$$

Then, (22) is transformed to the following system with respect to x, y:

$$\begin{cases} x(qz) = \lambda x(z) + y(z) + \displaystyle\sum_{i+j\geq 2} c_{ij} x(z)^i y(z)^j = X(x(z), y(z)), \\ y(qz) = \lambda y(z) + \displaystyle\sum_{i+j\geq 2} d_{ij} x(z)^i y(z)^j = Y(x(z), y(z)). \end{cases} \tag{27}$$

Suppose that (27) admits a solution $(x(z), y(z))$. If $\frac{dx}{dz}(z_0) \neq 0$ for some z_0, then we can write $z = \psi(x)$ in a neighborhood V_0 of $x_0 = x(z_0)$. Then

$$y = y(z) = y(\psi(x)) = \Psi(x) \tag{28}$$

is defined in V_0. The function $\Psi(x)$ can be continued analytically avoiding critical values of $x = x(z)$.

Suppose $x(q^n z), y(q^n z) \to 0$ as $n \to +\infty$, uniformly on any compact set on the z-plane. If for any compact set L on the z-plane, there is $n_0 = n_0(L) \in \mathbb{N}$ such that $\frac{dx}{dz}(q^n z) \neq 0$ ($z \in L$, $n \geq n_0$), then $y = \Psi(x)$ is defined in a punctured disc $\mathbb{D}_0 = \{x; 0 < |x| < \delta\}$, which may be multi-valued. If $\Psi(x)$ is single-valued in $\mathbb{D}_0$, then, since $y(q^n z) \to 0$ as $n \to \infty$, the function $\Psi(x)$ is bounded, and hence *holomorphic also at* $x = 0$. By (28), $x(z)$ satisfies $x(qz) = X(x(z), y(z)) = X(x(z), \Psi(x(z)))$, that is,

$$x(qz) = X(x(z), \Psi(x(z))), \tag{29}$$

where $\Psi(x)$ is holomorphic in a neighborhood $\mathbb{D} = \{x\colon |x| < \delta\}$ of $x = 0$. Since $y(qz) = Y(x(z), y(z))$ and $y(qz) = \Psi(x(qz)) = \Psi(X(x(z), \Psi(x(z)))$, the function $\Psi(x)$ satisfies the following functional equation:

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)). \tag{30}$$

In [14, Theorems 1 and 2], we proved that equation (30) admits a holomorphic solution $\Psi(x) = \sum_{n=2}^{\infty} \gamma_n x^n$, in a neighborhood of $x = 0$, which is uniquely determined supposed γ_2 is prescribed. Therefore, the function $y = \Psi(x)$, obtained by analytic continuation from $(x(t), y(t))$, coincides with the holomorphic solution of equation (30).

Conversely, suppose $\Psi(x)$ be a solution of (30) in $\mathbb{D}$. Let $x(z)$ be a solution of the first order equation (29), such that $x(q^n z), \Psi(x(q^n z)) \in \mathbb{D}$ for $n \geq n_0$, for sufficiently large $n_0 \in \mathbb{N}$. Here we rewrite $q^n z$ as in z and put $y(z) = \Psi(x(z))$. Then $y(qz) = \Psi(x(qz))$, and

$$y(qz) = \Psi(X(x(z), y(z)) = Y(x(z), y(z)), \quad x(qz) = X(x(z), y(z)). \tag{31}$$

Thus $(x(z), y(z))$ is a solution of (27). Therefore, the system of equations (27) is reduced to the single *first-order* difference equation (29).

Therefore, we have following result.

Proposition 3.1. *Suppose that* (27) *admits a solution* $(x(z), y(z))$ *such that* $x(q^n z) \to 0$, $y(q^n z) \to 0$ *as* $n \to +\infty$*, uniformly on any compact set on the* z*-plane. Further, suppose that for any compact set* L *on the* z*-plane, there exists* $n_0 = n_0(L) \in \mathbb{N}$ *such that* $\frac{dx}{dz}(q^n z) \neq 0$ $(z \in L, n \geq n_0)$*. Then a function* Ψ *is defined such that* $y = \Psi(x)$ *in a punctured disc* $\mathbb{D}_0 = \{x; 0 < |x| < \delta\}$*. If* $\Psi(x)$ *is single-valued in* $\mathbb{D}_0$*, then* $\Psi(x)$ *is holomorphic also at* $x = 0$ *and satisfies the functional equation* (30) *in a neighborhood of* $x = 0$*,* $\mathbb{D} = \{x\colon |x| < \delta\}$*.*

Conversely, suppose $\Psi(x)$ *be a solution of* (30) *in* $\mathbb{D}$*. Let* $x(z)$ *be a solution of the first-order equation* (29)*, such that* $x(q^n z), \Psi(x(q^n z)) \in \mathbb{D}$ *for* $n \geq n_0$*, for sufficiently* *large* $n_0 \in \mathbb{N}$. *Rewrite* $q^n z$ *as in* z, *and put* $y(z) = \Psi(x(z))$. *Then* $(x(z), y(z))$ *is a solution of* (27).

For the problems of nonlinear second order difference equations (see, e.g., [14, 15]), we considered this relationship of (30) and an equation $x(t+1) = X(x(z), \Psi(x(z)))$. For the nonlinear second order q-difference equations, Proposition 3.1 is a point of our method.

3.2 – *General solutions*

Hereafter, we consider solutions $f(z)$ of (1), which satisfy the following conditions (a), (aD), as well as the condition (ab). Further, we will make use of notations introduced in the last Section 3.1, i.e., the matrix P, the solutions $x(z)$, $y(z)$ of the system of equations (27), and the function $y = \Psi(x)$ defined there.

Let $f(z)$ be a solution of (1). Suppose that

$$f(q^n z) \to 0 \tag{a}$$

as $n \to +\infty$ uniformly on any compact set L on the z-plane. Further, we suppose that, for any compact set L on the z-plane, there is an $n_0 = n_0(L) \in \mathbb{N}$ such that

$$\frac{df}{dz}(q^n z) \neq 0, \quad \text{for } n \geq n_0, \tag{aD}$$

When f satisfies (a) and (aD), since

$$\begin{pmatrix} x \\ y \end{pmatrix} = P^{-1} \begin{pmatrix} f \\ g \end{pmatrix},$$

we have

$$x(q^n z) \to 0 \quad \text{as } n \to 0,$$

and

$$\frac{dx}{dz}(q^n z) \neq 0, \quad \text{for } n \geq n_0,$$

for any $z \in L$, where L is any compact subset on the z-plane. Then the function $\Psi(x)$ is defined as in (28) in a punctured neighborhood $\mathbb{D}_0 = \{x \mid 0 < |x| < \delta\}$. We further suppose that

$$\Psi(x) \text{ is single-valued in } \mathbb{D}_0. \tag{ab}$$

As seen in the last Section 3.1, $y = \Psi(x)$ is holomorphic in $\mathbb{D} = \{x \mid |x| < \delta\}$.

Then we will show the following Theorem 3.2.

THEOREM 3.2. *Let λ_1, λ_2 be roots of the characteristic equation of (1) such that $\lambda_1 = \lambda_2$. We assume that $q = \lambda = \lambda_1 = \lambda_2$, and suppose $|q| < 1$.*

Suppose that $f_1(z)$ is the solution of (1) constructed in Section 2 which have the expansions $f_1(z) = \sum_{k=1}^{\infty} a_k z^k$ in $S(\eta) = \{z; |z| < \eta\}$, with some constants $a_1 \neq 0$ and $\eta > 0$. Further, suppose that $\mathcal{F}(z)$ is an analytic solution of (1) which satisfies (a), (aD), and (ab), uniformly on any compact subset on the z-plane. Then, there is a constant π_0 such that

$$\mathcal{F}(z) = (q+1)\sum_{k=1}^{\infty} a_k(\pi_0 z)^k - \sum_{k=1}^{\infty} a_k(q\pi_0 z)^k + \Psi\Big((q+1)\sum_{k=1}^{\infty} a_k(\pi_0 z)^k - \sum_{k=1}^{\infty} a_k(q\pi_0 z)^k\Big), \tag{32}$$

in $S(\eta)$. Further, $\frac{\mathcal{F}(q^{n+1}z)}{\mathcal{F}(q^n z)} \to q$ as $n \to +\infty$. Where Ψ is a solution of (30).

Conversely, a function $\mathcal{F}(z)$ which is represented as in (32) in $S(\eta)$, for some $\eta > 0$ and a constant π_0, is a solution of (1) such that $\mathcal{F}(q^n z) \to 0$ and $\frac{\mathcal{F}(q^{n+1}z)}{\mathcal{F}(q^n z)} \to q$ as $n \to +\infty$.

PROOF. Suppose $\mathcal{F}(z)$ be a solution of (1) that satisfies (a), (aD), and (ab), uniformly on any compact subsets on the z-plane.

Put

$$\mathcal{G}(z) = \mathcal{F}(qz)$$

and

$$\begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} = P^{-1}\begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix}, \tag{33}$$

with P in (24). Then

$$\mathcal{X}(z) = (\lambda + 1)\mathcal{F}(z) - \mathcal{G}(z).$$

Since $\mathcal{F}(q^n z) \to 0$ and $\mathcal{G}(q^n z) \to 0$ as $n \to \infty$, we have

$$\mathcal{X}(q^n z) \to 0 \quad \text{as } n \to \infty. \tag{34}$$

Let $f_1(z)$ be a solution given in Section 2. By the assumptions, since $q = \lambda = \lambda_1 = \lambda_2$, we have $p = 1$ and

$$f_1(z) = \sum_{k=1}^{\infty} a_k z^k, \quad a_1 \neq 0, \tag{35}$$

in $S(\eta) = \{z; |z| < \eta\}$, such that $f_1(q^n z) \to 0$ as $n \to \infty$. Then since $\mathcal{G}(z) = \mathcal{F}(qz)$, we can write by (33) as

$$\begin{cases} x_1(z) = (\lambda + 1) f_1(z) - f_1(qz) = \displaystyle\sum_{k=1}^{\infty} a_k(\lambda + 1 - q^k) z^k, \\ y_1(z) = -\lambda f_1(z) + f_1(qz) = \displaystyle\sum_{k=1}^{\infty} a_k(-\lambda + q^k) z^k, \end{cases} \tag{36}$$

where x_1 is a function of z and $x_1(0) = 0$, $x_1'(0) = a_1(\lambda + 1 - q) = a_1 \neq 0$, hence x_1 is, by the supposition, a holomorphic function of z in $|z| < \eta$. Thus, z is holomorphic with respect to x_1 in $|x_1| < \tilde{\delta}$ with some $\tilde{\delta} > 0$. Therefore, $y_1(z)$ is holomorphic with respect to x_1, and can be written as $y_1 = \Psi(x_1)$, which is holomorphic at $x_1 = 0$. From Proposition 3.1, we see that $\Psi(x_1)$ satisfies the equation (30). Therefore, $x_1(z)$ is the solution of

$$x_1(qz) = X(x_1(z), \Psi(x_1(z))). \tag{37}$$

From assumption on $\mathcal{F}(z)$, similarly we have $\mathcal{X}(z)$, $\mathcal{Y}(z)$ such that $\mathcal{Y}(z) = \Psi(\mathcal{X}(z))$, and

$$\mathcal{X}(qz) = X(\mathcal{X}(z), \Psi(\mathcal{X}(z))). \tag{38}$$

Here we set a map $U\colon z \to \zeta = x_1(z)$ such that

$$U(z) = x_1(z) = \sum_{k=1}^{\infty} a_k(\lambda + 1 - q^k) z^k. \tag{39}$$

Then, since the map $U(z)$ is an open map for any $\delta_1 > 0$, there is a $\delta_2 > 0$ such that

$$U(\{|z| < \delta_1\}) \supset \{|\zeta| < \delta_2\}. \tag{40}$$

Since $\mathcal{X}(q^n z) \to 0$ as $n \to \infty$, if z belongs to a compact set L on the z-plane, then there is an $n_0 \in \mathbb{N}$ such that, for $z' \in L$,

$$|\mathcal{X}(q^n z')| < \delta_2 < \delta, \quad \text{for } n \geq n_0. \tag{41}$$

Thus, from (40), there is a z^* such that

$$\mathcal{X}(q^n z') = U(z^*). \tag{42}$$

From $\lambda_1 = q$, since $U'(0) = a_1 \neq 0$, by making use of the implict function theorem, we have a U^{-1} such that

$$z^* = U^{-1}(\mathcal{X}(q^n z')). \tag{43}$$

Put $z = q^n z'$, then $z^* = U^{-1}(\mathcal{X}(z))$, and we write

$$z^* = U^{-1}(\mathcal{X}(z)) = \ell(z). \tag{44}$$

From (40), since $\mathcal{X}(z) \in \mathbb{D} = \{z \mid |z| < \delta\}$, there is a holomorphic solution $\Psi(\mathcal{X})$ of (30) in this case (a), (aD), and (ab). Since $\mathcal{X}(z)$ is a solution of (27), $\mathcal{X}(z)$ is a solution of the first-order difference equation (38), further, from (39) and (42), we have

$$\begin{aligned}\mathcal{X}(qz) &= X(\mathcal{X}(z), \Psi(\mathcal{X}(z)))\\ &= X(U(z^*), \Psi(U(z^*)))\\ &= X(x_1(z^*), \Psi(x_1(z^*)))\\ &= x_1(qz^*) = U(qz^*).\end{aligned}$$

Hence, $qz^* = U^{-1}(\mathcal{X}(qz))$, and we have

$$qz^* = \ell(qz), \quad q\ell(z) = qz^* = \ell(qz). \tag{45}$$

If we put $\pi(z) = \ell(z)/z$, then we obtain

$$\pi(qz) = \ell(qz)/qz = (q\ell(z))/qz = \ell(z)/z = \pi(z),$$

and we can write

$$\ell(z) = z\pi(z), \tag{46}$$

where $\pi(z)$ is defined for a compact set L with $|z|$ sufficiently large. Furthermore, we can continue the $\pi(z)$ analytically as a periodic function with period q. Hence, we have $\pi(q^n z) = \pi(z)$ for all $n \in \mathbb{N}$, for any z in L. Since $|q| < 1$,

$$\pi(z) = \pi(q^n z) \to \pi(0) \quad \text{as } n \to \infty,$$

for any $z \in L$. Thus, the periodic function $\pi(z)$ is a constant function, i.e., we have a constant π_0 defended by

$$\frac{U^{-1}(\mathcal{X}(z))}{z} \to \pi_0 \quad \text{as } z \to 0, \tag{47}$$

and from (46) we can write $l(z) = \pi_0 z$. Thus, from (44) we have

$$z^* = \pi_0 z, \tag{48}$$

From (42) and (39), $\mathcal{X}(z)$ can be written as

$$\mathcal{X}(z) = U(z^*) = U(\pi_0 z) = x_1(\pi_0 z) = (\lambda + 1) f_1(\pi_0 z) - f_1(q\pi_0 z). \tag{49}$$

Since $\mathcal{Y} = \Psi(\mathcal{X})$, by making use of the equation (33), we have

$$\begin{aligned}
\mathcal{F}(z) &= \mathcal{X}(z) + \mathcal{Y}(z) \\
&= \mathcal{X}(z) + \Psi(\mathcal{X}(z)) \\
&= x_1(\pi_0 z) + \Psi(x_1(\pi_0 z)) \\
&= (\lambda + 1) f_1(\pi_0 z) - f_1(q\pi_0 z) \\
&\quad + \Psi\big((\lambda + 1) f_1(\pi_0 z) - f_1(q\pi_0 z)\big) \\
&= (q + 1) \sum_{k=1}^{\infty} a_k (\pi_0 z)^k - \sum_{k=1}^{\infty} a_k (q\pi_0 z)^k \\
&\quad + \Psi\Big((q + 1) \sum_{k=1}^{\infty} a_k (\pi_0 z)^k - \sum_{k=1}^{\infty} a_k (q\pi_0 z)^k\Big),
\end{aligned}$$

where π_0 is a constant defined by (47) for $z \in L$. Since L is arbitrary and $|q| < 1$, we can have the constant π_0 for any $z \in \mathbb{C}$, and Ψ is a solution of (30). By making use of [14, Theorem 2], in a neighborhood of $x = 0$, a non trivial solution Ψ is obtained in the form

$$\Psi(x) = \sum_{k=2}^{\infty} \gamma_k x^k. \tag{50}$$

In fact,

$$\begin{aligned}
&\Psi(X(x, \Psi(x))) \\
&\quad = \gamma_2 (X(x, \Psi(x)))^2 + \gamma_3 (X(x, \Psi(x)))^3 + \gamma_4 (X(x, \Psi(x)))^4 + \cdots \\
&\quad = \gamma_2 \Big(\lambda x + \Psi(x) + \sum_{i+j \geq 2} c_{ij} x^i \Psi(x)^j\Big)^2 + \gamma_3 \Big(\lambda x + \Psi(x) + \sum_{i+j \geq 2} c_{ij} x^i \Psi(x)^j\Big)^3 \\
&\qquad + \gamma_4 \Big(\lambda x + \Psi(x) + \sum_{i+j \geq 2} c_{ij} x^i \Psi(x)^j\Big)^4 + \cdots \\
&\quad = \gamma_2 \Big(\lambda x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \cdots \\
&\qquad\qquad + \sum_{i+j \geq 2} c_{ij} x^i (\gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \cdots)^j\Big)^2 \\
&\quad + \gamma_3 \Big(\lambda x + \gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \cdots \\
&\qquad\qquad + \sum_{i+j \geq 2} c_{ij} x^i (\gamma_2 x^2 + \gamma_3 x^3 + \gamma_4 x^4 + \cdots)^j\Big)^3
\end{aligned}$$

$$
\begin{aligned}
&+\gamma_4\Big(\lambda x+\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots\\
&\qquad+\sum_{i+j\ge 2}c_{ij}x^i(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^j\Big)^4+\cdots\\
&=\gamma_2\big(\lambda x+\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots\\
&\qquad+c_{20}x^2+c_{11}x(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)\\
&\qquad+c_{02}(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^2+\cdots\big)^2\\
&\quad+\gamma_3\big(\lambda x+\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots\\
&\qquad+c_{20}x^2+c_{11}x(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)\\
&\qquad+c_{02}(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^2+\cdots\big)^3\\
&\quad+\gamma_4\big(\lambda_1 x+\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots\\
&\qquad+c_{20}x^2+c_{11}x(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)\\
&\qquad+c_{02}(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^2+\cdots\big)^4\\
&\quad+\cdots
\end{aligned}
$$

and

$$
\begin{aligned}
&Y(x,\Psi(x(z)))\\
&=\lambda\Psi(x)+\sum_{i+j\ge 2}d_{ij}x^i\Psi(x)^j\\
&=\lambda(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)+\sum_{i+j\ge 2}d_{ij}x^i(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^j\\
&=\lambda(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)+d_{20}x^2\\
&\quad+d_{11}x(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)+d_{02}(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^2\\
&\quad+d_{30}x^3+d_{21}x^2(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)\\
&\quad+d_{12}x(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4\cdots)^2+d_{03}(\gamma_2x^2+\gamma_3x^3+\gamma_4x^4+\cdots)^3+\cdots.
\end{aligned}
$$

Hence,

$$\gamma_2(\lambda^2-\lambda)=d_{20}.$$

From the assumptions $q=\lambda$ and $|q|<1$, we have $\lambda^2-\lambda\neq 0$ and $\gamma_2=d_{20}/(\lambda^2-\lambda)$, that is, the expansion of $\Psi(x)$ begins with x^2. From $\mathcal{X}(qz)=X(\mathcal{X}(z),\Psi(\mathcal{X}(z)))$, we have

$$\mathcal{X}(qz)=\lambda\mathcal{X}(z)+\Psi(\mathcal{X}(z))+\sum_{i+j\ge 2}c_{ij}\,\mathcal{X}(z)^i\,\Psi(\mathcal{X}(z))^j,$$

and
$$\frac{\mathcal{X}(qz)}{\mathcal{X}(z)} = \lambda + \frac{\Psi(\mathcal{X}(z))}{\mathcal{X}(z)} + \sum_{i+j\geq 2} c_{ij}\,\mathcal{X}(z)^{i-1}\Psi(\mathcal{X}(z))^j.$$

From (34), since $\mathcal{X}(q^n z) \to 0$ as $n \to +\infty$, and by (50),
$$\Psi(\mathcal{X}(q^n z)) \to 0, \qquad \frac{\Psi(\mathcal{X}(q^n z))}{\mathcal{X}(q^n z)} \to 0,$$

and
$$\frac{\mathcal{X}(q^{n+1} z)}{\mathcal{X}(q^n z)} = \lambda + \frac{\Psi(\mathcal{X}(q^n z))}{\mathcal{X}(q^n z)} + \sum_{i+j\geq 2} c_{ij}\,\mathcal{X}(q^n z)^{i-1}\Psi(\mathcal{X}(q^n z))^j \to \lambda$$

as $n \to +\infty$. From $\mathcal{F}(z) = \mathcal{X}(z) + \Psi(\mathcal{X}(z))$, we have
$$\begin{aligned}\frac{\mathcal{F}(q^{n+1} z)}{\mathcal{F}(q^n z)} &= \frac{\mathcal{X}(q^{n+1} z) + \Psi(\mathcal{X}(q^{n+1} z))}{\mathcal{X}(q^n z) + \Psi(\mathcal{X}(q^n z))} \\ &= \frac{\frac{\mathcal{X}(q^{n+1} z)}{\mathcal{X}(q^n z)} + \frac{\Psi(\mathcal{X}(q^{n+1} z))}{\mathcal{X}(q^{n+1} z)} \cdot \frac{\mathcal{X}(q^{n+1} z)}{\mathcal{X}(q^n z)}}{1 + \frac{\Psi(\mathcal{X}(q^n z))}{\mathcal{X}(q^n z)}} \to \lambda = q \quad \text{as } n \to +\infty.\end{aligned}$$

Conversely, if we put $\mathcal{F}(z)$ as (32), where π_0 is a constant and Ψ is a solution of (30), then $\mathcal{F}(z)$ is a solution of (1) such that $\mathcal{F}(q^n z) \to 0$ as $n \to +\infty$. Then we have a solution $\mathcal{X}$ of (27) such that
$$\mathcal{F}(z) = \mathcal{X}(z) + \Psi(\mathcal{X}(z)),$$

where $\mathcal{X}(q^n z) \to 0$ as $n \to +\infty$. Hence, $\frac{\mathcal{F}(q^{n+1} z)}{\mathcal{F}(q^n z)} \to \lambda = q$ as $n \to +\infty$. ∎

4. An example for Theorem 3.2

In this section we derive analytic general solutions of equation (1) in an example.

Example 4.1. We consider the following q-difference equation with $q = 1/2$:

$$f(q^2 z) = -\frac{1}{4} f(z) + f(qz) + f(z)^2 f(qz), \tag{51}$$

and an analytic solution such that $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $f(q^n z) \to 0$ as $n \to \infty$.

From the characteristic equation $D(\lambda) = \lambda^2 - \lambda + \frac{1}{4} = (\lambda - \frac{1}{2})^2 = 0$, we have $\lambda = \lambda_1 = \lambda_2 = 1/2 = q$. Set $p = 1$; we have $q^k \neq \lambda$ for any $k > p = 1$. From the function $v(x, y) = x^2 y$, we have $b_{i,j} = 0$ ($i \neq 2$ or $j \neq 1$), $b_{2,1} = 1$. From (10), we can take any value as a_1, and, from (7),

$$a_2 q^4 = -\frac{1}{4} a_2 + a_2 q^2,$$

i.e., $a_2((q^2)^2 - a_2 q^2 + \frac{1}{4}) = 0$. From $D(q^2) = D(1/4) \neq 0$, we have $a_2 = 0$. Further we have $a_3 q^6 = -\frac{1}{4} a_3 + a_3 q^3 + a_1^2 a_1 q$, i.e., $a_3((q^3)^2 - q^3 + \frac{1}{4}) = a_3 D(q^3) = q a_1^3$. $D(q^3) = D(1/8) \neq 0$, we can have $a_3 = q a_1^3 / D(q^3) \neq 0$. We can determine a_k for $k \geq 4$, and have a non trivial formal solution.

Making use of Theorem 2.1, we have the existence of an analytic solution f_1 such that

$$f_1(z) = \sum_{k=1}^{\infty} a_k z^k, \tag{52}$$

in a $S(\eta)$. Set

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = P \begin{pmatrix} x_1 \\ y_1 \end{pmatrix};$$

since from (25)

$$P^{-1} = \begin{pmatrix} 3/2 & -1 \\ -1/2 & 1 \end{pmatrix},$$

then we have

$$\begin{cases} x_1(qz) = \frac{1}{2} x_1(z) + y_1(z) - v(f_1(z), g_1(z)) \\ \qquad = \frac{1}{2} x_1(z) + y_1(z) - v(x_1(z) + y_1(z), \frac{1}{2} x_1(z) + \frac{3}{2} y_1(z)) \\ \qquad = \frac{1}{2} x_1(z) + y_1(z) - (x_1(z) + y_1(z))^2 \Big(\frac{1}{2} x_1(z) + \frac{3}{2} y_1(z)\Big) \\ \qquad = X(x_1(z), y_1(z)), \\ y_1(qz) = \frac{1}{2} y_1(z) + v(f_1(z), g_1(z)) \\ \qquad = \frac{1}{2} y_1(z) + v(x_1(z) + y_1(z), \frac{1}{2} x_1(z) + \frac{3}{2} y_1(z)) \\ \qquad = \frac{1}{2} y_1(z) + (x_1(z) + y_1(z))^2 \Big(\frac{1}{2} x_1(z) + \frac{3}{2} y_1(z)\Big) \\ \qquad = Y(x_1(z), y_1(z)), \end{cases} \tag{53}$$

and

$$x_1(z) = \frac{3}{2} f_1(z) - g_1(z) = \frac{3}{2} f_1(z) - f_1(qz). \tag{54}$$

Since $y_1(z) = \Psi(x_1(z))$, equation (30) is written as

$$(55) \quad \Psi\Big(\frac{1}{2}x_1(z) + \Psi(x_1(z)) - (x_1(z) + \Psi(x_1(z)))^2\Big(\frac{1}{2}x_1(z) + \frac{3}{2}\Psi(x_1(z))\Big)\Big) = \frac{1}{2}\Psi(x_1(z)) + \big(x_1(z) + \Psi(x_1(z))\big)^2\Big(\frac{1}{2}x_1(z) + \frac{3}{2}\Psi(x_1(z))\Big).$$

Making use of [14, Theorem 2], we have a holomorphic solution Ψ_1 of (55) in a domain $\mathbb{D} = \{z \mid |z| < \delta\}$,

Next, suppose $\mathcal{F}(z)$ is an analytic solution of (51) which satisfies (a), (aD), and (ab). From Theorem 3.2, we can write $\mathcal{F}(z)$ as

$$(56) \quad \mathcal{F}(z) = \frac{3}{2}\sum_{k=1}^{\infty} a_k(\pi_0 z)^k - \sum_{k=1}^{\infty} a_k\Big(\frac{1}{2}\pi_0 z\Big)^k + \Psi_1\Big(\frac{3}{2}\sum_{k=1}^{\infty} a_k(\pi_0 z)^k - \sum_{k=1}^{\infty} a_k\Big(\frac{1}{2}\pi_0\Big)^k\Big),$$

in $S(\eta)$ for a some $\eta > 0$, and we have

$$\frac{\mathcal{F}(q^{n+1}z)}{\mathcal{F}(q^z)} \to \frac{1}{2}.$$

Conversely, write a function $\mathcal{F}(z)$ as in (56) and let $\Psi_1(z)$ be a solution of (55) in a $\mathbb{D} = \{z \mid |z| < \delta\}$. Set $\mathcal{F}(qz) = \mathcal{G}(z)$, $\Psi_1(\mathcal{X}(z)) = \mathcal{Y}(z)$, and

$$\begin{pmatrix} \mathcal{F}(z) \\ \mathcal{G}(z) \end{pmatrix} = P\begin{pmatrix} \mathcal{X}(z) \\ \mathcal{Y}(z) \end{pmatrix}.$$

From (54),

$$\begin{aligned} \mathcal{F}(z) &= \frac{3}{2}f_1(\pi_0 z) - f_1(q\pi_0 z) + \Psi_1\Big(\frac{3}{2}f_1(\pi_0 z) - f_1(q\pi_0 z)\Big) \\ &= x_1(\pi_0 z) + \Psi_1(x_1(\pi_0 z)). \end{aligned}$$

Further, we can write $x_1(\pi_0 z) = \mathcal{X}(z)$ and

$$\mathcal{F}(z) = \mathcal{X}(z) + \mathcal{Y}(z) = \mathcal{X}(z) + \Psi_1(\mathcal{X}(z)).$$

From the assumptions on Ψ_1, $\mathcal{X}(z)$, and $\mathcal{Y}(z)$, the function $\mathcal{X}$, $\mathcal{Y}$ satisfies equation (53). That is, $\mathcal{F}(z) = \mathcal{X}(z) + \mathcal{Y}(z)$ is a solution of equation (51).

5. Conclusion

We summarise the contents of this paper as follows.

(1) In Theorem 2.1, we have shown the existence of an analytic solution of equation (1) when the characteristic values λ_1, λ_2 are equal, $\lambda_1 = \lambda_2 \neq 0$, and $|\lambda_1| = |\lambda_2| \neq 1$.

(2) We have general solutions of equation (1) when $q = \lambda_1 = \lambda_2$ in Theorem 3.2.

In the near future, we will study the nonlinear q-difference equation

$$f(q^2 z) = u(z, f(z), f(qz)).$$

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